PRACTICAL STABILITY AND VECTOR-LYAPUNOV FUNCTIONS FOR IMPULSIVE DIFFERENTIAL EQUATIONS WITH "SUPREMUM"

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Abstract. *Stability of nonlinear impulsive differential equations with "supremum" is studied. A special type of stability, combining two different measures and a dot product on a cone, is defined. Perturbing cone-valued piecewise continuous Lyapunov functions have been applied. Method of Razumikhin as well as comparison method for scalar impulsive ordinary differential equations have been employed.*

Keywords: stability in two measures, cone-valued piecewise continuous Lyapunov functions, impulsive differential equations with "supremum"

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1. Introduction

Many problems in the control theory correspond to the maximal deviation of the regulated quantity. Such kind of problems could be adequately modeled by differential equations that contain the maxima operator. At the same time many real processes are characterized by instantaneous changes of their state at certain moments and impulses are involved into the models. In the case when the equations contain maxima operator as well as impulses the equation is called impulsive differential equation with "supremum". Recently some stability problems in terms of two measures for impulsive equations are studied by Lyapunov's second method employing appropriate piecewise continuous Lyapunov's functions.

In the present paper the practical stability of the solutions of impulsive differential equations with "supremum" is studied. A new type of practical stability, combining two different measures ([4]) and dot product on a cone ([1]), is defined. In this paper, differently than the existing up to date results, two different measures are applied to both the given system and the comparison scalar equation. Cone-valued perturbing Lyapunov functions are employed as well as comparison results for scalar impulsive differential equations.

2. Preliminary notes and definitions

Let $\mathbb{R}^n$ be n-dimensional Euclidean space with a norm $||.||$ and $\mathbb{R}_+ = [0,\infty)$. Let $\{\tau_k\}_1^\infty$ be a sequence of fixed points in $\mathbb{R}_+$ such that $\tau_{k+1} > \tau_k$ and $\lim_{k\to\infty} \tau_k = \infty$.

Let $r > 0$ be a fixed constant.

Denote by $PC(X,Y)$ $(X\subset\mathbb{R}, Y\subset\mathbb{R}^n)$ the set of all functions $u: X\to Y$ that are piecewise continuous in X with points of discontinuity $\tau_k\in X$ and $u(\tau_k)=u(\tau_k-0)$.

We denote by $PC^1(X,Y)$ the set of all functions $u\in PC(X,Y)$ that are continuously differentiable for $t\in X,\ \ t\neq\tau_k$.

Consider the system of nonlinear impulsive differential equations with "supremum"

$$x'=f\Big(t,\,x(t),sup_{s\in[t-r,\,t]}x(s)\Big)\quad\text{for}\quad t\geq t_0,\ \ t\neq\tau_k, \tag{1}$$

$$x(\tau_k+0)=I_k\big(x(\tau_k-0)\big)\qquad\text{for}\quad k=1,\,2,\ldots, \tag{2}$$

where $x\in\mathbb{R}^n$, $t_0\in\mathbb{R}_+$, $f:\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$, $I_k:\mathbb{R}^n\to\mathbb{R}^n$, $k=1,\,2,\,3,\ldots$.

Let $\phi\in PC([t_0-r,t_0],\mathbb{R}^n)$. We denote by $x(t;t_0,\phi)$ the solution of system (1), (2) with an initial condition

$$x(t;t_0,\phi)=\phi(t-t_0),\ \ t\in[t_0-r,t_0],\qquad x(t_0+0;t_0,\phi)=\phi(0). \tag{3}$$

Will assume in our further inevestigations that for any initial function $\phi\in PC([t_0-r,t_0],\mathbb{R}^n)$ the solution of the initial value problem (1), (2), (3) exists on $[t_0-r,\infty)$.

Let $x,y\in\mathbb{R}^n$. Denote by $(x\bullet y)$ the dot product of both vectors x and y.

Let $\mathcal{K}\subset\mathbb{R}^n$ be a cone. Consider the set

$$\mathcal{K}^*=\Big\{\varphi\in\mathbb{R}^n:\ (\varphi\bullet x)\geq 0 \text{ for any }\ x\in\mathcal{K}\Big\}.$$

We note $\mathcal{K}^*$ is a cone.

