ALONG THE TRACES OF THE CONIC SECTIONS

Toni Chehlarova, Evgenia Sendova

Abstract. Based on a geometry problem by Ivan Salabashev we demonstrate an exploratory process leading to conic sections as a locus of points. We work for the purpose with the GeoGebra *dynamic software making use of different objects' representations for explorations related with conic sections.*

Introduction

The notions *circumference* and *circle* are introduced in the primary school, in 6th grade the formulae for the *length of the circumference* and *area of a circle* are studied, and in 8th grade – the mutual position of circles, type of angles in a circle are also introduced. The ellipses are introduced in the physics classes in 10th grade in the context of the planets' movement around the Sun. The parabola, on the other hand, is studied in 10th grade as the graph of the quadratic function $y = ax^2 + bx + c,\ a \neq 0$.

The notions of *ellipse* and *hyperbola*, as well as their canonic equations together with their graphical depictions are considered in the selective classes in the 12th grade.

The importance of using multiple representations of a scientific concept so as to overcome its (sometimes) apparent complexity has been emphasized by many researchers: *different representations show a concept in a different light, highlighting some of its aspects and hiding others* [1]. Or put it another way, *experience with a wide variety of things refines your category system and allows you to make incisive, abstract connections based on deep-shared qualities* [2].

Although there are various representations of the conic sections in the school curriculum their interrelations are often not seen by the students due to the lack of more systematic approach.

The dynamic software provokes new ideas for organizing the mathematics education so that this problem could be overcome. A promising approach in this direction is to create conditions for exploratory activities. This approach, is often referred to as "inquiry based learning" and is a key notion in two recent European projects - *InnoMathEd и Fibonacci* [3, 4]. The main goals of these projects is the development and the dissemination of innovative ideas, learning environments supporting the exploratory style in the mathematics education.

The process of solving an old geometry problem by means of *GeoGebra* [5] lead us to the idea to use it as an illustration of the exploration activities behind looking for a locus which turned to be a conic section.

(The problem was formulated more than 100 years ago by Ivan Salabashev, brought to our attention by a math teacher and first explored by means of the language-based computer environment *Geomland* [6].)

The problem of salabashev read:

Given two circles find the locus of the centers of the circles which are tangent simultaneously to the given ones.

Here below you could see three results obtained at different mutual positions of the given circles:
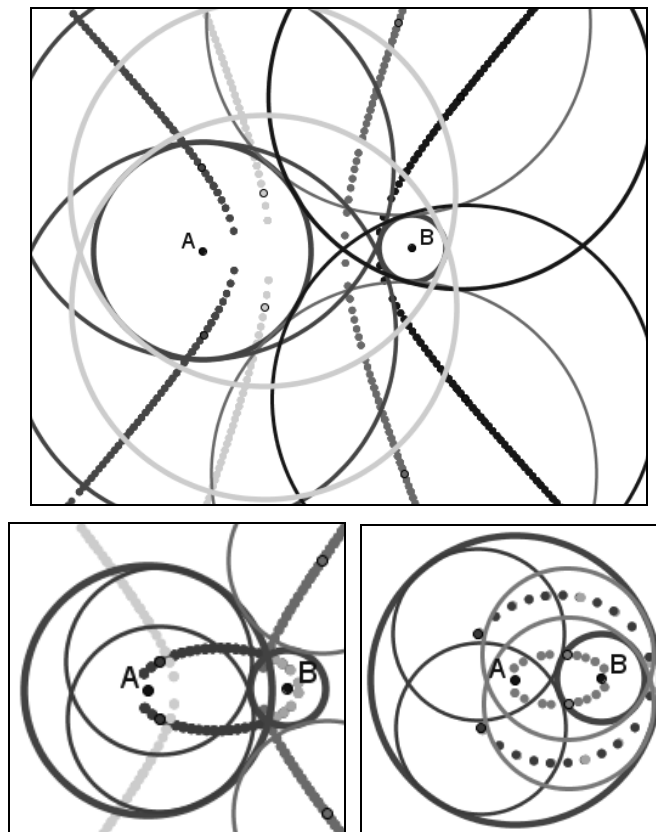
The consecutive steps of the process and some interesting partial cases (e.g. concentric or tangent circles) are described in [7].

The exploratory process is facilitated if one observes just part of the possible cases while hiding the others.

Let us consider one of the curves obtained as a trace of the moving point (the center of the circle tangent to the given two circles). It looks like a conic section.

