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ON GENERALIZED WEYL FRACTIONAL q-INTEGRAL OPERATOR INVOLVING GENERALIZED BASIC HYPERGEOMETRIC FUNCTIONS

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Abstract

Fractional q-integral operators of generalized Weyl type, involving generalized basic hypergeometric functions and a basic analogue of Fox's H-function have been investigated. A number of integrals involving various q-functions have been evaluated as applications of the main results.

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1. Introduction

Al-Salam [3] introduced the generalized Weyl fractional q-integral operator in the following manner:

$$K_q^{\eta,\mu}\{f(x)\} = \frac{q^{-\eta}x^{\eta}}{\Gamma_q(\mu)} \int_x^{\infty} (y-x)_{\mu-1}\, y^{-\eta-\mu} f(yq^{1-\mu}) d(y;q), \tag{1.1}$$

where $Re(\mu) > 0$, η is arbitrary and the basic integration, cf. Gasper and Rahman [5], is defined as:

$$\int_x^{\infty} f(t)d(t;q) = x(1-q)\sum_{k=1}^{\infty} q^{-k} f(xq^{-k}). \tag{1.2}$$

In view of relation (1.2), operator (1.1) can be expressed as:

$$K_q^{\eta,\mu}\{f(x)\} = (1-q)^{\mu}\sum_{k=0}^{\infty} q^{k\eta}\frac{(q^{\mu};q)_k}{(q;q)_k}\ f(xq^{-\mu-k}), \tag{1.3}$$

where $Re(\mu) > 0$ and η being an arbitrary complex quantity.

In the sequel we shall use the following notations and definitions:

For real or complex a and $|q| < 1$, the q-shifted factorial is defined as:

$$(a;q)_n = \begin{cases} 1 & ;\text{if} \quad n = 0 \\ (1-a)(1-aq)\cdots(1-aq^{n-1}) & ;\text{if} \quad n \in N. \end{cases} \tag{1.4}$$

In terms of the q-gamma function, (1.4) can be expressed as

$$(a;q)_n \ = \ \frac{\Gamma_q(a+n)(1-q)^n}{\Gamma_q(a)}, \quad n > 0, \tag{1.5}$$

where the q-gamma function (cf. Gasper and Rahman [5]), is given by

$$\Gamma_q(a) \ = \ \frac{(q;q)_\infty}{(q^a;q)_\infty(1-q)^{a-1}} = \frac{(q;q)_{a-1}}{(1-q)^{a-1}}, \tag{1.6}$$

where $a \neq 0, -1, -2, \cdots$. Also,

$$(x-y)_\nu = x^\nu \prod_{n=0}^{\infty}\left[\frac{1-(y/x)q^n}{1-(y/x)q^{\nu+n}}\right] = x^\nu\ {}_1\Phi_0\left[\begin{array}{ccc} q^{-\nu} & ; & \\ & & q, yq^\nu/x \\ - & ; & \end{array}\right]. \tag{1.7}$$

The generalized basic hypergeometric series, cf. Gasper and Rahman [5], is given by

$${}_r\Phi_s\left[\begin{array}{ccc} a_1,\cdots,a_r & ; & \\ & & q, x \\ b_1,\cdots,b_s & ; & \end{array}\right] = \sum_{n=0}^{\infty}\frac{(a_1,\cdots,a_r;q)_n}{(q,b_1,\cdots,b_s;q)_n}\ x^n\ \left\{(-1)^n q^{n(n-1)/2}\right\}^{(1+s-r)}, \tag{1.8}$$

where for convergence, we have $|q| < 1$ and $|x| < 1$ if $r = s+1$, and for any x if $r \leq s$. The abnormal type of generalized basic hypergeometric series ${}_r\Phi_s(\cdot)$ is defined as

$${}_r\Phi_s\left[\begin{array}{ll} a_1,\cdots,a_r & ; q, x \\ b_1,\cdots,b_s & ; q^\lambda \end{array}\right] = \sum_{n=0}^{\infty}\frac{(a_1,\cdots,a_r;q)_n}{(q,b_1,\cdots,b_s;q)_n}\ x^n q^{\lambda n(n+1)/2}, \tag{1.9}$$

where $\lambda > 0$ and $|q| < 1$.

