

# A DICHOTOMY FOR ORDINARY DIFFERENTIAL EQUATIONS IN A BANACH SPACE

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**Abstract.** *A dichotomy similar property for a class of homogeneous differential equations in an arbitrary Banach space is introduced. By help of them, existence of quasi bounded solutions of the appropriate nonhomogeneous equation is proved.*

**Keywords:** Ordinary Differential Equations, Dichotomy

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## 1. Introduction

The notion of exponential and ordinary dichotomy is fundamental in the qualitative theory of ordinary differential equations. It is considered in the last 20 years in detail for example in the monographs [2]-[6].

In the given paper we introduce a dichotomy  $(D_1, D_2, M, N,)$  which is a generalisation of all dichotomies known by the authors. The aim of these paper is the research of this dichotomy and the connection between it and the existence of a special kind of solutions of the related nonhomogeneous differential equation.

An example for equation, who is  $(D_1, D_2, M, N)$  dichotomous but not classical dichotomous is given.

## 2. Problem statement

Let  $X$  be an arbitrary Banach space with norm  $|\cdot|$  and identity  $I$  and let be  $J = [c, \infty)$  where  $c \in \mathbb{R}$ . Let  $L(X)$  be the space of all linear bounded operators acting in  $X$  with the norm  $\|\cdot\|$ .

We consider the linear equation

$$(1) \quad \frac{dx}{dt} = A(t)x$$

where  $A(t) \in L(X), t \in J$ .

By  $V(t)$  we will denote the Cauchy operator of (1).

We consider the nonhomogeneous equation

$$(2) \quad \frac{dx}{dt} = A(t)x + f(t)$$

We say that the operators  $A_1(t), A_2(t) : X \rightarrow X (t \in J)$  satisfy the condition (H1) if the following relation is fulfilled:

$$(H1) \quad A_1(t) = R_1(t)V^{-1}(t), \quad A_2(t) = R_2(t)V^{-1}(t), \quad \text{where } R_1(t) + R_2(t) = I$$

**Lemma 1.** *Let the condition (H1) holds. Then the function*

$$(3) \quad x(t) = \int_c^t V(t)A_1(s)f(s)ds - \int_t^\infty V(t)A_2(s)f(s)ds$$

*is a solution of the equation (2) if the integrals in (3) exist.*

*Proof.*

$$x'(t) = A(t) \int_c^t V(t)A_1(s)f(s)ds + V(t)A_1(t)f(t) + V(t)A_2(t)f(t) - A(t) \int_t^\infty V(t)A_2(s)f(s)ds$$

$$x'(t) = A(t)x(t) + V(t)A_1(t)f(t) + V(t)A_2(t)f(t)$$

$$x'(t) = A(t)x(t) + (V(t)A_1(t) + V(t)A_2(t))f(t)$$

$$x'(t) = A(t)x(t) + V(t)(A_1(t) + A_2(t))f(t)$$

$$x'(t) = A(t)x(t) + f(t) \quad \square$$

We introduce the following conditions

$$(H2). \quad |V(t)A_1(s)z| \leq M(t, s, z), t \geq s, z \in X$$

$$(H3). \quad |V(t)A_2(s)z| \leq N(t, s, z), t < s, z \in X$$

**Definition 1.** *We call the equation (1) be a  $(A_1, A_2, M, N,)$  dichotomous if the conditions (H2), (H3) are fulfilled.*

Let  $a(t)$  is an arbitrary positive scalar function. We consider following Banach spaces :

$$K_a = \{g : J \rightarrow X : \sup_{t \in J} a(t) \int_c^t M(t, s, g(s))ds < \infty\}$$

$$L_a = \{g : J \rightarrow X : \sup_{t \in J} a(t) \int_t^\infty N(t, s, g(s))ds < \infty\}$$

$$C_a = \{g : J \rightarrow X : \sup_{t \in J} a(t) |g(t)| < \infty\}$$

### 3. Main results

#### 3.1. Estimates of the solution of (1).

**Theorem 1.** *Let the equation (1) be  $(A_1, A_2, M, N,)$  - dichotomous and let  $A_1(t), A_2(t)$  fulfilled the condition (H1). Then for every function  $f \in K_a \cap L_a$  the equation (2) has a solution in the space  $C_a$ .*

*Proof.* From Lemma 1 it follows

$$|a^{-1}(t)x(t)| \leq \int_c^t |V(t)D_1(s)f(s)|ds + \int_t^\infty |V(t)D_2(s)f(s)|ds,$$

$$|a^{-1}(t)x(t)| \leq \int_c^t M(t, s, f(s))ds + \int_t^\infty N(t, s, f(s))ds < \infty$$

□

For many important cases the right hand part of (H2) and (H3) has the form

$$(4) \quad \begin{cases} M(t, s, z) = \varphi_1(t)\varphi_2(s) |z|, & (t \geq s), z \in X \\ N(t, s, z) = \psi_1(t)\psi_2(s) |z|, & (t < s), z \in X \end{cases}$$

