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INVERSION FORMULAS FOR THE q-RIEMANN-LIOUVILLE AND q-WEYL TRANSFORMS USING WAVELETS

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Abstract

This paper aims to study the q-wavelets and the continuous q-wavelet transforms, associated with the q-Bessel operator for a fixed $q \in]0,1[$. Using the q-Riemann-Liouville and the q-Weyl transforms, we give some relations between the continuous q-wavelet transform, studied in [3], and the continuous q-wavelet transform associated with the q-Bessel operator, and we deduce formulas which give the inverse operators of the q-Riemann-Liouville and the q-Weyl transforms.

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Key Words and Phrases: q-Bessel operator, q-wavelet, q-Riemann-Liouville and q-Weyl operators

1. Introduction

In [7], A. Fitouhi and K. Trimèche generalized the theory of continuous wavelet transforms as presented by T. H. Koornwinder [11] and studied the generalized wavelets and the generalized continuous wavelet transforms associated with a class of singular differential operators. This class contains, in particular, the so called Bessel operator, which was studied extensively by K. Trimèche in [13].

In [1], F. Bouzeffour studied fractional transforms associated with the q-Bessel operator and as an application, he gave inversion formulas for the q-Riemann-Liouville and q-Weyl transforms, introduced in [6].

In this paper, we try to generalize our results from [3] by studying wavelets and continuous wavelet transforms associated with the q-Bessel operator, studied in [6]. The basic tool in this work are some elements of q-harmonic analysis related to the just mentioned operator. Next, using the q-Riemann-Liouville and the q-Weyl operators, we will give some relations between the continuous q-wavelet transform, studied in [3], and the continuous q-wavelet transform associated with the q-Bessel operator, and we deduce other formulas which give the inverse operators of the q-Riemann-Liouville and the q-Weyl transforms. These formulas are better than those given in [6] and [1] because they are simple and we have a large choice of q-wavelets associated with the q-Bessel operator, that can be used in these formulas.

We are not in a position to claim that all our results here are new, but the methods used are direct and constructive, and have a good resemblance with the classical ones. Our approach in this paper is very similar to the classical picture developed in [7] and [13].

The paper is organized as follows: in Section 2, we present some q-harmonic results associated with the q-Bessel operator. In Section 3, we define the q-wavelets and the continuous q-wavelet transforms associated with the q-Bessel operator. In Section 4, we give a characterization of the image set of the q-wavelet transform associated with the q-Bessel operator. Section 5 is devoted to give some inversion formulas of the q-Riemann-Liouville and the q-Weyl transforms. Finally, in Section 6, we give the inversion formulas for the q-Riemann-Liouville and the q-Weyl transforms using wavelets.

2. Preliminaries on q-harmonic analysis related to the q-Bessel operator

Throughout this paper, we fix $q \in]0,1[$ such that $\frac{Log(1-q)}{Logq} \in \mathbb{Z}$ and $\alpha > -\frac{1}{2}$. We refer to [8] and [9] for the definitions, notations and properties of the q-shifted factorials, the q-hypergeometric functions, the Jackson's q-derivative and the Jackson's q-integrals. For the definitions and proprieties of the special functions used here, we refer to the papers ([12], [6], [4], [2], [1]). The main sets and functional spaces used are:

- $\mathbb{R}_q = \{\pm q^n : n \in \mathbb{Z}\} \cup \{0\}$, $\mathbb{R}_{q,+} = \{q^n : n \in \mathbb{Z}\}$ and $\widetilde{\mathbb{R}}_{q,+} = \mathbb{R}_{q,+} \cup \{0\}$.
- $\mathcal{D}_{*q}(\mathbb{R}_q)$ the space of restrictions on $\mathbb{R}_q$ of even infinitely q-differentiable functions on $\mathbb{R}$ with compact supports.

• $\mathcal{C}_{*q,0}(\mathbb{R}_q)$ the space of restrictions on $\mathbb{R}_q$ of even smooth functions, continued in 0 and vanishing at ∞.
• $\mathcal{S}_{*q}(\mathbb{R}_q)$ the space of restrictions on $\mathbb{R}_q$ of infinitely q-differentiable and even functions satisfying: $\sup_{x\in\mathbb{R}_q;0\leq k\leq n} \mid (1+x^2)^m D_q^k f(x) \mid < +\infty, \;\; n,m\in\mathbb{N}$.
• For $p>0$, $L^p_{q,\alpha}(\mathbb{R}_{q,+}) = \left\{ f : \|f\|_{p,\alpha,q} = \left(\int_0^\infty |f(x)|^p x^{2\alpha+1} d_q x \right)^{\frac{1}{p}} < \infty \right\}$.

