CONJUGATE COMPOSITIONS IN EVEN-DIMENSIONAL AFFINELY CONNECTED SPACES WITHOUT A TORSION

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Abstract. *Let in even-dimensional affinely connected space without a torsion A_{2m} be given a composition $X_m \times \overline{X}_m$ by the affinor a^β_α. The affinor b^β_α, determined with the help of the eigen-vectors of the matrix (a^β_α), defines the second composition $Y_m \times \overline{Y}_m$. Conjugate compositions are introduced by the condition: the affinors of any of both compositions transform the vectors from the one position of the composition, generated by the other affinor, in the vectors from the another its position. It is proved that the compositions define by affinors a^β_α and b^β_α are conjugate. It is proved also that if the composition $X_m \times \overline{X}_m$ is Cartesian and composition $Y_m \times \overline{Y}_m$ is Cartesian or chebyshevian, or geodesic than the space A_{2m} is affine.*

Keywords: affinely connected space, net, composition,conjugate, chebyshevian, Cartesian, geodesic compositions

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1. Preliminary

Let A_N be an affinely connected space without a torsion, i.e. with a symmetric affinely connectedness, define by the coefficients $\Gamma^\sigma_{\alpha\beta}$. According to [2] the space A_N assumes a composition $X_n \times X_m$ of two base manyfolds X_n and X_m $(n+m=N)$ if and only if there exists an affinor a^β_α, such that

$$a^\beta_\alpha a^\sigma_\beta = \delta^\sigma_\alpha. \tag{1}$$

This space will be denoted $A_N(X_n \times X_m)$ and a^β_α will be called the affinor of the composition $A_N(X_n \times X_m)$ [2]. Two positions $P(X_n), P(X_m)$ of the base manyfolds pass through any point of $A_N(X_n \times X_m)$.

We shall consider an affinely connected spaces $A_N(X_n \times X_m)$ with integrable structure of the compositions. According to [3], [5] the integrability condition of the structure is characterized with the equality

$$a^\sigma_\beta \nabla_{[\alpha} a^\nu_{\sigma]} - a^\sigma_\alpha \nabla_{[\beta} a^\nu_{\sigma]} = 0. \tag{2}$$

For the projecting affinors $\overset{n}{a}{}^{\beta}_{\alpha}$, $\overset{m}{a}{}^{\beta}_{\alpha}$, define by the conditions $\overset{n}{a}{}^{\beta}_{\alpha} = \frac{1}{2}(\delta^\beta_\alpha + a^\beta_\alpha)$, $\overset{m}{a}{}^{\beta}_{\alpha} = \frac{1}{2}(\delta^\beta_\alpha - a^\beta_\alpha)$, the following equalities are fulfilled: $\overset{n}{a}{}^{\beta}_{\alpha}\overset{n}{a}{}^{\sigma}_{\beta} = \overset{n}{a}{}^{\sigma}_{\alpha}$, $\overset{m}{a}{}^{\beta}_{\alpha}\overset{m}{a}{}^{\sigma}_{\beta} = \overset{m}{a}{}^{\sigma}_{\alpha}$, $\overset{n}{a}{}^{\beta}_{\alpha}\overset{m}{a}{}^{\sigma}_{\beta} = \overset{m}{a}{}^{\beta}_{\alpha}\overset{n}{a}{}^{\sigma}_{\beta} = 0$ [3], [4] .

According to [4] for an arbitrary vector $v^\alpha \in A_N$ we have $v^\alpha = \overset{n}{a}{}^{\alpha}_{\sigma} v^\sigma + \overset{m}{a}{}^{\alpha}_{\sigma} v^\sigma = \overset{n}{V}{}^{\alpha} + \overset{m}{V}{}^{\alpha}$, where $\overset{n}{V}{}^{\alpha} = \overset{n}{a}{}^{\alpha}_{\sigma} v^\sigma \in P(X_n)$, $\overset{m}{V}{}^{\alpha} = \overset{m}{a}{}^{\alpha}_{\sigma} v^\sigma \in P(X_m)$.