We select 5 points - A_1, B_1, C_2, D_2, E_2 - on the trace and use the button for a conic section through 5 points. We see that the trace seems to coincide with a part of the hyperbola being obtained.
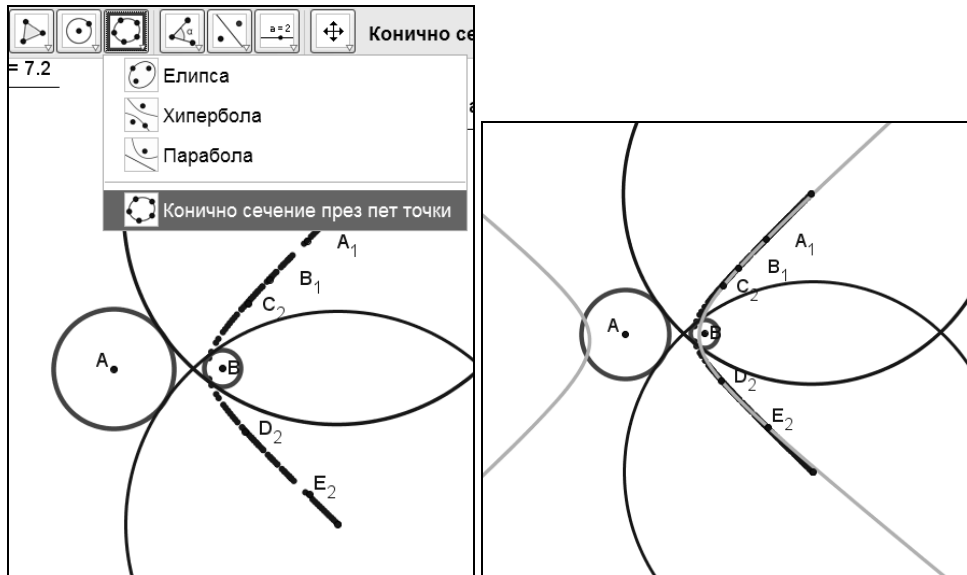

We could use the points A_1, B_1, C_2, D_2, E_2 and move them on another trace. The results enhance our intuition that the loci we are looking for are conic sections, their type depending on the initial mutual position of the given circles.
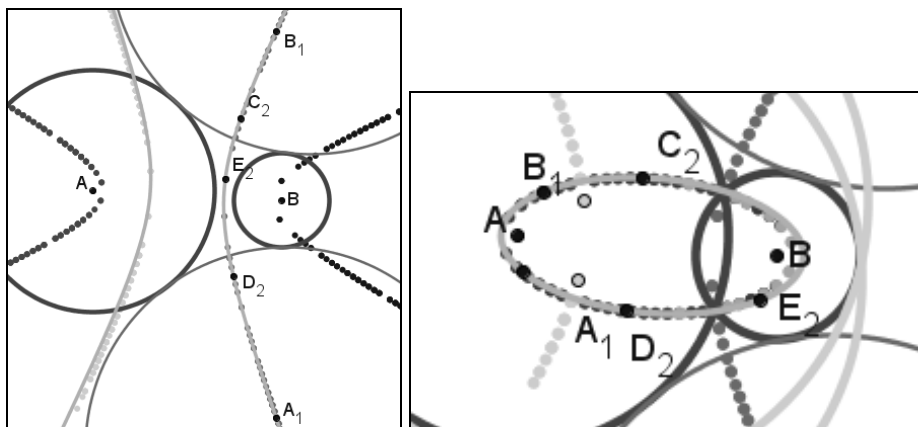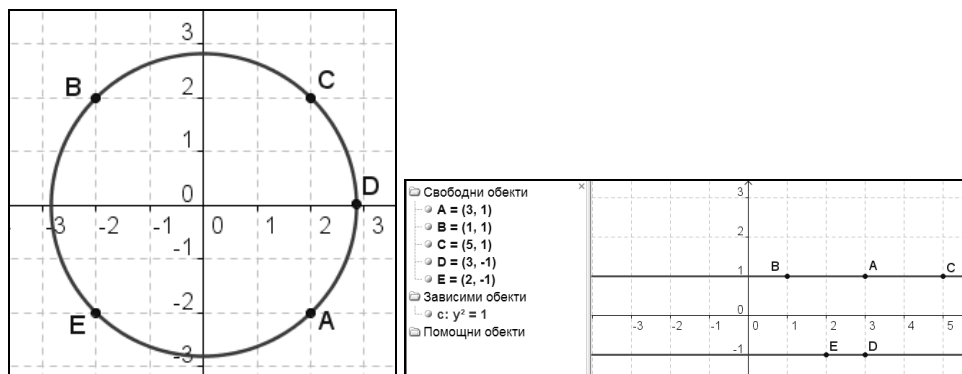