The q-Bessel operator is defined and studied in [6] by

$$\Delta_{\alpha,q} f(z) = \left(\frac{1}{x^{2\alpha+1}} D_q [x^{2\alpha+1} D_q f] \right) \left(q^{-1} z\right).$$

We recall (see [6]) that for $\lambda \in \mathbb{C}$, the problem

$$\begin{cases} \Delta_{\alpha,q} u(x) = -\lambda^2 u(x), \\ u(0) = 1, u'(0) = 0 \end{cases} \tag{1}$$

has as unique solution the function $z \mapsto j_\alpha^{(3)}(\lambda z; q^2)$, where $j_\alpha^{(3)}(.; q^2)$ is the normalized q-Bessel function.

The generalized q-Bessel translation operator $T^\alpha_{q,x}$, $\;x\in\mathbb{R}_{q,+}$ is defined (see [6]) on $\mathcal{D}_{*q}(\mathbb{R}_q)$ by

$$T^\alpha_{q,x}(f)(y) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^2, q^{2\alpha+2}; q^2)_n} \left(\frac{x}{y}\right)^{2n} \sum_{k=-n}^{n} (-1)^{n-k} U_k(n) f(q^k y),$$

$y\in\mathbb{R}_{q,+}$ and $T^\alpha_{q,0}(f) = f$, where

$$U_k(n) = q^{k(k-1)+2n(k+\alpha)} \sum_{p=0}^{k} \begin{bmatrix} n \\ p \end{bmatrix}_{q^2} \begin{bmatrix} n \\ n+k-p \end{bmatrix}_{q^2} q^{-2p(n+k+\alpha-p)}$$

is the q-Bessel-Binomial coefficient associated with the q-Bessel operator [6]. It verifies, in particular

$$\int_0^\infty T^\alpha_{q,x}(f)(y) g(y) y^{2\alpha+1} d_q y = \int_0^\infty f(y) T^\alpha_{q,x}(g)(y) y^{2\alpha+1} d_q y, \;\; x\in\widetilde{\mathbb{R}}_{q,+}, \tag{2}$$

and

$$T^\alpha_{q,x} j_\alpha^{(3)}(ty; q^2) = j_\alpha^{(3)}(tx; q^2) j_\alpha^{(3)}(ty; q^2), \quad x,y,t\in\widetilde{\mathbb{R}}_{q,+}. \tag{3}$$

The q-Bessel Fourier transform and the q-convolution product are defined (see [6]) for $f,g\in\mathcal{D}_{*q}(\mathbb{R}_q)$, by

$$\mathcal{F}_{\alpha,q}(f)(\lambda) = c_{\alpha,q} \int_0^\infty f(x) j_\alpha^{(3)}(\lambda x; q^2) x^{2\alpha+1} d_q x, \tag{4}$$

$$f *_B g(x) = c_{\alpha,q} \int_0^\infty T^\alpha_{q,x} f(y) g(y) y^{2\alpha+1} d_q y, \tag{5}$$

where $c_{\alpha,q} = \dfrac{(1+q^{-1})^{-\alpha}}{\Gamma_{q^2}(\alpha+1)}$.

Using the properties of the generalized q-Bessel translation and the normalized q-Bessel function, one can prove easily the following results.

THEOREM 1. *For $f, g \in \mathcal{D}_{*q}(\mathbb{R}_q)$, we have*

$$\mathcal{F}_{\alpha,q}(f *_B g) = \mathcal{F}_{\alpha,q}(f)\mathcal{F}_{\alpha,q}(g), \tag{6}$$

$$\mathcal{F}_{\alpha,q}(T^\alpha_{q,x} f)(\lambda) = j^{(3)}_\alpha(\lambda x; q^2)\mathcal{F}_{\alpha,q}(f)(\lambda), \quad x \in \widetilde{\mathbb{R}}_{q,+}, \quad \lambda \in \mathbb{R}_{q,+} \tag{7}$$

and

$$\mathcal{F}_{\alpha,q}(\Delta_{\alpha,q} f)(\lambda) = -\frac{\lambda^2}{q^{2\alpha+1}}\mathcal{F}_{\alpha,q}(f)(\lambda), \quad \lambda \in \mathbb{C}. \tag{8}$$