The q-exponential series is

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n}. \tag{1.10}$$

The q-binomial series is given by

$$_1\Phi_0 \left[\begin{array}{cc} \alpha & ; \\ & q, x \\ - & ; \end{array} \right] = \frac{(\alpha x; q)_\infty}{(x, q)_\infty}. \tag{1.11}$$

The basic analogue of the Sine and Cosine functions are

$$\sin_q(ax) = \frac{1}{2i} \left\{ e_q(iax) - e_q(-iax) \right\}, \tag{1.12}$$

and

$$\cos_q(ax) = \frac{1}{2} \left\{ e_q(iax) + e_q(-iax) \right\}. \tag{1.13}$$

Similarly, we have the q-Laguerre polynomial:

$$L_n^{(\alpha)}(x;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n} \, {}_1\Phi_1 \left[\begin{array}{cc} q^{-n} & ; \\ & q, -xq^n \\ q^{\alpha+1} & ; \end{array} \right]; \tag{1.14}$$

the little q-Jacobi polynomial:

$$P_n^{(\alpha,\beta)}(x;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n} \, {}_2\Phi_1 \left[\begin{array}{cc} q^{-n}, q^{\alpha+\beta+n+1} & ; \\ & q, xq \\ q^{\alpha+1} & ; \end{array} \right]; \tag{1.15}$$

the Wall polynomial (or little q-Laguerre polynomial)

$$W_n(x;b,q) = (-1)^n (b;q)_n \; q^{n(n+1)/2} \, {}_2\Phi_1 \left[\begin{array}{cc} q^{-n}, 0 & ; \\ & q, x \\ b & ; \end{array} \right]; \tag{1.16}$$

and the Stieltjes-Wigert polynomial:

$$s_n(x;q) = \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q;q)_k} q^{k(k+1)/2} \, (-x)^k = {}_1\Phi_0 \left[\begin{array}{cc} q^{-n} & ; q, -x \\ - & ; q^1 \end{array} \right]. \tag{1.17}$$

Saxena, Modi and Kalla [9], introduced a basic analogue of the H-function in terms of the Mellin-Barnes type basic contour integral in the following manner:

$$H_{A,B}^{m,n} \left[x; q \,\middle|\, \begin{array}{c} (a, \alpha) \\ (b, \beta) \end{array} \right]$$

$$= \frac{1}{2\pi i} \int_C \frac{\prod_{j=1}^{m} G(q^{b_j-\beta_j s}) \prod_{j=1}^{n} G(q^{1-a_j+\alpha_j s}) \pi x^s}{\prod_{j=m+1}^{B} G(q^{1-b_j+\beta_j s}) \prod_{j=n+1}^{A} G(q^{a_j-\alpha_j s})\ G(q^{1-s}) \sin \pi s} ds, \quad (1.18)$$

where

$$G(q^\alpha) = \prod_{n=0}^{\infty} \left\{(1-q^{\alpha+n})\right\}^{-1} = \frac{1}{(q^\alpha;q)_\infty}, \quad (1.19)$$

and $0 \le m \le B$; $0 \le n \le A$; α_j and β_j are all positive integers. The contour C is a line parallel to $Re(\omega s) = 0$, with indentations, if necessary, in such a manner that all the poles of $G(q^{b_j-\beta_j s})$; $1 \le j \le m$, are to its right, and those of $G(q^{1-a_j-\alpha_j s})$; $1 \le j \le n$, are to the left of C. The basic integral converges if $Re\,[s \log(x) - \log \sin \pi x] < 0$, for large values of $|s|$ on the contour C, that is if $\left|\left\{\arg(x) - \omega_2\ \omega_1^{-1} - \log |x|\right\}\right| < \pi$, where $|q| < 1$, $\log q = -\omega = -(\omega_1 + i\omega_2)$, $\omega,\ \omega_1$ and ω_2 are definite quantities. ω_1 and ω_2 being real.

For $\alpha_j = \beta_i = 1,\ j = 1, \cdots, A;\ i = 1, \cdots, B$, the definition (1.18) reduces to the q-analogue of the Meijer G-function due to Saxena, Modi and Kalla [9], namely:

$$G_{A,B}^{m,n}\left[x;q \,\middle|\, \begin{matrix} a_1, \cdots, a_A \\ b_1, \cdots, b_B \end{matrix}\right]$$

$$= \frac{1}{2\pi i} \int_C \frac{\prod_{j=1}^{m} G(q^{b_j-s}) \prod_{j=1}^{n} G(q^{1-a_j+s}) \pi x^s}{\prod_{j=m+1}^{B} G(q^{1-b_j+s}) \prod_{j=n+1}^{A} G(q^{a_j-s})\ G(q^{1-s}) \sin \pi s} ds, \quad (1.20)$$

where $0 \le m \le B$; $0 \le n \le A$ and $Re\,[s \log(x) - \log \sin \pi x] < 0$.