We will define the following sets of measures:

$$\begin{aligned}
\Gamma &= \big\{h\in C\big([-r,\infty)\times\mathbb{R}^n,\mathcal{K}\big):\ \min_{x\in\mathbb{R}^n}h(t,x)=0 \text{ for each } t\in[-r,\infty)\big\},\\
\bar{\Gamma} &= \big\{h\in C\big(\mathbb{R},\mathbb{R}_+\big):\min_{u\in\mathbb{R}}h(u)=0\ \big\},\\
\tilde{\Gamma} &= \big\{h\in C\big(\mathbb{R}_+\times\mathbb{R},\mathbb{R}_+\big):\min_{u\in\mathbb{R}}h(t,u)=0 \text{ for each } t\in\mathbb{R}_+ \text{ and}\\
&\qquad h(t,u_1)\leq h(t,u_2)\ \text{ for }\ |\,u_1|\leq|\,u_2|,\ t\in\mathbb{R}_+\big\}.
\end{aligned}$$

Remark 1. Note that any norm in $\mathbb{R}^n$ is a function from Γ and any norm in $\mathbb{R}$ is from the both classes $\tilde{\Gamma}$ and $\bar{\Gamma}$. For example, the function $h^*(t,u)=e^{-t}|u|\in\tilde{\Gamma}$.

Let $h_0\in\Gamma$, $\varphi_0\in\mathcal{K}^*$, $t_0\in\mathbb{R}_+$ and $\phi\in PC\big([t_0-r,t_0],\mathbb{R}^n\big)$. Define

$$H_0(t_0,\phi,\varphi_0)=\sup\big\{\big(\varphi_0\bullet h_0(s,\phi(s))\big):\quad s\in[t_0-r,t_0]\big\}. \tag{4}$$

We will study practical stability in terms of two measures of impulsive differential equations with "supremum". In the case when cone-valued Lyapunov functions are applied both measures are from the set Γ. In this case we will introduce the definition of a new type of stability, that combines the ideas of stability in terms of two measures ([4]) and dot product on a cone.

Definition 1. Let the vector $\varphi_0 \in \mathcal{K}^*$, the measures $h, h_0 \in \Gamma$, and the positive constants λ, A be given. System of impulsive differential equations with "supremum" (1),(2) is said to be

(S1) *d-practically stable* in terms of both measures (h_0, h) with respect to (λ, A) if there exists $t_0 \in \mathbb{R}_+$ such that for any $\phi \in PC\big([t_0 - r, t_0], \mathbb{R}^n\big)$ inequality $H_0(t_0, \phi, \varphi_0) < \lambda$ implies $\big(\varphi_0 \bullet h(t, x(t; t_0, \phi))\big) < A$ for $t \geq t_0$, where the function H_0 is defined by (4), and $x(t; t_0, \phi)$ is a solution of (1), (2), (3);

(S2) *uniformly d-practically stable* in terms of both measures (h_0, h) with respect to (λ, A), if (S1) is satisfied for all $t_0 \in \mathbb{R}_+$.

In our further investigations we will use the following initial value problem for the comparison scalar impulsive ordinary differential equation:

$$u' = g(t, u), \quad t \neq \tau_k, \qquad u(\tau_k + 0) = \xi_k(u(\tau_k)), \quad k = 1, 2, \ldots, \tag{5}$$

$$u(t_0) = u_0, \tag{6}$$

where $u, u_0 \in \mathbb{R}$, $g: \ \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $\xi_k: \ \mathbb{R} \to \mathbb{R}$ for $k = 1, 2, \ldots$.

In our further inevestigations we will assume that for any initial point u_0 the solution $u(t; t_0, u_0)$ of the initial value problem (5), (6) exists on $[t_0, \infty)$.

We will give the definition for practical stability in terms of two measures for the comparison scalar impulsive ordinary differential equation (5).