Following [3] we will write the known characteristics for the affinor of some special compositions $X_n \times X_m$:

Proposition 1. The positions $P(X_n)$ and $P(X_m)$ of the c, c-composition (Cartesian) $X_n \times X_m$ are parallelly translated along any line in the space if and only if $\nabla_\alpha a_\beta^\sigma = 0$.

Proposition 2. The positions $P(X_n)$ and $P(X_m)$ of the ch, ch-composition (Chebyshevian) $X_n \times X_m$ are parallelly translated along the lines of X_m and X_n, respectively if and only if $\nabla_{[\alpha} a_{\beta]}^\sigma = 0$.

Proposition 3. The positions $P(X_n)$ and $P(X_m)$ of the g, g-composition (geodesic) $X_n \times X_m$ are parallelly translated along the lines of X_n and X_m, respectively if and only if $a_\alpha^\sigma \; \nabla_\sigma a_\beta^\nu + a_\beta^\sigma \; \nabla_\alpha a_\sigma^\nu = 0$.

2. Conjugate compositions in affinely connected spaces without a torsion A_{2m}

Let the affinor a_α^β defines a composition $X_m \times \overline{X}_m$ in affinely connected spaces without a torsion A_{2m}.

Let accept:

(3)
$$\alpha, \beta, \gamma, \sigma, \nu \in \{1, 2, \ldots, 2m\}; \qquad i,\, j,\, k,\, p,\, q,\, r,\, s \in \{1, 2, \ldots, m\};$$
$$\bar{i},\, \bar{j},\, \bar{k},\, \bar{p},\, \bar{q},\, \bar{r},\, \bar{s} \in \{m+1, m+2, \ldots, 2m\}.$$

Let $\underset{1}{v}^{\alpha}, \underset{2}{v}^{\alpha}, \ldots, \underset{m}{v}^{\alpha}, \ldots, \underset{2m}{v}^{\alpha}$ be the eigen-vectors of the matrix (a_α^β), as

$$a_\alpha^\beta \, \underset{s}{v}^{\alpha} = \underset{s}{v}^{\beta}, \quad a_\alpha^\beta \, \underset{\bar{s}}{v}^{\alpha} = -\underset{\bar{s}}{v}^{\beta} \, . \tag{4}$$

They define the net $\left(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{2m}{v}\right)$.

The reciprocal covectors $\overset{\alpha}{v}_\sigma (\alpha = 1,\, 2,\, \ldots,\, 2m)$ are defined by the equalities

$$\underset{\alpha}{v}^{\beta} \, \overset{\alpha}{v}_\sigma = \delta_\sigma^\beta \iff \underset{\alpha}{v}^{\beta} \, \overset{\sigma}{v}_\beta = \delta_\alpha^\sigma \, . \tag{5}$$

Following the paper [6], we can consider the affinor a_α^β of the composition $X_m \times \overline{X}_m$ as an affinor, associated with the net $\left(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{2m}{v}\right)$. Therefore a_α^β has the presentation

$$a_\alpha^\beta = \underset{1}{v}^{\beta} \, \overset{1}{v}_\alpha + \cdots + \underset{m}{v}^{\beta} \, \overset{m}{v}_\alpha - \underset{m+1}{v}^{\beta} \, \overset{m+1}{v}_{\alpha} - \cdots - \underset{2m}{v}^{\beta} \, \overset{2m}{v}_{\alpha} = \underset{i}{v}^{\beta} \, \overset{i}{v}_\alpha - \underset{\bar{i}}{v}^{\beta} \, \overset{\bar{i}}{v}_\alpha \, . \tag{6}$$