Further, if we set $n = 0$ and $m = B$ in the equation (1.20), we get the basic analogue of MacRobert's E-function due to Agarwal [1], namely

$$G_{A,B}^{B,0}\left[x;q \,\middle|\, \begin{matrix} a_1, \cdots, a_A \\ b_1, \cdots, b_B \end{matrix}\right] \equiv E_q\,[B; b_j : A; a_j : x]$$

$$= \frac{1}{2\pi i} \int_C \frac{\prod_{j=1}^{B} G(q^{b_j-s}) \pi x^s}{\prod_{j=1}^{A} G(q^{a_j-s})\ G(q^{1-s}) \sin \pi s} ds, \quad (1.21)$$

where $Re\,[s \log(x) - \log \sin \pi x] < 0$.

Saxena and Kumar [8], introduced the basic analogues of $J_\nu(x)$, $Y_\nu(x)$, $K_\nu(x)$, $H_\nu(x)$ in terms of $H_q(\cdot)$ function as follows:

$$J_\nu(x;q) = \{G(q)\}^2 H_{0,3}^{1,0}\left[\frac{x^2(1-q)^2}{4};q \,\middle|\, \begin{matrix} - \\ (\frac{\nu}{2},1),\ (\frac{-\nu}{2},1),\ (1,1) \end{matrix}\right], \quad (1.22)$$

where $J_\nu(x;q)$ denotes the q-analogue of Bessel function of first kind $J_\nu(x)$;

$$Y_\nu(x;q) = \{G(q)\}^2 H_{1,4}^{2,0}\left[\frac{x^2(1-q)^2}{4};q \,\middle|\, \begin{matrix} (\frac{-\nu-1}{2},1) \\ (\frac{\nu}{2},1),\ (\frac{-\nu}{2},1),\ (\frac{-\nu-1}{2},1),\ (1,1) \end{matrix}\right], \quad (1.23)$$

where $Y_\nu(x;q)$ denotes the q-analogue of the Bessel function $Y_\nu(x)$;

$$K_\nu(x;q) = (1-q) H_{0,3}^{2,0}\left[\frac{x^2(1-q)^2}{4};q \,\middle|\, \begin{matrix} - \\ (\frac{\nu}{2},1),\ (\frac{-\nu}{2},1),\ (1,1) \end{matrix}\right], \quad (1.24)$$

where $K_\nu(x;q)$ denotes the basic analogue of the Bessel function of the third kind $K_\nu(x)$;

$$H_\nu(x;q) = \left(\frac{1-q}{2}\right)^{1-\alpha} H_{1,4}^{3,1}\left[\frac{x^2(1-q)^2}{4};q \,\middle|\, \begin{matrix} (\frac{1+\alpha}{2},1) \\ (\frac{\nu}{2},1),\ (\frac{-\nu}{2},1),\ (\frac{1+\alpha}{2},1),\ (1,1) \end{matrix}\right], \quad (1.25)$$

where $H_\nu(x;q)$ is the basic analogue of Struve's function $H_\nu(x)$.

Following Saxena and Kumar [8], Mathai and Saxena [6], [7], we have the following q-extensions of certain elementary functions in terms of a basic analogue of the Meijer G-function as:

$$e_q(-x) = G(q) H_{0,2}^{1,0}\left[x(1-q);q \,\middle|\, \begin{matrix} - \\ (0,1),(1,1) \end{matrix}\right], \quad (1.26)$$

$$\sin_q(x) = \sqrt{\pi}(1-q)^{-1/2}\{G(q)\}^2 H_{0,3}^{1,0}\left[\frac{x^2(1-q)^2}{4};q \,\middle|\, \begin{matrix} - \\ (\frac{1}{2},1),(0,1),(1,1) \end{matrix}\right]; \quad (1.27)$$

$$\cos_q(x)=\sqrt{\pi}(1-q)^{-1/2}\{G(q)\}^2 H_{0,3}^{1,0}\left[\frac{x^2(1-q)^2}{4};q \;\middle|\; \begin{matrix} - \\ (0,1),(\frac{1}{2},1),(1,1) \end{matrix}\right]; \tag{1.28}$$