Using the special button for a conic section through 5 points makes it easier to see the relationship between the different conic sections.
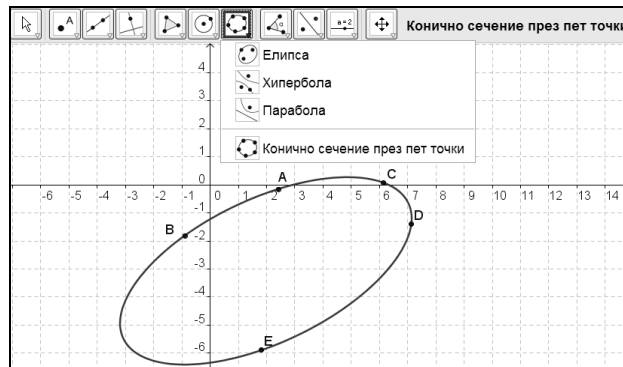
By moving one of the points we get various conic sections.
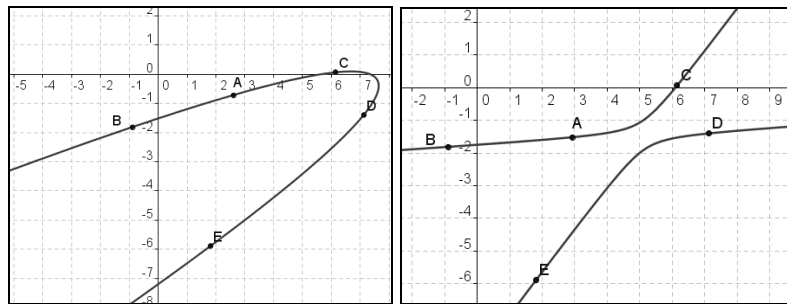
Some partial cases and generated cases can be easily obtained and reflected upon.


Let us come back to the options *GeoGebra* provides for depicting conic sections. Even trying out the different options could enhance the understanding of important properties and applications of the conic sections.