Now according to [6] for the projecting affinors we have $\overset{m}{a}{}_\alpha^\beta = \underset{i}{v}^{\beta} \, \overset{i}{v}_\alpha$, $\overset{\overline{m}}{a}{}_\alpha^\beta = \underset{\bar{i}}{v}^{\beta} \, \overset{\bar{i}}{v}_\alpha$. Let net $\left(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{2m}{v}\right)$ be chosen as a coordinate one. Then we have

$$\underset{1}{v}^{\sigma}(1, 0, \ldots, 0), \; \underset{2}{v}^{\sigma}(0, 1, \ldots, 0), \ldots, \; \underset{2m}{v}^{\sigma}(0, 0, \ldots, 0, 1). \tag{7}$$

Let consider the vectors

$$\underset{i}{w}^{\alpha} = \underset{i}{v}^{\alpha} + \underset{m+i}{v}^{\alpha}, \quad \underset{m+i}{w}^{\alpha} = \underset{i}{v}^{\alpha} - \underset{m+i}{v}^{\alpha}. \tag{8}$$

The reciprocal covectors $\overset{\alpha}{w}_{\sigma}(\alpha = 1, 2, \dots, 2m)$ are defined by the equalities

$$\underset{\alpha}{w}^{\beta} \overset{\alpha}{w}_{\sigma} = \delta_{\sigma}^{\beta} \iff \underset{\alpha}{w}^{\beta} \overset{\sigma}{w}_{\beta} = \delta_{\alpha}^{\sigma} . \tag{9}$$

Let introduce the affinor

$$b_{\alpha}^{\beta} = \underset{i}{w}^{\beta} \overset{i}{w}_{\alpha} - \underset{\bar{i}}{w}^{\beta} \overset{\bar{i}}{w}_{\alpha}. \tag{10}$$

From (9), (10) we obtain $b_{\alpha}^{\beta} b_{\beta}^{\sigma} = \delta_{\alpha}^{\sigma}$. Hence the affinor b_{α}^{β} defines a composition $Y_m \times \overline{Y}_m$. We denote by $P(Y_m)$ and $P(\overline{Y}_m)$ the positions of this composition. Using (9), (10) we establish

$$b_{\alpha}^{\beta} \underset{s}{w}^{\alpha} = \underset{s}{w}^{\beta}, \quad b_{\alpha}^{\beta} \underset{\bar{s}}{w}^{\alpha} = -\underset{\bar{s}}{w}^{\beta}, \tag{11}$$

from where it follows that $\underset{s}{w}^{\alpha}$ and $\underset{\bar{s}}{w}^{\alpha}$ are the eigen-vectors of the matrix (b_{α}^{β}). According to [6] the projecting affinors of the composition $Y_m \times \overline{Y}_m$ have the following form $\overset{m}{b}{}_{\alpha}^{\beta} = \underset{i}{w}^{\beta} \overset{i}{w}_{\alpha}$, $\overset{\overline{m}}{b}{}_{\alpha}^{\beta} = \underset{\bar{i}}{w}^{\beta} \overset{\bar{i}}{w}_{\alpha}$.

Definition 1. The compositions $X_m \times \overline{X}_m$ and $Y_m \times \overline{Y}_m$ be called conjugate if

1) for arbitrary vectors $v^{\alpha} \in P(X_m)$ and $\overline{v}^{\alpha} \in P(\overline{X}_m)$ are fulfilled $b_{\alpha}^{\beta} v^{\alpha} \in P(\overline{X}_m)$ and $b_{\alpha}^{\beta} \overline{v}^{\alpha} \in P(X_m)$;

2) for arbitrary vectors $u^{\alpha} \in P(Y_m)$ and $\overline{u}^{\alpha} \in P(\overline{Y}_m)$ are fulfilled $a_{\alpha}^{\beta} u^{\alpha} \in P(\overline{Y}_m)$ and $a_{\alpha}^{\beta} \overline{u}^{\alpha} \in P(Y_m)$.

Theorem 1. The compositions $X_m \times \overline{X}_m$, define by the affinor (6) and associated with the net $\left(\underset{1}{v}, \underset{2}{v}, \dots, \underset{2m}{v}\right)$ and the composition $Y_m \times \overline{Y}_m$, define by the affinors (10) are conjugate.