$$\sinh_q(x)=\frac{\sqrt{\pi}}{i}(1-q)^{-1/2}\{G(q)\}^2 H_{0,3}^{1,0}\left[-\frac{x^2(1-q)^2}{4};q \;\middle|\; \begin{matrix} - \\ (\frac{1}{2},1),(0,1),(1,1) \end{matrix}\right]; \tag{1.29}$$

$$\cosh_q(x)=\sqrt{\pi}(1-q)^{-1/2}\{G(q)\}^2 H_{0,3}^{1,0}\left[-\frac{x^2(1-q)^2}{4};q \;\middle|\; \begin{matrix} - \\ (0,1),(\frac{1}{2},1),(1,1) \end{matrix}\right]. \tag{1.30}$$

A detailed account of various classical special functions expressible in terms of Meijer's G-function or Fox's H-function can be found in research monographs by Mathai and Saxena [6] and [7].

The main motive of the present paper is to investigate the generalized Weyl fractional q-integral operator involving basic hypergeometric functions including the basic analogue of the H-function. Certain interesting special cases have also been derived as the applications of the main results.

2. Main results

In this section, we shall evaluate the following fractional q-integrals of generalized Weyl type involving basic hypergeometric function ${}_r\Phi_s(\cdot)$ and basic analogue of Fox's H-function. The main results are presented in the following theorems.

THEOREM 1. *If $Re(\eta-\lambda)>0$ and ρ is any complex number, then the generalized Weyl fractional q-integral of x^λ-weighted basic hypergeometric function ${}_r\Phi_s(\cdot)$, is given by*

$$K_q^{\eta,\mu}\left\{x^\lambda\ {}_r\Phi_s\left[\begin{matrix} a_1,\cdots,a_r & ; & \\ & & q,\rho x \\ b_1,\cdots,b_s & ; & \end{matrix}\right]\right\} = \frac{x^\lambda q^{-\mu\lambda}\Gamma_q(\eta-\lambda)}{\Gamma_q(\eta-\lambda+\mu)}$$

$$ {}_{r+1}\Phi_{s+1}\left[\begin{matrix} a_1,\cdots,a_r,q^{1-\eta+\lambda-\mu} & ; & \\ & & q,\rho x \\ b_1,\cdots,b_s,q^{1-\eta+\lambda} & ; & \end{matrix}\right]. \tag{2.1}$$

P r o o f. In view of relations (1.3) and (1.8), the left hand side of (2.1) becomes

$$(1-q)^\mu\sum_{k=0}^{\infty} q^{\eta k}\frac{(q^\mu;q)_k}{(q;q)_k}\ (xq^{-\mu-k})^\lambda\sum_{n=0}^{\infty}\frac{(a_1,\cdots,a_r;q)_n}{(q,b_1,\cdots,b_s;q)_n}$$
$$\times\ \left\{(-1)^n q^{n(n-1)/2}\right\}^{(1+s-r)}\ (\rho x q^{-\mu-k})^n,$$

and, interchanging the order of summations, we further obtain

$$x^\lambda(1-q)^\mu q^{-\mu\lambda}\sum_{n=0}^{\infty}\frac{(a_1,\cdots,a_r;q)_n}{(q,b_1,\cdots,b_s;q)_n}\ \left\{(-1)^n q^{n(n-1)/2}\right\}^{(1+s-r)}\ (\rho x q^{-\mu})^n$$
$$\times\sum_{k=0}^{\infty}\frac{(q^\mu;q)_k}{(q;q)_k}\ q^{k(\eta-\lambda-n)}.$$

On summing the inner ${}_1\Phi_0(\cdot)$ series with the help of the equation (1.11), the above expression reduces to

$$x^\lambda(1-q)^\mu q^{-\mu\lambda}\sum_{n=0}^{\infty}\frac{(a_1,\cdots,a_r;q)_n}{(q,b_1,\cdots,b_s;q)_n(q^{\eta-\lambda-n};q)_\mu}$$
$$\times\ \left\{(-1)^n q^{n(n-1)/2}\right\}^{(1+s-r)}\ (\rho x q^{-\mu})^n,$$

and by further simplification, the above expression yields to the right hand side of (2.1). ■