Definition 2. Let the measures $h^* \in \tilde{\Gamma}, h_0^* \in \bar{\Gamma}$, and the positive constants λ, A be given. Impulsive differential equation (5) is said to be

(S3) *practically stable* in terms of both measures (h_0^*, h^*) with respect to (λ, A) if there exists $t_0 \in \mathbb{R}_+$ such that for any $u_0 \in \mathbb{R}$ inequality $h_0^*(u_0) < \lambda$ implies $h^*(t, u(t; t_0, u_0)) < A$ for $t \geq t_0$, where $u(t; t_0, u_0)$ is a solution of (5), (6);

(S4) *uniformly practically stable* in terms of both measures (h_0^*, h^*) with respect to (λ, A) if (S3) is satisfied for all $t_0 \in \mathbb{R}_+$.

We will introduce the following class of Lyapunov's functions:

Definition 3. We will say that the function $V(t, x) : [-r, \infty) \times \mathbb{R}^n \to \mathcal{K}$, $V = (V_1, V_2, \ldots, V_n)$, belongs to the class $\mathcal{L}$ if:

1. $V(t, x) \in PC^1([-r, \infty) \times \mathbb{R}^n, \mathcal{K})$;
2. For each $k = 1, 2, \ldots$ and $x \in \mathbb{R}^n$ there exist the finite limits

$$V(\tau_k, x) = V(\tau_k - 0, x) = \lim_{t \uparrow \tau_k} V(t, x), \qquad V(\tau_k + 0, x) = \lim_{t \downarrow \tau_k} V(t, x);$$

3. There exist constants $M_i > 0$, $i = 1, 2, \ldots, n$, such that for any $t \in \mathbb{R}_+$, $x, y \in \mathbb{R}^n$ the following inequality $|V_i(t, x) - V_i(t, y)| \leq M_i ||x - y||$ holds.

Let $t \neq \tau_k$, $(k = 1, 2, \ldots)$, function $V \in \mathcal{L}$, and $\phi \in PC([t-r, t], \mathbb{R}^n)$. We define a derivative $\mathcal{D}_{(1),(2)} V(t, x)$ of the function V along the system (1), (2) by the equalities

$$\begin{aligned}\mathcal{D}_{(1),(2)}V_i(t,\phi(t)) \quad = \quad & \frac{\partial V_i(t,\phi(t))}{\partial t} \\ & + \sum_{j=1}^{n} \frac{\partial V_i(t,\phi(t))}{\partial x_j} f_j\Big(t,\phi(t), sup_{s\in[-r,0]}\phi(t+s)\Big) \\ & \qquad\qquad i=1,2,\dots,n,\end{aligned}$$

where $\mathcal{D}_{(1),(2)}V(t,x) = (\mathcal{D}_{(1),(2)}V_1(t,x), \mathcal{D}_{(1),(2)}V_2(t,x), \dots, \mathcal{D}_{(1),(2)}V_n(t,x))$.

Consider following sets

$$K = \big\{a \in C[\mathbb{R}_+,\mathbb{R}_+] \ : \ a(s) \text{ is strictly increasing and } a(0)=0\big\};$$
$$K_1 = \big\{a \in C[\mathbb{R}_+,\mathbb{R}_+] \ : \ a \in K \ \text{ and } a(s) \geq s\big\}.$$

Let $\rho = const > 0$, $\varphi_0 \in \mathcal{K}^*$, $h \in \Gamma$. Consider the following set:

$$\mathcal{S}(h,\rho,\varphi_0) = \{(t,x) \in [0,\infty) \times \mathbb{R}^n : \quad (\varphi_0 \bullet h(t,x)) < \rho\}.$$

In the further investigations we will use the following comparison result:

Lemma 1. (Hristova [2]). Let the following conditions be fulfilled:

1. The vector $\varphi_0 \in \mathcal{K}^*$ and function $V \in \mathcal{L}$ are such that

(i) for any number $t \geq 0 : \ t \neq \tau_k$ and any function $\psi \in PC([t-r,t],\mathbb{R}^n)$ such that $(\varphi_0 \bullet V(t,\psi(t))) \geq (\varphi_0 \bullet V(t+s,\psi(t+s)))$ for $s \in [-r,0)$ the inequality

$$\Big(\varphi_0 \bullet \mathcal{D}_{(1),(2)}V(t,\psi(t))\Big) \leq g(t,(\varphi_0 \bullet V(t,\psi(t))))$$

holds, where $g \in PC(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$.