Proof: With the help of (4), (6) and (8) we find

$$\begin{aligned} a_{\alpha}^{\beta} \underset{s}{w}^{\alpha} &= a_{\alpha}^{\beta} \left(\underset{s}{v}^{\alpha} + \underset{s+m}{v}^{\alpha} \right) = \underset{s}{v}^{\alpha} - \underset{s+m}{v}^{\alpha} = \underset{\bar{s}}{w}^{\alpha} , \\ a_{\alpha}^{\beta} \underset{\bar{s}}{w}^{\alpha} &= a_{\alpha}^{\beta} \left(\underset{\bar{s}-m}{v}^{\alpha} - \underset{\bar{s}}{v}^{\alpha} \right) = \underset{\bar{s}-m}{v}^{\alpha} + \underset{\bar{s}}{v}^{\alpha} = \underset{s}{w}^{\alpha} . \end{aligned} \tag{12}$$

Now if an arbitrary vector $v^{\alpha} \in P(Y_m)$, then $v^{\alpha} = \overset{i}{\lambda} \underset{i}{w}^{\alpha}$, where $\overset{i}{\lambda}$ are functions of the point and $a_{\alpha}^{\beta} v^{\alpha} = \overset{i}{\lambda} a_{\alpha}^{\beta} \underset{i}{w}^{\alpha}$. Taking into account (12) we can write $a^{\ \beta}_{\alpha}\, v^{\alpha} = \overset{1}{\lambda} \underset{m+1}{w}{}^{\beta} + \overset{2}{\lambda} \underset{m+2}{w}{}^{\beta} + \ldots, \overset{2m}{\lambda} \underset{2m}{w}{}^{\beta} = \overset{i}{\lambda} \underset{m+i}{w}{}^{\beta}$, which means that $a^{\ \beta}_{\alpha}\, v^{\alpha} \in P(\overline{Y}_m)$.

So we proved that from $v^{\alpha} \in P(Y_m)$ it follows $a^{\ \beta}_{\alpha}\, v^{\alpha} \in P(\overline{Y}_m)$. The proof of the proposition - from $v^{\alpha} \in P(\overline{Y}_m)$ it follows $a^{\ \beta}_{\alpha}\, v^{\alpha} \in P(Y_m)$ - is similar. From (5), (8) and (9) we obtain
(13)

$$\begin{aligned} &\overset{s}{w}_{\alpha}\, \underset{s}{v}^{\alpha} = \frac{1}{2}\ , \quad \overset{s}{w}_{\alpha}\, \underset{s+m}{v}{}^{\alpha} = \frac{1}{2}\ , \quad \overset{s}{w}_{\alpha}\, \underset{k}{v}^{\alpha} = 0\ , \quad \overset{s}{w}_{\alpha}\, \underset{k+m}{v}{}^{\alpha} = 0\ , \quad s \neq k\ ; \\ &\overset{\bar{s}}{w}_{\alpha}\, \underset{\bar{s}}{v}^{\alpha} = -\frac{1}{2}\ , \quad \overset{\bar{s}}{w}_{\alpha}\, \underset{\bar{s}-m}{v}{}^{\alpha} = \frac{1}{2}\ , \quad \overset{\bar{s}}{w}_{\alpha}\, \underset{\bar{k}}{v}^{\alpha} = 0\ , \quad \overset{\bar{s}}{w}_{\alpha}\, \underset{\bar{k}-m}{v}{}^{\alpha} = 0\ , \quad \bar{s} \neq \bar{k}\ . \end{aligned}$$