THEOREM 2. *Let $Re(\mu)>0$, $\lambda\in I$, ρ be any complex number, then the following generalized Weyl fractional q-integral of $H_q(.)$ function for $\lambda\geq 0$ and $\lambda<0$ holds:*

$$K_q^{\eta,\mu}\left\{H_{A,B}^{m,n}\left[\rho x^\lambda;q\left|\begin{matrix}(a,\alpha)\\(b,\beta)\end{matrix}\right.\right]\right\}$$

$$= (1-q)^{\mu} H^{m+1,n}_{A+1,B+1}\left[\rho(xq^{-\mu})^{\lambda};q \,\middle|\, \begin{matrix}(a,\alpha),(\mu+\eta,\lambda)\\(\eta,\lambda),(b,\beta)\end{matrix}\right], \qquad \lambda \geq 0 \tag{2.2}$$

$$= (1-q)^{\mu} H^{m,n+1}_{A+1,B+1}\left[\rho(xq^{-\mu})^{\lambda};q \,\middle|\, \begin{matrix}(1-\eta,-\lambda),(a,\alpha)\\(b,\beta),(1-\mu-\eta,-\lambda)\end{matrix}\right], \quad \lambda < 0, \tag{2.3}$$

where $0 \leq m \leq B$, $0 \leq n \leq A$ *and* $Re\,[s\log(x) - \log\sin\pi x] < 0$.

P r o o f. To prove the theorem, we consider the left hand side of (2.2) and use definitions (1.3) and (1.18) to obtain

$$\frac{(1-q)^{\mu}}{2\pi i}\sum_{k=0}^{\infty} q^{\eta k}\frac{(q^{\mu};q)_k}{(q;q)_k}$$
$$\times \int_C \frac{\prod_{j=1}^{m} G(q^{b_j-\beta_j s}) \prod_{j=1}^{n} G(q^{1-a_j+\alpha_j s})\pi(\rho x^{\lambda} q^{-\mu\lambda-k\lambda})^s}{\prod_{j=m+1}^{B} G(q^{1-b_j+\beta_j s}) \prod_{j=n+1}^{A} G(q^{a_j-\alpha_j s})G(q^{1-s})\sin\pi s}\,ds.$$

On interchanging the order of summation and integration, valid under the conditions given with equation (1.18), the above expression reduces to

$$\frac{(1-q)^{\mu}}{2\pi i}\int_C \frac{\prod_{j=1}^{m} G(q^{b_j-\beta_j s}) \prod_{j=1}^{n} G(q^{1-a_j+\alpha_j s})\pi(\rho x^{\lambda} q^{-\mu\lambda})^s}{\prod_{j=m+1}^{B} G(q^{1-b_j+\beta_j s}) \prod_{j=n+1}^{A} G(q^{a_j-\alpha_j s})G(q^{1-s})\sin\pi s}$$
$$\times \sum_{k=0}^{\infty}\frac{(q^{\mu};q)_k}{(q;q)_k} q^{k(\eta-\lambda s)}ds.$$

On summing the inner ${}_1\Phi_0(\cdot)$ series, with the help of equation (1.11) and on using definition (1.19), the left hand side of (2.2) finally reduces to

$$\frac{(1-q)^{\mu}}{2\pi i}\int_C \frac{\prod_{j=1}^{m} G(q^{b_j-\beta_j s})G(q^{\eta-\lambda s}) \prod_{j=1}^{n} G(q^{1-a_j+\alpha_j s})\pi(\rho x^{\lambda} q^{-\mu\lambda})^s}{\prod_{j=m+1}^{B} G(q^{1-b_j+\beta_j s}) \prod_{j=n+1}^{A} G(q^{a_j-\alpha_j s})G(q^{\mu+\eta-\lambda s})G(q^{1-s})\sin\pi s}ds.$$

Interpreting the above expression in view of definition (1.18), we obtain the right hand side of (2.2). The second part of the theorem follows similarly. ■

3. Applications of the main results

In this section, we evaluate some basic integrals of generalized Weyl type, involving basic hypergeometric functions and various elementary basic functions expressible in terms of a basic analogue of Fox's H-function, as applications of the theorems from the previous section. These results are presented in the table that follows.

For the sake of brevity, we mention here the proofs of a few results given in the table. For example the results (3.1) to (3.6) have been derived by assigning appropriate values to the parameters r, s and ρ in (2.1), Theorem 1, keeping in view of the definitions given by equations (1.7)-(1.17).