(ii) $\Big(\varphi_0 \bullet V(\tau_k+0, I_k(x))\Big) \leq \xi_k(\varphi_0 \bullet V(\tau_k,x)), \quad k=1,2,\dots, x \in \mathbb{R}^n$, where functions $\xi_k \in K_1$.

2. The solution $x(t) = x(t;t_0,\phi)$ of the initial value problem (1), (2), (3) is defined for $t \in [t_0-r,T]$, where $\phi \in PC([t_0-r,t_0],\mathbb{R}^n)$.

3. The function $u^*(t) = u^*(t;t_0,u_0)$ is the maximal solution of (5) with initial condition $u^*(t_0) = u_0$, which is defined for $t \in [t_0,T]$.

Then the inequality $\sup_{s\in[-r,0]}(\varphi_0 \bullet V(t_0+s,\phi(t_0+s))) \leq u_0$ implies the inequality $(\varphi_0 \bullet V(t,x(t))) \leq u^*(t)$ for $t \in [t_0,T]$.

3. Main results

We will obtain sufficient conditions for d-practical stability in terms of two measures of systems of impulsive differential equations with "supremum".

Theorem 1. Let the following conditions be fulfilled:

1. The function $f \in PC(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $f(t,0,0) \equiv 0$.
2. The functions $I_k \in C(\mathbb{R}^n,\mathbb{R}^n)$, $I_k(0)=0$, $(k=1,2,\dots)$.
3. The functions $h_0, h \in \Gamma$, $h^* \in \tilde{\Gamma}$, $h_0^* \in \bar{\Gamma}$.
4. The vector $\varphi_0 \in \mathcal{K}^*$.
5. There exists a function $V \in \mathcal{L}$ such that

(i) $b((\varphi_0 \bullet h(t,x)) \leq h^*(t,(\varphi_0 \bullet V(t,x)))$ and $h_0^*((\varphi_0 \bullet V(t,x))) \leq a((\varphi_0 \bullet h_0(t,x))$ for $(t,x) \in [-r,\infty) \times \mathbb{R}^n$, where $a, b \in K$;

(ii) for any number $t \geq 0 : t \neq \tau_k$ and any function $\psi \in PC([t-r,t],\mathbb{R}^n)$ such that $(\varphi_0 \bullet V(t,\psi(t))) \geq (\varphi_0 \bullet V(t+s,\psi(t+s)))$ for $s \in [-r,0)$ the inequality

$$\Big(\varphi_0 \bullet \mathcal{D}_{(1),(2)} V(t,\psi(t))\Big) \leq g\big(t,(\varphi_0 \bullet V(t,\psi(t)))\big)$$

holds, where $g \in PC(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$, $g(t,0) \equiv 0$.

(iii) $\Big(\varphi_0 \bullet V(\tau_k+0, I_k(x))\Big) \leq \xi_k((\varphi_0 \bullet V(\tau_k,x)))$, for $k = 1,2,\ldots,$ and $x \in \mathbb{R}^n$, where functions $\xi_k \in K_1$.

Then the (uniform) practical stability in terms of both measures (h_0^*, h^*) with respect to $(a(\lambda), b(A))$ of scalar impulsive differential equation (5) implies (uniform) d-practical stability in terms of both measures (h_0, h) with respect to (λ, A) of the system of impulsive differential equations with "supremum" (1), (2) where the positive constants λ, A are given.