Dual representation of objects

The conic sections could be constructed not only by buttons but also by commands with specific syntax, e.g. here are the corresponding commands for construction an **Ellipse** in *GeoGebra*:

```
Syntax:
Ellipse[ <Focal Point>, <Focal Point>, <Semimajor Axis Length> ]
Ellipse[ <Focal Point>, <Focal Point>, <Segment> ]
Ellipse[ <Point>, <Point>, <Point> ]
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When constructing an ellipse by two foci A and B and the sum AC and BC, where C is an arbitrary point from the ellipse, it is appropriate to use a parameter - a scroller m for the semi-sum so that the students could observe its invariance when exploring the movement of C along the curve for different instances of the ellipse.
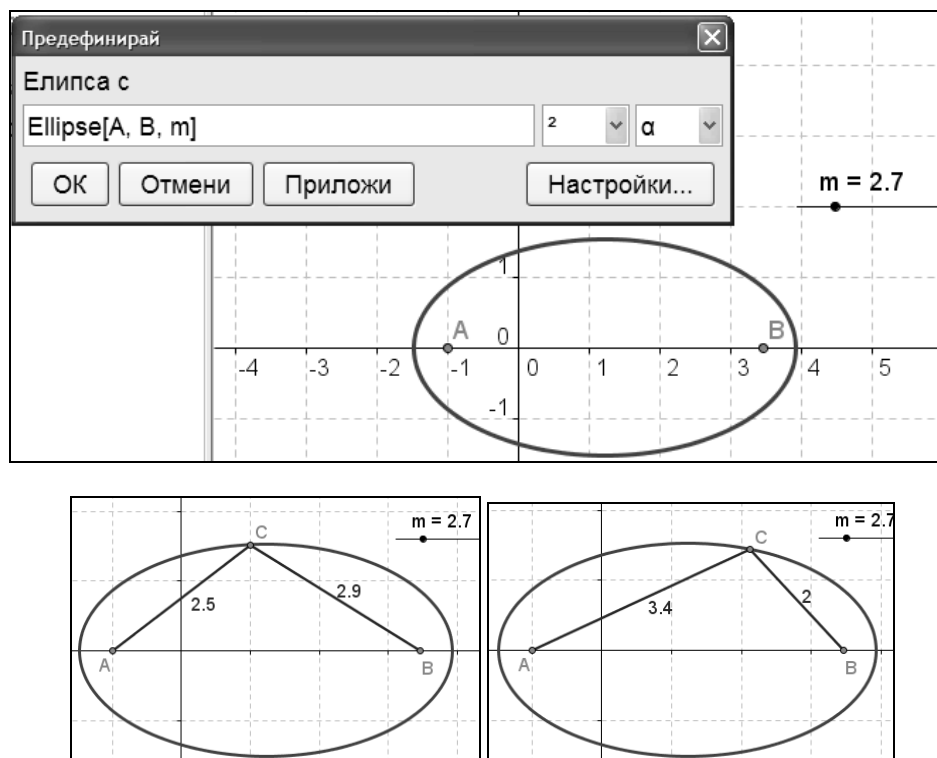
Yet another way to construct an ellipse is to enter its canonic equation in the command line:
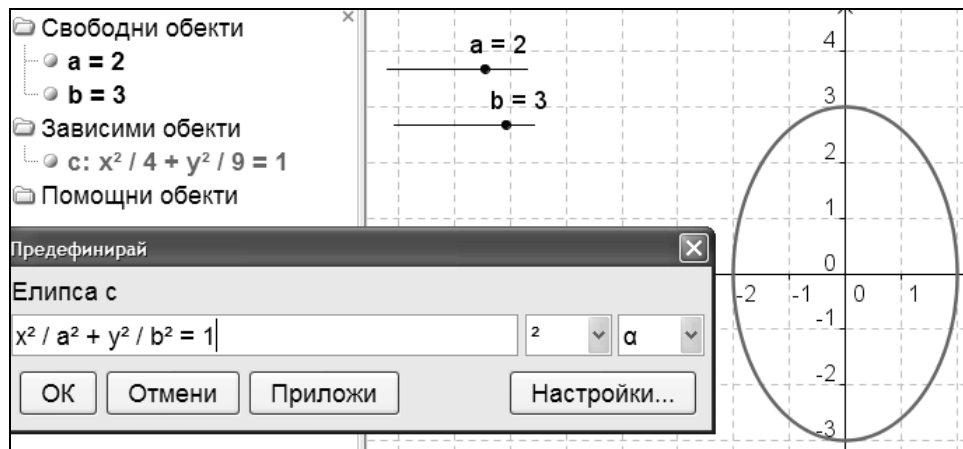
Using parameters-scrollers provides again opportunities for constructing classes of ellipses and carrying out experiments by varying the parameters.
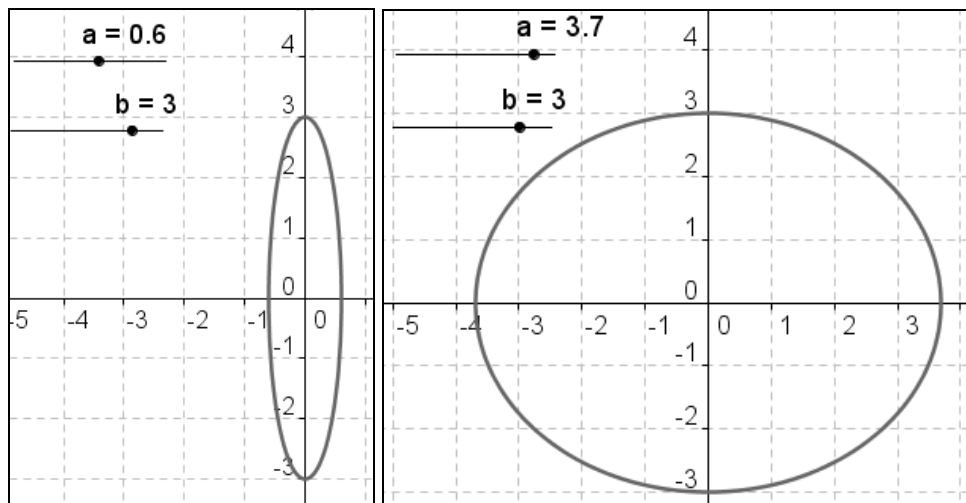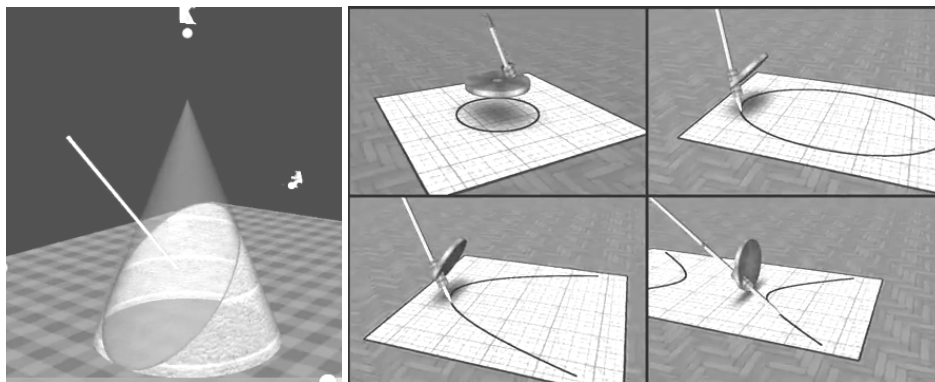

The circle appears as a partial case of an ellipse - by setting $a = b$ in the canonic representation of the ellipse, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ or making the foci A and B coincide.