If we set $r = s = 0$ and $\rho = 1$ in (2.1), it reduces to (3.2). Further, on making use of the result (3.2), we can easily prove the results (3.7) and (3.8). The proof of the result (3.9) is similar to the result (2.1).

The proofs of the results (3.11) and (3.12) follow directly from (2.2) with $\rho = \lambda = 1$ and on using the definitions (1.20)-(1.21), respectively.

While, if we assign $m = 1, n = A = 0, B = 3, b_1 = \nu/2, b_2 = -\nu/2, b_3 = 1, \lambda = 2$ and $\rho = \frac{(1-q)^2}{4}$ in (2.1), we obtain

$$K_q^{\eta,\mu}\left\{H_{0,3}^{1,0}\left[\frac{x^2(1-q)^2}{4};q\,\middle|\,\begin{matrix}-\\(\frac{\nu}{2},1),(-\frac{\nu}{2},1),(1,1)\end{matrix}\right]\right\} = (1-q)^{\mu}$$
$$\times H_{1,4}^{2,0}\left[\frac{x^2(1-q)^2}{4q^{2\mu}};q\,\middle|\,\begin{matrix}(\mu+\eta,2)\\(\eta,2),(\frac{\nu}{2},1),(-\frac{\nu}{2},1),(1,1)\end{matrix}\right]. \tag{3.22}$$

In view of of definition (1.22), the above equation (3.22) reduces to the result (3.13). The results (3.14)-(3.21) can be proved similarly by assigning particular values to the parameters m, n, A, B, λ and ρ, keeping in mind definitions (1.23)-(1.30), respectively.

The results deduced in the present paper aim to contribute to the theory of basic hypergeometric series and q-fractional calculus. They are expected to find some applications to the solutions of fractional q-differ-integral equations. We intend to take up this aspect in a next contribution.

Eq. No.	$f(x)$	$K_q^{\eta,\mu}\{f(x)\} = \dfrac{q^{-\eta}x^{\eta}}{\Gamma_q(\mu)}\displaystyle\int_x^{\infty}(y-x)_{\mu-1}y^{-\eta-\mu}f(yq^{1-\mu})d(y;q)$ $Re(\mu)>0$ and for any η
3.1	x^{λ}	$\dfrac{x^{\lambda}q^{-\mu\lambda}\Gamma_q(\eta-\lambda)}{\Gamma_q(\eta-\lambda+\mu)}$, $Re(\eta-\lambda)>0$
3.2	$x^{\lambda}e_q(x)$	$\dfrac{x^{\lambda}q^{-\mu\lambda}\Gamma_q(\eta-\lambda)}{\Gamma_q(\eta-\lambda+\mu)}\,{}_1\Phi_1\left[\begin{array}{ll} q^{1-\eta+\lambda-\mu} & ; \\ & q,x \\ q^{1-\eta+\lambda} & ; \end{array}\right]$, $Re(\eta-\lambda)>0$
3.3	$x^{\lambda}(1-xq^{\alpha})_{-\alpha}$	$\dfrac{x^{\lambda}q^{-\mu\lambda}\Gamma_q(\eta-\lambda)}{\Gamma_q(\eta-\lambda+\mu)}\,{}_2\Phi_1\left[\begin{array}{ll} q^{\alpha},q^{1-\eta+\lambda-\mu} & ; \\ & q,x \\ q^{1-\eta+\lambda} & ; \end{array}\right]$, $Re(\eta-\lambda)>0$
3.4	$x^{\lambda}L_n^{(\alpha)}(x;q)$	$\dfrac{x^{\lambda}q^{-\mu\lambda}(q^{\alpha+1};q)_n\Gamma_q(\eta-\lambda)}{(q;q)_n\Gamma_q(\eta-\lambda+\mu)}$ $\times\,{}_2\Phi_2\left[\begin{array}{ll} q^{-n},q^{1-\eta+\lambda-\mu} & ; \\ & q,xq^{-n} \\ q^{\alpha+1},q^{1-\eta+\lambda} & ; \end{array}\right]$, $Re(\eta-\lambda)>0$
3.5	$x^{\lambda}P_n^{(\alpha,\beta)}(x;q)$	$\dfrac{x^{\lambda}q^{-\mu\lambda}(q^{\alpha+1};q)_n\Gamma_q(\eta-\lambda)}{(q;q)_n\Gamma_q(\eta-\lambda+\mu)}$ $\times\,{}_3\Phi_2\left[\begin{array}{ll} q^{-n},q^{\alpha+\beta+n-1},q^{1-\eta+\lambda-\mu} & ; \\ & q,xq \\ q^{\alpha+1},q^{1-\eta+\lambda} & ; \end{array}\right]$, $Re(\eta-\lambda)>0$
3.6	$x^{\lambda}W_n(x;b,q)$	$\dfrac{x^{\lambda}q^{-\mu\lambda+n(n+1)/2}(b;q)_n\Gamma_q(\eta-\lambda)}{(q;q)_n\Gamma_q(\eta-\lambda+\mu)}$ $\times\,{}_3\Phi_2\left[\begin{array}{ll} q^{-n},0,q^{1-\eta+\lambda-\mu} & ; \\ & q,xq \\ b,q^{1-\eta+\lambda} & ; \end{array}\right]$, $Re(\eta-\lambda)>0$