Proof. Let scalar impulsive differential equation (5) be practically stable in terms of both measures (h_0^*, h^*) with respect to $(a(\lambda), b(A))$ Therefore there exists a point $t_0 \geq 0$ such that $h_0^*(u_0) < a(\lambda)$ implies

$$h^*(t, u(t;t_0,u_0)) < b(A) \qquad \text{for} \quad t \geq t_0. \tag{7}$$

Choose a function $\phi \in PC([t_0-r,t_0],\mathbb{R}^n)$ such that

$$H_0(t_0,\phi,\varphi_0) < \lambda. \tag{8}$$

Let $u_0 = \sup_{s\in[-r,0]}(\varphi_0 \bullet V(t_0+s,\phi(t_0+s)))$. From Lemma 1 for $T = \infty$ and $\Omega = \mathbb{R}^n$ it follows

$$(\varphi_0 \bullet V(t, x(t;t_0,\phi))) \leq u^*(t;t_0,u_0) \qquad \text{for} \quad t \geq t_0, \tag{9}$$

From condition 5(i) we obtain for all $s \in [-r,0]$

$$h_0^*((\varphi_0 \bullet V(t_0+s,\phi(t_0+s)))) \leq a((\varphi_0 \bullet h_0(t_0+s,\phi(t_0+s)))) < a(\lambda). \tag{10}$$

From inequalities (10) and

$$h_0^*(\sup_{s\in[-r,0]}(\varphi_0 \bullet V(t_0+s,\phi(t_0+s)))) \leq \sup_{s\in[-r,0]} h_0^*((\varphi_0 \bullet V(t_0+s,\phi(t_0+s)))$$

we obtain

$$h_0^*(u_0) < a(\lambda). \tag{11}$$

From condition 5(i) and inequalities (7), (9), (11) we get for $t \geq t_0$

$$\begin{aligned} b((\varphi_0 \bullet h(t,x(t;t_0,\phi)))) &\leq h^*(t,(\varphi_0 \bullet V(t,x(t;t_0,\phi)))) \\ &\leq h^*(t,u^*(t;t_0,u_0)) < b(A), \end{aligned}$$

or

$$(\varphi_0 \bullet h(t,x(t;t_0,\phi))) < A.$$

□

In the case when Lyapunov function does not satisfy globally the conditions 5(ii) and 5(iii) of Theorem 1, we obtain the following sufficient conditions:

Theorem 2. Let the following conditions be fulfilled:

1. The conditions 1 and 2 of Theorem 1 are satisfied.
2. The functions $h_0, h \in \Gamma$, $h_0^* \in \bar{\Gamma}$, $h^* \in \tilde{\Gamma}$, there exist positive constants λ, A and a function $\Psi \in K$, $\Psi(x) \leq x$ such that $(\varphi_0 \bullet h(t,x)) \leq \Psi((\varphi_0 \bullet h_0(t,x)))$ for $(t,x) \in \mathcal{S}(h_0,\lambda,\varphi_0)$, and $(\varphi_0 \bullet h(\tau_k,x)) < A$ implies $(\varphi_0 \bullet h(\tau_k, I_k(x))) < A$ for $x \in \mathbb{R}^n$, $k = 1,2,\dots$.
3. There exists a function $V(t,x) : \mathcal{S}(h,A,\varphi_0) \to \mathbb{R}_+$ with $V \in \Lambda$ such that
   - (i) $b((\varphi_0 \bullet h(t,x))) \leq h^*(t,(\varphi_0 \bullet V(t,x)))$ and $h_0^*((\varphi_0 \bullet V(t,x))) \leq a((\varphi_0 \bullet h_0(t,x)))$ for $(t,x) \in \mathcal{S}(h,A,\varphi_0)$, where $a, b \in K$;
   - (ii) for any number $t \in \mathbb{R}_+ :\ t \neq \tau_k,\ k = 1,2,\dots$ and any function $\psi \in PC([t-r,t],\mathbb{R}^n) :\ (t,\psi(t)) \in \mathcal{S}(h,A,\varphi_0)$ such that
   $$(\varphi_0 \bullet V(t,\psi(t))) > (\varphi_0 \bullet V(t+s,\psi(t+s)))$$
   for $s \in [-r,0)$ the inequality
   $$(\varphi_0 \bullet D_{(1),(2)}V(t,\psi(t))) \leq g(t,(\varphi_0 \bullet V(t,\psi(t))))$$
   holds, where $g \in PC(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$ and $g(t,0) \equiv 0$;
   - (iii) $(\varphi_0 \bullet V(\tau_k+0, I_k(x))) \leq \xi_k((\varphi_0 \bullet V(\tau_k,x)))$ for $(\tau_k,x) \in \mathcal{S}(h,A,\varphi_0)$, and $k = 1,2,\dots$, where $\xi_k \in \mathcal{K}_1$.