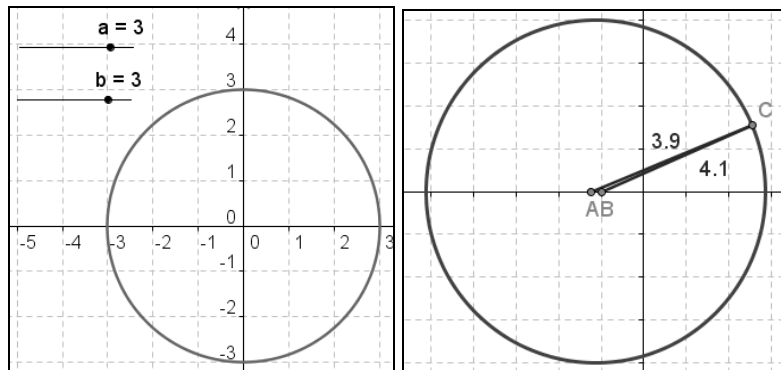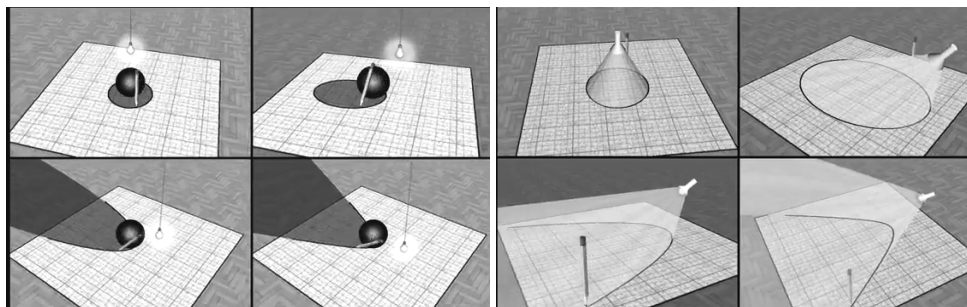
It is appropriate to show the conic sections as intersections of a regular conic surface with a plane, or as virtual mechanisms with a rolling disc;
(see http://www.youtube.com/watch?v=dMRUXTUDyx8
http://www.youtube.com/watch?v=vflwXI5X7_o&NR=1),

with a *light source*
(http://www.youtube.com/watch?v=xeKYNbVGEpw&feature=related
http://www.youtube.com/watch?v=a55XTKRWHfQ&NR=1)

or with a *thread* (http://www.youtube.com/watch?v=z8mU-onxnEw) – all these *Elica* applications of Pavel Boytchev [8].
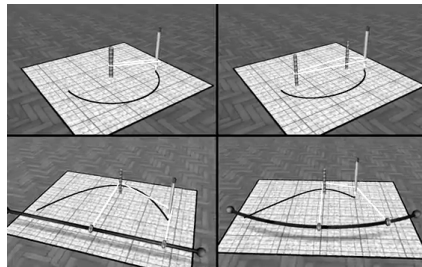
Conclusions

New technologies in education not only require a new competence from the teachers but provide them with environments rich enough to enhance the explorer in them and in their students alike. The explorations combined with multiple representations of a concept can contribute to a better understanding, since *if you understand something in only one way, then you don't really understand it at all* [9].

References

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Toni Chehlarova   Evgenia Sendova
Institute of Mathematics and Informatics   Institute of Mathematics and Informatics
Bulgarian Academy of Sciences   Bulgarian Academy of Sciences
Sofia 1113, Acad. G. Bonchev Str., Block 8   Sofia 1113, Acad. G. Bonchev Str., Block 8
toni.chehlarova@gmail.com   jenny.sendova@gmail.com