3.7	$x^{\lambda}sin_q(ax)$	$\frac{x^{\lambda}q^{-\mu\lambda}\Gamma_q(\eta-\lambda)}{2i\Gamma_q(\eta-\lambda+\mu)}\,{}_1\Phi_1\left[\begin{matrix} q^{1-\eta+\lambda-\mu} & ; & \\ & & q, iax \\ q^{1-\eta+\lambda} & ; & \end{matrix}\right]$ $-\ \frac{x^{\lambda}q^{-\mu\lambda}\Gamma_q(\eta-\lambda)}{2i\Gamma_q(\eta-\lambda+\mu)}$ $\times\,{}_1\Phi_1\left[\begin{matrix} q^{1-\eta+\lambda-\mu} & ; & \\ & & q, -iax \\ q^{1-\eta+\lambda} & ; & \end{matrix}\right],$ $Re(\eta-\lambda)>0$
3.8	$x^{\lambda}cos_q(ax)$	$\frac{x^{\lambda}q^{-\mu\lambda}\Gamma_q(\eta-\lambda)}{2\Gamma_q(\eta-\lambda+\mu)}\,{}_1\Phi_1\left[\begin{matrix} q^{1-\eta+\lambda-\mu} & ; & \\ & & q, iax \\ q^{1-\eta+\lambda} & ; & \end{matrix}\right]$ $+\ \frac{x^{\lambda}q^{-\mu\lambda}\Gamma_q(\eta-\lambda)}{2\Gamma_q(\eta-\lambda+\mu)}$ $\times\,{}_1\Phi_1\left[\begin{matrix} q^{1-\eta+\lambda-\mu} & ; & \\ & & q, -iax \\ q^{1-\eta+\lambda} & ; & \end{matrix}\right],$ $Re(\eta-\lambda)>0$
3.9	$x^{\lambda}\,{}_r\Phi_s\left[\begin{matrix} a_1,\cdots,a_r & ;q,x \\ b_1,\cdots,b_s & ;q^{\delta}\end{matrix}\right]$ $\delta>0$	$\frac{x^{\lambda}q^{-\mu\lambda}\Gamma_q(\eta-\lambda)}{\Gamma_q(\eta-\lambda+\mu)}\,{}_{r+1}\Phi_{s+1}\left[\begin{matrix} a_1,\cdots,a_r, & q^{1-\eta+\lambda-\mu} & ;q,x \\ b_1,\cdots,b_s, & q^{1-\eta+\lambda} & ;q^{\delta}\end{matrix}\right],$ $Re(\lambda-\mu)>0$
3.10	$x^{\lambda}s_n(x;q)$	$\frac{x^{\lambda}q^{-\mu\lambda}\Gamma_q(\eta-\lambda)}{\Gamma_q(\eta-\lambda+\mu)}$ $\times\,{}_2\Phi_1\left[\begin{matrix} q^{-n}, q^{1-\eta+\lambda-\mu} & ;q,-x \\ q^{1-\eta+\lambda} & ;q^{1}\end{matrix}\right],$ $Re(\lambda-\mu)>0$
3.11	$G^{m,n}_{A,B}\left[x;q\left\vert\begin{matrix} a_1,\cdots,a_A \\ b_1,\cdots,b_B\end{matrix}\right.\right]$	$(1-q)^{\mu}$ $\times G^{m+1,n}_{A+1,B+1}\left[xq^{-\mu};q\left\vert\begin{matrix} a_1,\cdots,a_A,\mu+\eta \\ \eta, b_1,\cdots,b_B\end{matrix}\right.\right],$ $0\le m\le B$, $0\le n\le A$ and $Re[s\log(x)-\log\sin\pi s]<0$