With the help of (8), (10) and (13) we find

$$(14)\ \ b^{\ \beta}_{\alpha}\, \underset{s}{v}^{\alpha} = \frac{1}{2}\, \underset{s}{w}^{\beta} - \frac{1}{2}\, \underset{s+m}{w}{}^{\beta} = \underset{s+m}{v}{}^{\beta}\ , \quad b^{\ \beta}_{\alpha}\, \underset{\bar{s}}{v}^{\alpha} = \frac{1}{2}\, \underset{\bar{s}-m}{w}{}^{\beta} + \frac{1}{2}\, \underset{\bar{s}}{w}^{\beta} = \underset{\bar{s}-m}{v}{}^{\beta}\ .$$

Now if an arbitrary vector $v^{\alpha} \in P(X_m)$, then $v^{\alpha} = \overset{i}{\mu}\, \underset{i}{v}^{\alpha}$ where $\overset{i}{\mu}$ are functions of the point and $b^{\ \beta}_{\alpha}\, v^{\alpha} = \overset{i}{\mu}\, b^{\ \beta}_{\alpha}\, \underset{i}{v}^{\alpha}$. Taking into account (14) we can write $b^{\ \beta}_{\alpha}\, v^{\alpha} = \overset{i}{\mu}\, \underset{m+i}{v}{}^{\alpha}$, which means $b^{\ \beta}_{\alpha}\, v^{\alpha} \in P(\overline{X}_m)$.

The proof of the proposition - from $v^{\alpha} \in P(\overline{X}_m)$ it follows $b^{\ \beta}_{\alpha}\, v^{\alpha} \in P(X_m)$ - is similar. □

Let consider the affinor $c^{\ \beta}_{\alpha} = -a^{\ \beta}_{\sigma}\, b^{\ \sigma}_{\alpha}$. From (5), (6), (8) and (10) we obtain

$$(15) \qquad c^{\ \beta}_{\alpha} = -a^{\ \beta}_{\sigma}\, b^{\ \sigma}_{\alpha} = \underset{i}{w}^{\beta}\, \overset{m+i}{w}_{\alpha} - \underset{m+i}{w}{}^{\beta}\, \overset{i}{w}_{\alpha}.$$

Since according to (9) and (15) $c^{\ \beta}_{\sigma}\, c^{\ \sigma}_{\alpha} = -\underset{\sigma}{w}^{\beta}\, \overset{\sigma}{w}_{\alpha} = -\delta^{\ \beta}_{\alpha}$, the affinor $c^{\ \beta}_{\alpha}$ defines an elliptic composition as, while the affinors $a^{\ \beta}_{\alpha}$ and $b^{\ \beta}_{\alpha}$ define hyperbolic compositions. If z^{α} is an eigen-vector of the matrix $(c^{\ \beta}_{\alpha})$, then $c^{\ \beta}_{\alpha}\, z^{\alpha} = \pm\, i\, z^{\beta}$, where $i^2 = -1$.
From the equalities $a^{\ \beta}_{\alpha}\, a^{\ \alpha}_{\sigma} = \delta^{\ \beta}_{\sigma},\ b^{\ \beta}_{\alpha}\, b^{\ \alpha}_{\sigma} = \delta^{\ \beta}_{\sigma},\ c^{\ \beta}_{\alpha}\, c^{\ \alpha}_{\sigma} = -\delta^{\ \beta}_{\sigma},\ a^{\ \beta}_{\alpha}\, b^{\ \alpha}_{\sigma} = -c^{\ \beta}_{\sigma}$ easily follow

$$(16) \qquad \begin{aligned} &a^{\ \beta}_{\alpha}\, b^{\ \alpha}_{\sigma} = -b^{\ \beta}_{\alpha}\, a^{\ \alpha}_{\sigma} = -c^{\ \beta}_{\sigma}\ , \quad b^{\ \beta}_{\alpha}\, c^{\ \alpha}_{\sigma} = -c^{\ \beta}_{\alpha}\, b^{\ \alpha}_{\sigma} = a^{\ \beta}_{\sigma}\ , \\ &c^{\ \beta}_{\alpha}\, a^{\ \alpha}_{\sigma} = -a^{\ \beta}_{\alpha}\, c^{\ \alpha}_{\sigma} = b^{\ \beta}_{\sigma}\ . \end{aligned}$$

