COMPUTATION OF SOME DIFFERENTIAL OPERATORS IN TORIC COORDINATES[1]

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Abstract. *Toric coordinates and toric vector field have been introduced in* [2]*. Let* $\boldsymbol{A}$ *be an arbitrary vector field. We obtain formulae for the* $\mathrm{div}\boldsymbol{A}$, $\mathrm{rot}\boldsymbol{A}$ *and the Laplace operator in toric coordinates.*

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1. Introduction

First we remind the formulae for the toric coordinates:

$$(1)\qquad \left|\begin{array}{ll} x = (a + r\cos(u+v))\cos u & \\ y = (a + r\cos(u+v))\sin u & \\ z = r\sin(u+v), & u, v \in [0, 2\pi];\ 0 < r < a \end{array}\right. .$$

The introduction of toric coordinates is motivated by the exploration of toric magnetics in electro-techniques. The vector field $\vec{t}$ is is defined as a toric vector field, if $dr(\vec{t}) = 0$. Toric coordinates could also be used as a transformation of differential equations.

We will denote $a + r\cos(u+v)$ by ρ. The toric coordinate system is not orthogonal. Let the local unit basic vectors be $\mathbf{t^0}$ (tangent to the u-lines), $\mathbf{e_v}$ and $\mathbf{e_r}$. The coordinates of $\mathbf{t}$ are

$$(2)\qquad \left|\begin{array}{l} x = -a\sin u - r\sin(2u+v) \\ y = a\cos u + r\cos(2u+v), \qquad |\mathbf{t}| = \sqrt{r^2+\rho^2}\ . \\ z = r\cos(u+v) \end{array}\right.$$

We will use the helpful vector $\mathbf{e_p}$ which is tangent to the parallel in the point M. This vector is orthogonal to $\mathbf{e_v}$. The coordinates in the system $\mathbf{e_p}$, $\mathbf{e_v}$, $\mathbf{e_r}$ will be denoted by prime above. Our first result is that the projections of the vector $\mathbf{t}$ onto $\mathbf{e_p}$ and $\mathbf{e_v}$ are respectively ρ and r. So we have

$$(3)\qquad \mathbf{t} = \rho\,\mathbf{e_p} + r\,\mathbf{e_v}\ , \quad \left(\mathbf{e_p} = \frac{1}{\rho}\,\mathbf{t} - \frac{r}{\rho}\,\mathbf{e_v}\right).$$

We can immediately write an expression for $div\mathbf{A}$ and $rot\mathbf{A}$ in the system $(\mathbf{e_p}, \mathbf{e_v}, \mathbf{e_r})$ of crooked-line coordinates. Then using a suitable transformation and (2) we obtain our formulae.

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2. Computation of $div\mathbf{A}$

As we wrote above, let U, V, r be the usual orthogonal toric coordinates and $\mathbf{A}\left(A_1^{'}, A_2^{'}, A_3^{'}\right)$ be a vector field.

Instead of (1) we have $\left| \begin{array}{l} x = (a + r\cos V)\cos U \\ y = (a + r\cos V)\sin U \\ z = r\sin V \end{array} \right.$.

We obtain $H_1^{'} = \rho$, $H_2^{'} = r$, $H_3^{'} = 1$ for the coefficients of Lammè, [1]. Therefore,

$$div\mathbf{A} = \frac{1}{r\rho}\left[\frac{\partial(A_1^{'}r)}{\partial U} + \frac{\partial(A_2^{'}\rho)}{\partial V} + \frac{\partial(A_3^{'}r\rho)}{\partial r}\right]. \tag{4}$$

We must do the substitution $U = u$, $V = v + u$. The derivative $\dfrac{\partial}{\partial U}$ must be replaced by $\dfrac{\partial}{\partial u} - \dfrac{\partial}{\partial v}$.

For the coordinates of $\mathbf{A}$ we obtain $A_1^{'} = A_1\rho$, $A_2^{'} = A_2 + rA_1$. Then

$$div\mathbf{A} = \frac{1}{r\rho}\left[\frac{\partial(A_1 r\rho)}{\partial U} + \frac{\partial(A_2\rho + r\rho A_1)}{\partial V} + \frac{\partial(A_3 r\rho)}{\partial r}\right] =$$

$$= \frac{1}{r\rho}\left[\frac{\partial(A_1 r\rho)}{\partial u} + \frac{\partial(A_2\rho)}{\partial v} + \frac{\partial(A_3 r\rho)}{\partial r}\right].$$

3. Computation of $rot\mathbf{A}$

We will again use the coordinates U, V, r. As a private case of crooked-line coordinates we obtain that

$$\left(rot\,\mathbf{A}\right)_1^{'} = \frac{1}{H_2^{'}H_3^{'}}\left[\frac{\partial(A_3^{'}H_3^{'})}{\partial q_2} - \frac{\partial(A_2^{'}H_2^{'})}{\partial q_3}\right] = \tag{5}$$

$$= \frac{1}{r}\left[\frac{\partial(A_3^{'})}{\partial v} - \frac{\partial(A_2^{'}r)}{\partial r}\right] = \frac{1}{r}\left[\frac{\partial(A_3)}{\partial v} - \frac{\partial(A_2 + rA_1)r}{\partial r}\right],$$

$$\left(rot\,\mathbf{A}\right)_2^{'} = \frac{1}{\rho}\left[\frac{\partial(A_1^{'}\rho)}{\partial r} - \frac{\partial A_3}{\partial U}\right] = \frac{1}{\rho}\left[\frac{\partial(A_1\rho^2)}{\partial r} - \frac{\partial A_3}{\partial u} + \frac{\partial A_3}{\partial v}\right],$$

$$\left(rot\,\mathbf{A}\right)_3^{'} = \frac{1}{r\rho}\left[\frac{\partial(A_2^{'}r)}{\partial U} - \frac{\partial(A_1^{'}\rho)}{\partial v}\right] =$$

$$= \frac{1}{r\rho}\left[\frac{\partial(A_2 + rA_1)r}{\partial u} - \frac{\partial(A_2 + rA_1)r}{\partial v} - \frac{\partial(A_1\rho^2)}{\partial v}\right], \text{ see [1].}$$