Then the (uniform) practical stability in terms of both measures (h_0^*, h^*) with respect to $(a(\lambda), b(A))$ of scalar impulsive differential equation (5) implies (uniform) d-practical stability in terms of both measures (h_0, h) with respect to (λ, A) of the system of impulsive differential equations with "supremum" (1), (2).

Proof. Let scalar impulsive differential equation (5) be practically stable in terms of both measures (h_0^*, h^*) with respect to $(a(\lambda), b(A))$. Therefore there exists a point $t_0 \geq 0$ such that $h_0^*(u_0) < a(\lambda)$ implies inequality (7).

Choose a function $\phi \in PC([t_0 - r, t_0], \mathbb{R}^n)$ such that (8) holds.

We will prove that for $t \geq t_0$

$$(\varphi_0 \bullet h(t, x(t;t_0,\phi))) < A. \tag{12}$$

From inclusion $(t,\phi(t)) \in \mathcal{S}(h_0,\lambda,\varphi_0)$ for $t \in [t_0 - r, t_0]$ and conditions 2 and 3(i) we get for $s \in [t_0 - r, t_0]$ the inequalities

$$(\varphi_0 \bullet h(s,\phi(s))) \leq \Psi\big((\varphi_0 \bullet h_0(s,\phi(s)))\big) \leq \Psi(H_0(t_0,\phi,\varphi_0)) < \Psi(\lambda) \leq \lambda < A.$$

Assume (12) does not hold for $t > t_0$.

Consider the following two cases:

Case 1. Let there exists a point $t^* > t_0$, $t^* \neq \tau_k$, $k = 1,2,\dots$ such that

$$(\varphi_0 \bullet h(t^*, x(t^*;t_0,\phi))) = A \text{ and } (\varphi_0 \bullet h(t, x(t;t_0,\phi))) < A,\ \ t \in [t_0 - r, t^*). \tag{13}$$

Let $u_0 = \sup_{s\in[-r,0]}(\varphi_0 \bullet V(t_0+s,\phi(t_0+s)))$. From Lemma 1 for the function $V(t,x)$ defined on the set $\{(t,x) \in [t_0,t^*] \times \mathbb{R}^n : (\varphi_0 \bullet h(t,x)) \leq A\}$ it follows the validity of the inequality

$$(\varphi_0 \bullet V(t,x(t;t_0,\phi))) \leq u^*(t;t_0,u_0) \quad \text{for } t \in [t_0,t^*]. \tag{14}$$

From condition $3(i)$ we obtain

$$h_0^*((\varphi_0 \bullet V(t_0+s,\phi(t_0+s)))) \leq a((\varphi_0 \bullet h_0(t_0+s,\phi(t_0+s)))) < a(\lambda), \quad s \in [-r,0]$$

or

$$h_0^*(u_0) < a(\lambda). \tag{15}$$

From inequalities (14), (15), the choice of the point t^*, and condition $3(i)$ we get

$$\begin{aligned} b(A) &= b((\varphi_0 \bullet h(t^*,x(t^*;t_0,\phi))) \leq h^*(t,(\varphi_0 \bullet V(t^*,x(t^*;t_0,\phi))) \\ &\leq h^*(t,u^*(t^*;t_0,u_0)) < b(A). \end{aligned}$$

The obtained contradiction proves the validity of (12) for $t > t_0$.

Case 2. Let there exists a natural number k such that the inequality $(\varphi_0 \bullet h(t,x(t;t_0,\phi))) < A$ hods for $t \in [t_0-r,\tau_k)$ and $(\varphi_0 \bullet h(\tau_k,x(\tau_k;t_0,\phi))) = A$. Then as in Case 1 for $t^* = \tau_k$ we obtain a contradiction.

The obtained contradictions prove the validity of (12) for $t > t_0$.

$\square$

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