It is not hard to check that in case (4) the following estimate holds

$$\begin{aligned} \sup_{t \in J} \alpha^{-1}(t) |x(t)| &\leq \int_c^t \varphi_2(s) |f(s)| ds + \int_t^\infty \psi_2(s) |f(s)| ds = \\ &= \int_c^t \beta(s) |f(s)| ds + \int_t^\infty \beta(s) |f(s)| ds < \infty \end{aligned}$$

where

$$\alpha(t) = \max_{t \in J} \{\varphi_1(t), \psi_1(t)\}, \quad \beta(t) = \max_{t \in J} \{\varphi_2(t), \psi_2(t)\}$$

**Theorem 2.** *Let the following conditions are fulfilled:*

1. *The equation (1) be  $(A_1, A_2, M, N)$  - dichotomous.*
2. *The operators  $A_1(t), A_2(t)$  satisfy the condition (H1).*

*Then following estimates hold*

$$(5) \quad |x_1(t)| \leq M(t, s, x_1(s)), \quad t \geq s$$

for all solutions  $x_1(t)$  of (1), which started in the set

$$\bigcap_{s \in J} \text{Fix } R(s)$$

and

$$(6) \quad |x_2(t)| \leq N(t, s, x_2(s)), \quad t < s$$

for all solutions  $x_2(t)$  of (1), which started in the set

$$\bigcap_{s \in J} \text{Fix}(I - R(s))$$

*Proof.* The equalities

$$x_1(t) = V(t)x_1(0) = V(t)R(s)x_1(0) = V(t)R(s)V^{-1}(s)x_1(s)$$

imply following estimates

$$|x_1(t)| = |V(t)R(s)V^{-1}(s)x_1(s)| \leq M(t, s, x_1(s)), \quad (t \geq s)$$

The proof of (6) is analogously.

□

### 3.2. Considerations.

**3.2.1.** For  $R(s) = P(s \in J)$  where  $P : X \rightarrow X$  is a projector and  $M(t, s, z) = Ke^{-\delta(t-s)}$ ,  $t \geq s$ ,  $N(t, s, z) = Ke^{-\delta(t-s)}$ ,  $s > t$  we obtain the estimates 3.6a and 3.6b, chapter IV, §3 [2].

**3.2.2.** Let  $A_1(t) = PV^{-1}(t)$ ,  $A_2(t) = (I - P)V^{-1}(t)$  and  $R(t) = P$ ,  $P : X \rightarrow X$  is a projector.

For  $M(t, s, z) = Ke^{-\int_s^t \delta(\tau) d\tau}$  ( $t \geq s$ ) and  $N(t, s, z) = Ke^{-\int_t^s \delta(\tau) d\tau}$  ( $s > t$ ) we obtain the exponential dichotomy of [2] and [3] :

$$\|V(t)PV^{-1}(s)\| \leq Ke^{-\int_s^t \delta(\tau) d\tau} \quad (t \geq s)$$

and

$$\|V(t)(I - P)V^{-1}(s)\| \leq Ke^{-\int_t^s \delta(\tau) d\tau} \quad (s > t).$$

For  $\delta(t) = 0$  ( $0 \leq t < \infty$ ) we obtain the ordinary dichotomy of [2].

**3.2.3.** Let  $M(t, s, z) = Kh^{-1}(t)h(s)$  ( $t \geq s \geq 0$ ) and  $N(t, s, z) = Kk^{-1}(t)k(s)$  ( $0 \leq t \leq s$ )

For

$$\|V(t)PV^{-1}(s)\| \leq Kh^{-1}(t)h(s) \quad , (t \geq s \geq 0)$$

and

$$\|V(t)(I - P)V^{-1}(s)\| \leq Kk^{-1}(t)k(s) \quad , (0 \leq t \leq s)$$

we obtain the dichotomy of [6]-[8].

It may be also noted, that the dichotomies [1],[5],[6],[7],[8] are a generalisation of the dichotomy in [3].

### 3.4. Example.

Let be  $X = \mathbb{R}$ ,  $x' = a(t)x$ ,  $a(t)$  is a piecewise continuous function. In this case we have

$$V(t) = \exp \int_{t_0}^t a(s) ds.$$

It is well known, that this equation in several cases is not dichotomous or exponential dichotomous.

The conditions (H2) and (H3) have respectively the forms

$$|V(t)r_1(s)V^{-1}(s)\xi| \leq \alpha(t)\beta(s)|\xi| \quad (t \geq s),$$

$$|V(t)r_2(s)V^{-1}(s)\xi| \leq \alpha(t)\beta(s)|\xi| \quad (t < s).$$

We set  $r_1(t) + r_2(t) = 1$ , ( $t \geq 0$ ). Now we have

$$|r_1(s)V^{-1}(s)\xi| \leq \beta(s)|\xi|, \quad |r_2(s)V^{-1}(s)\xi| \leq \beta(s)|\xi|,$$

i.e.  $\beta(t) \geq \max\{r_1(t)V^{-1}(t), r_2(t)V^{-1}(t)\} \geq V^{-1}(t)$ .

We take  $\alpha(t) \geq |V(t)|$ , ( $t \geq c$ ). Because condition (H1) holds,

$$x(t) = \int_c^t V(t)r_1(s)V^{-1}(s)ds - \int_t^\infty V(t)r_2(s)V^{-1}(s)ds$$

is a solution of the nonhomogeneous equation

$$\frac{dx}{dt} = a(t)x .$$

The function  $x(t)$  belongs to the space  $C_a$  for every  $f \in L_a = K_a$ .

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