3.12	$E_q[B;b_j:A;a_j:x]$	$(1-q)^{\mu} \times G^{B+1,0}_{A+1,B+1}\left[xq^{-\mu};q \middle\vert \begin{matrix} a_1,\cdots,a_A,\mu+\eta \\ \eta,b_1,\cdots,b_B \end{matrix} \right]$
3.13	$J_\nu(x;q)$	$(1-q)^{\mu}\{G(q)\}^2 \times H^{2,0}_{1,4}\left[\frac{x^2(1-q)^2}{4q^{2\mu}};q \middle\vert \begin{matrix} (\mu+\eta,2) \\ (\eta,2),(\frac{\nu}{2},1),(\frac{-\nu}{2},1),(1,1) \end{matrix} \right]$
3.14	$Y_\nu(x;q)$	$(1-q)^{\mu}\{G(q)\}^2\ H^{3,0}_{2,5}\left[\frac{x^2(1-q)^2}{4q^{2\mu}};q \middle\vert \begin{matrix} (\frac{-\nu-1}{2},1),(\mu+\eta,2) \\ (\eta,2),(\frac{\nu}{2},1),(\frac{-\nu}{2},1),(\frac{-\nu-1}{2},1),(1,1) \end{matrix} \right]$
3.15	$K_\nu(x;q)$	$(1-q)^{\mu+1} \times H^{3,0}_{1,4}\left[\frac{x^2(1-q)^2}{4q^{2\mu}};q \middle\vert \begin{matrix} (\mu+\eta,2) \\ (\eta,2),(\frac{\nu}{2},1),(\frac{-\nu}{2},1),(1,1) \end{matrix} \right]$
3.16	$H_\nu(x;q)$	$(1-q)^{\mu+1-\alpha}2^{\alpha-1}\ H^{4,1}_{2,5}\left[\frac{x^2(1-q)^2}{4q^{2\mu}};q \middle\vert \begin{matrix} (\frac{\alpha+1}{2},1),(\mu+\eta,2) \\ (\eta,2),(\frac{\nu}{2},1),(\frac{-\nu}{2},1),(\frac{\alpha+1}{2},1),(1,1) \end{matrix} \right]$
3.17	$e_q(-x)$	$(1-q)^{\mu}G(q)G^{2,0}_{1,3}\left[xq^{-\mu}(1-q);q \middle\vert \begin{matrix} \mu+\eta \\ \eta,\ 0,\ 1 \end{matrix} \right]$
3.18	$\sin_q(x)$	$\sqrt{\pi}(1-q)^{\mu-1/2}\{G(q)\}^2 \times H^{2,0}_{1,4}\left[\frac{x^2(1-q)^2}{4q^{2\mu}};q \middle\vert \begin{matrix} (\mu+\eta,2) \\ (\eta,2),(\frac{1}{2},1),(0,1),(1,1) \end{matrix} \right]$

3.19	$\cos_q(x)$	$\sqrt{\pi}(1-q)^{\mu-1/2}\{G(q)\}^2 \; H_{1,4}^{2,0}\left[\frac{x^2(1-q)^2}{4q^{2\mu}};q \,\middle\vert\, \begin{matrix}(\mu+\eta,2)\\ (\eta,2),(0,1),(\frac{1}{2},1),(1,1)\end{matrix}\right]$
3.20	$\sinh_q(x)$	$\frac{\sqrt{\pi}}{i}(1-q)^{\mu-1/2}\{G(q)\}^2 \; H_{1,4}^{2,0}\left[\frac{-x^2(1-q)^2}{4q^{2\mu}};q \,\middle\vert\, \begin{matrix}(\mu+\eta,2)\\ (\eta,2),(\frac{1}{2},1),(0,1),(1,1)\end{matrix}\right]$
3.21	$\cosh_q(x)$	$\sqrt{\pi}(1-q)^{\mu-1/2}\{G(q)\}^2 \; H_{1,4}^{2,0}\left[\frac{-x^2(1-q)^2}{4q^{2\mu}};q \,\middle\vert\, \begin{matrix}(\mu+\eta,2)\\ (\eta,2),(0,1),(\frac{1}{2},1),(1,1)\end{matrix}\right]$

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