Because of (5), (6), (9), (10) and (15) we have $a^{\ \alpha}_{\alpha} = b^{\ \alpha}_{\alpha} = c^{\ \alpha}_{\alpha} = 0$, from where we obtain $a^{\ \beta}_{\alpha}\, a^{\ \alpha}_{\beta} = b^{\ \beta}_{\alpha}\, b^{\ \alpha}_{\beta} = c^{\ \beta}_{\alpha}\, c^{\ \alpha}_{\beta} = 0$. Then from (4), (5), (6), (7), (9), (10) and (15) it follows that in the parameters of the chosen coordinate system the matrix $(a^{\ \beta}_{\alpha})$, $(b^{\ \beta}_{\alpha})$, $(c^{\ \beta}_{\alpha})$, have the following form

$$(a^{\ \beta}_{\alpha}) = \begin{pmatrix} \delta^{\ j}_{i} & 0 \\ 0 & -\delta^{\ \bar j}_{\bar i} \end{pmatrix}, \qquad (b^{\ \beta}_{\alpha}) = \left(\begin{array}{cccc|cccc} & & & & 1 & & & \\ & \bigcirc & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \\ \hline 1 & & & & & & & \\ & 1 & & & & \bigcirc & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \end{array}\right),$$

$$(17) \qquad (c^{\ \beta}_{\alpha}) = \left(\begin{array}{cccc|cccc} & & & & 1 & & & \\ & \bigcirc & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \\ \hline -1 & & & & & & & \\ & -1 & & & & \bigcirc & & \\ & & \ddots & & & & & \\ & & & -1 & & & & \end{array}\right).$$

Following [3] let introduce the notations $A^{\sigma}_{\alpha\beta} = \nabla_{\alpha}\, a^{\ \sigma}_{\beta}$, $B^{\sigma}_{\alpha\beta} = \nabla_{\alpha}\, b^{\ \sigma}_{\beta}$, $C^{\sigma}_{\alpha\beta} = \nabla_{\alpha}\, c^{\ \sigma}_{\beta}$. In the chosen coordinate system, which is adapted with the composition $X_m \times \overline{X}_m$, we have [3]

$$(18) \qquad \begin{gathered} A^{s}_{ik} = 0\ , \quad A^{s}_{\bar i k} = 0\ , \quad A^{s}_{i \bar k} = -2\Gamma^{s}_{i \bar k}\ , \quad A^{s}_{\bar i \bar k} = -2\Gamma^{s}_{\bar i \bar k}\ , \\ A^{\bar s}_{i \bar k} = 0\ , \quad A^{\bar s}_{\bar i \bar k} = 0\ , \quad A^{\bar s}_{ik} = 2\Gamma^{\bar s}_{ik}\ , \quad A^{\bar s}_{\bar i k} = 2\Gamma^{\bar s}_{\bar i k}\ . \end{gathered}$$

According to (17) we establish in the chosen coordinate system the following equalities for $B^{\sigma}_{\alpha\beta}$ and $C^{\sigma}_{\alpha\beta}$

$$(19) \qquad \begin{aligned} & B^{s}_{ik} = \Gamma^{s}_{i\ k+m} - \Gamma^{s+m}_{ik}, \quad B^{s}_{\bar i k} = \Gamma^{s}_{\bar i\ k+m} - \Gamma^{s+m}_{\bar i k}, \quad B^{s}_{i \bar k} = \Gamma^{s}_{i\ \bar k-m} - \Gamma^{s+m}_{i \bar k}, \\ & B^{s}_{\bar i \bar k} = \Gamma^{s}_{\bar i\ \bar k-m} - \Gamma^{s+m}_{\bar i \bar k}, \quad B^{\bar s}_{ik} = \Gamma^{\bar s}_{i\ k+m} - \Gamma^{\bar s-m}_{ik}, \quad B^{\bar s}_{\bar i k} = \Gamma^{\bar s}_{\bar i\ k+m} - \Gamma^{\bar s-m}_{\bar i k}; \\ & B^{\bar s}_{i \bar k} = \Gamma^{\bar s}_{i\ \bar k-m} - \Gamma^{\bar s-m}_{i \bar k}, \quad B^{\bar s}_{\bar i \bar k} = \Gamma^{\bar s}_{\bar i\ \bar k-m} - \Gamma^{\bar s-m}_{\bar i \bar k}, \end{aligned}$$