To obtain the final result for $rot\,\mathbf{A}$ we must multiply the above coordinates with $\dfrac{1}{\rho}\,\mathbf{t} - \dfrac{r}{\rho}\,\mathbf{e_v}$, $\mathbf{e_v}$ and $\mathbf{e_r}$, respectively, and put them in order. Some summands disappear. Here are the calculations:

$$\frac{1}{r}\left[\frac{\partial(A_3)}{\partial v} - \frac{\partial(A_2 + rA_1)r}{\partial r}\right]\left(\frac{1}{\rho}\,\mathbf{t} - \frac{r}{\rho}\,\mathbf{e_v}\right) +$$

$$+\frac{1}{\rho}\left[\frac{\partial(A_1\rho^2)}{\partial r} - \frac{\partial A_3}{\partial u} + \frac{\partial A_3}{\partial v}\right]\mathbf{e_v} +$$

$$+\frac{1}{r\rho}\left[\frac{\partial(A_2 + rA_1)r}{\partial u} - \frac{\partial(A_2 + rA_1)r}{\partial v} - \frac{\partial(A_1\rho^2)}{\partial v}\right]\mathbf{e_r} =$$

$$= \frac{1}{r}\left[\frac{\partial A_3}{\partial v} - \frac{\partial(A_2 + rA_1)r}{\partial r}\right]\frac{1}{\rho}\,\mathbf{t} +$$

$$+\frac{1}{\rho}\left[\frac{\partial(A_2 + rA_1)r}{\partial r} + \frac{\partial(A_1\rho^2)}{\partial r} - \frac{\partial A_3}{\partial u}\right]\mathbf{e_v} +$$

$$+\frac{1}{r\rho}\left[\frac{\partial(A_2 + rA_1)r}{\partial u} - \frac{\partial(A_2 + rA_1)r}{\partial v} - \frac{\partial(A_1\rho^2)}{\partial v}\right]\mathbf{e_r},$$

which is the final formula for $rot\,\mathbf{A}$ in toric coordinates.

Remark 3.1. If the vector field $\mathbf{A}$ is a toric one, then $A_3 \equiv 0$ and (5) can be simplified. For example, we can compute the $rot\,\mathbf{T}$ of the field of the tangent vectors $\mathbf{t}$, given by (2). Obviously, $\mathbf{T}$ is the field $(1, 0, 0)$ (in toric coordinates).

By (5) we obtain that

$$rot\,\mathbf{T} = -\frac{2}{\rho}\,\mathbf{t} + \frac{1}{\rho}\Big(2r + 2\rho\cos(u+v)\Big)\,\mathbf{e_v} - \frac{2}{r}\,\cos(u+v)\,\mathbf{e_r}.$$

4. Computation of the Laplace's operator

We must do the substitution (1) in the expression $w''_{xx} + w''_{yy} + w''_{zz}$ where $w = w(x, y, z)$.

First step. We will substitute x and y by ρ and u using the first two formulae in (1). As this is done in polar coordinates we can immediately write

$$w''_{xx} + w''_{yy} = \frac{1}{\rho^2}\,w''_{uu} + \frac{1}{\rho}\,w'_{\rho} + w''_{\rho\rho}. \tag{6}$$

Second step. We must replace ρ and z by r and $u + v$ using the formulae

$$\left|\begin{array}{l} \rho = a + r\cos(u+v) \\ z = r\sin(u+v) \end{array}\right. .$$

We obtain

(7) $$w''_{\rho\rho} + w''_{zz} = w''_{rr} + \frac{1}{r}\, w'_r + \frac{1}{r^2}\, w''_{vv} = \frac{1}{r}\left(rw'_r\right)'_r + \frac{1}{r^2}\, w''_{vv}.$$

Then we sum (6) and (7):

(8) $$w''_{xx} + w''_{yy} + w''_{zz} = \frac{1}{\rho^2}\, w''_{uu} + \frac{1}{\rho}\, w'_\rho + \frac{1}{r}\left(rw'_r\right)'_r + \frac{1}{r^2}\, w''_{vv}.$$

Third step. Computation of the term $\frac{1}{\rho}\, w'_\rho$:

We differentiate $w(u, v, r)$ by ρ, $(u = const)$

(9) $$w'_\rho = w'_r r'_\rho + w'_v (u+v)'_\rho\,,$$

where $r'_\rho = \cos(u+v)$ and $(u+v)'_\rho = -\dfrac{sin(u+v)}{r}$.

So,

$$\Delta w = \frac{1}{\rho^2}\, w''_{uu} + \frac{1}{r}\left(rw'_r\right)'_r + w'_r\, \frac{\cos(u+v)}{\rho} - w'_v\, \frac{\sin(u+v)}{r\rho} + \frac{1}{r^2}\, w''_{vv}.$$

The last equality can be finally written in the following way

$$\Delta w = \frac{1}{\rho^2}\, w''_{uu} + \frac{1}{r}\left(rw'_r\right)'_r + w'_r\, \frac{\cos(u+v)}{\rho} + \frac{1}{r^2\rho}\left(\rho w'_v\right)'_v.$$

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