$$(20) \qquad \begin{aligned} & C^{s}_{ik} = \Gamma^{s}_{i\ k+m} + \Gamma^{s+m}_{ik}, \quad C^{s}_{\bar i k} = \Gamma^{s}_{\bar i\ k+m} + \Gamma^{s+m}_{\bar i k}, \quad C^{s}_{i \bar k} = -\Gamma^{s}_{i\ \bar k-m} + \Gamma^{s+m}_{i \bar k}, \\ & C^{s}_{\bar i \bar k} = -\Gamma^{s}_{\bar i\ \bar k-m} + \Gamma^{s+m}_{\bar i \bar k}, \quad C^{\bar s}_{ik} = \Gamma^{\bar s}_{i\ k+m} - \Gamma^{\bar s-m}_{ik}, \quad C^{\bar s}_{\bar i k} = \Gamma^{\bar s}_{\bar i\ k+m} - \Gamma^{\bar s-m}_{\bar i k}, \\ & C^{\bar s}_{i \bar k} = -\Gamma^{\bar s}_{i\ \bar k-m} - \Gamma^{\bar s-m}_{i \bar k}, \quad C^{\bar s}_{\bar i \bar k} = -\Gamma^{\bar s}_{\bar i\ \bar k-m} - \Gamma^{\bar s-m}_{\bar i \bar k}. \end{aligned}$$

Theorem 2. If the composition $X_m \times \overline{X}_m$ is c, c - composition and its conjugate composition $Y_m \times \overline{Y}_m$ is of the kind (c, c) or (g, g) or (ch, ch), then the space A_{2m} is affine.

Proof: Let $X_m \times \overline{X}_m$ be c, c - composition. According to Proposition 1 and [3] $A^{\sigma}_{\alpha\beta} = \nabla_{\alpha}\, a^{\ \sigma}_{\beta} = 0$. In the chosen coordinate system these conditions accept the form [3]

$$\Gamma^{s}_{i\,\bar{k}} = \Gamma^{s}_{\bar{i}\,k} = \Gamma^{\bar{s}}_{ik} = \Gamma^{\bar{s}}_{\bar{i}\,k} = 0\ . \tag{21}$$

Using (21) we can write (19) properly

$$\begin{aligned} &B^{s}_{ik} = B^{s}_{\bar{i}\,k} = 0, \quad B^{s}_{i\,\bar{k}} = \Gamma^{s}_{i\,\bar{k}-m}, \quad B^{s}_{\bar{i}\,\bar{k}} = -\Gamma^{s+m}_{\bar{i}\,\bar{k}}, \\ &B^{\bar{s}}_{i\,\bar{k}} = B^{\bar{s}}_{\bar{i}\,\bar{k}} = 0, \quad B^{\bar{s}}_{ik} = -\Gamma^{\bar{s}-m}_{ik}, \quad B^{\bar{s}}_{\bar{i}\,k} = \Gamma^{\bar{s}}_{\bar{i}\,k+m}. \end{aligned} \tag{22}$$

1. Now let $Y_m \times \overline{Y}_m$ be d, d - composition. From Proposition 1 it follows $B^{\sigma}_{\alpha\beta} = \nabla_{\alpha}\, b^{\ \sigma}_{\ \beta} = 0$. Substituting in (22) we obtain $\Gamma^{s}_{i\,\bar{k}-m} = \Gamma^{s+m}_{\bar{i}\,\bar{k}} = \Gamma^{\bar{s}-m}_{ik} = \Gamma^{\bar{s}}_{\bar{i}\,k+m} = 0$. So the last results and (21) show us that $\Gamma^{\sigma}_{\alpha\beta} = 0$ for any α, β, σ.

2. Now let $Y_m \times \overline{Y}_m$ be ch, ch - composition. From Proposition 2 it follows $B^{\sigma}_{[\alpha\beta]} = \nabla_{[\alpha}\, b^{\ \sigma}_{\ \beta]} = 0$. Substituting in (22) we obtain $\Gamma^{s}_{i\,\bar{k}-m} = \Gamma^{\bar{s}}_{\bar{i}\,k+m} = 0$. So the last results and (21) show us that $\Gamma^{\sigma}_{\alpha\beta} = 0$ for any α, β, σ.

3. Now let $Y_m \times \overline{Y}_m$ be g, g - composition. Let consider the tensor $M^{\nu}_{\alpha\beta} = b^{\ \sigma}_{\ \alpha}\, B^{\nu}_{\beta\sigma} + b^{\ \sigma}_{\ \beta}\, B^{\nu}_{\sigma\alpha}$. Taking into account (17) and (22) for the components of the tensor $M^{\nu}_{\alpha\beta}$ in the chosen coordinate system we have

$$\begin{aligned} &M^{s}_{ik} = \Gamma^{s}_{ki}, \quad M^{s}_{\bar{i}\,k} = -\Gamma^{s+m}_{k+m\ i}, \quad M^{s}_{i\,\bar{k}} = \Gamma^{s}_{\bar{k}\ i}, \quad M^{s}_{\bar{i}\,\bar{k}} = \Gamma^{s}_{\bar{k}-m\ \bar{i}-m}, \\ &M^{\bar{s}}_{\bar{i}\,\bar{k}} = \Gamma^{\bar{s}}_{\bar{k}\ \bar{i}}, \quad M^{\bar{s}}_{\bar{i}\,k} = -\Gamma^{\bar{s}-m}_{k\ \bar{i}-m}, \quad M^{s}_{i\,\bar{k}} = -\Gamma^{\bar{s}-m}_{\bar{k}-m\ i}, \quad M^{\bar{s}}_{ik} = \Gamma^{\bar{s}}_{k+m\ i+m}\ . \end{aligned} \tag{23}$$

But according to Proposition 3 $Y_m \times \overline{Y}_m$ is an g, g - composition if and only if $M^{\nu}_{\alpha\beta} = 0$. Consequently $Y_m \times \overline{Y}_m$ is an g, g - composition if and only if $\Gamma^{s}_{ki} = \Gamma^{s+m}_{k+m\ i} = \Gamma^{s}_{\bar{k}\ i} = \Gamma^{s}_{\bar{k}-m\ \bar{i}-m} = \Gamma^{\bar{s}}_{\bar{k}\ \bar{i}} = \Gamma^{\bar{s}-m}_{k\ \bar{i}-m} = \Gamma^{\bar{s}-m}_{\bar{k}-m\ i} = \Gamma^{\bar{s}}_{k+m\ i+m} = 0$. So the last results and (21) show us that $\Gamma^{\sigma}_{\alpha\beta} = 0$ for any α, β, σ.

Obviously, in any of the above three cases the tensor of the curvature $R^{\sigma}_{\alpha\beta\gamma} = \partial_{\alpha}\Gamma^{\sigma}_{\beta\gamma} - \partial_{\beta}\Gamma^{\sigma}_{\alpha\gamma} + \Gamma^{\sigma}_{\alpha\nu}\Gamma^{\nu}_{\beta\ \gamma} - \Gamma^{\sigma}_{\beta\nu}\Gamma^{\nu}_{\alpha\ \gamma} = 0$, which means - the space A_{2m} is affine [1]. □

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