THEOREM 2. *For $f \in L^1_{q,\alpha}(\mathbb{R}_{q,+})$, we have*

$$\mathcal{F}_{\alpha,q}(f) \in \mathcal{C}_{*q,0}(\mathbb{R}_q) \quad \textit{and} \quad \|\mathcal{F}_{\alpha,q}(f)\|_{\mathcal{C}_{*q,0}(\mathbb{R}_q)} \le \frac{c_{\alpha,q}}{(q;q^2)^2_\infty}\|f\|_{1,\alpha,q}. \tag{9}$$

THEOREM 3. *$\mathcal{F}_{\alpha,q}$ is an isomorphism of $S_{*,q}(\mathbb{R}_q)$ onto itself and can be extended continuously to an isomorphism of $L^2_{q,\alpha}(\mathbb{R}_{q,+})$ onto itself, $\mathcal{F}^{-1}_{\alpha,q} = q^{4\alpha+2}\mathcal{F}_{\alpha,q}$ and*

$$\forall f \in L^2_{q,\alpha}(\mathbb{R}_{q,+}), \quad \|\mathcal{F}_{\alpha,q}(f)\|_{2,\alpha,q} = q^{2\alpha+1}\|f\|_{2,\alpha,q}. \tag{10}$$

PROPOSITION 1. *Let f and g be in $L^2_q(\mathbb{R}_{q,+}, x^{2\alpha+1}d_q x)$, then:*
1) $f *_B g \in L^2_{q,\alpha}(\mathbb{R}_{q,+})$ iff $\mathcal{F}_{\alpha,q}(f)\mathcal{F}_{\alpha,q}(g) \in L^2_{q,\alpha}(\mathbb{R}_{q,+})$.
2)

$$q^{4\alpha+2}\int_0^\infty |\, f *_B g(x)\,|^2\, x^{2\alpha+1} d_q x = \int_0^\infty |\,\mathcal{F}_{\alpha,q}(f)(x)\,|^2 |\,\mathcal{F}_{\alpha,q}(g)(x)\,|^2\, x^{2\alpha+1} d_q x, \tag{11}$$

where both sides are finite or infinite.

REMARK 1. Using Theorem 3, the relation (7) and the fact that $\sup_{x\in\mathbb{R}_q} |j^{(3)}_\alpha(x;q^2)| \le \frac{1}{(q;q^2)^2_\infty}$, one can see that, for $f \in L^2_{q,\alpha}(\mathbb{R}_{q,+})$ (resp. $\mathcal{S}_{*q}(\mathbb{R}_q)$), we have for $x \in \widetilde{\mathbb{R}}_{q,+}$, $T^\alpha_{q,x} f$ is in $L^2_{q,\alpha}(\mathbb{R}_{q,+})$ (resp. $\mathcal{S}_{*q}(\mathbb{R}_q)$) and

$$\|T^\alpha_{q,x} f\|_{2,\alpha,q} \le \frac{1}{(q;q^2)^2_\infty}\|f\|_{2,\alpha,q}. \tag{12}$$

3. q-Wavelet transforms associated with the q-Bessel operator

Definition 1. A q-wavelet associated with the q-Bessel operator is an even function $g \in L^2_{q,\alpha}(\mathbb{R}_{q,+})$ satisfying the following admissibility condition:

$$0 < C_{\alpha,g} = \int_0^\infty \mid \mathcal{F}_{\alpha,q}(g)(a) \mid^2 \frac{d_q a}{a} < \infty. \tag{13}$$

Remarks:

1) For all $\lambda \in \mathbb{R}_{q,+}$, we have $\quad C_{\alpha,g} = \int_0^\infty |\mathcal{F}_{\alpha,q}(g)(a\lambda)|^2 \frac{d_q a}{a}.$

2) Let f be a nonzero function in $\mathcal{S}_{*q}(\mathbb{R}_q)$ (resp. $\mathcal{D}_{*q}(\mathbb{R}_q)$). Then $g = \Delta_{\alpha,q} f$ is a q-wavelet associated with the q-Bessel operator, in $\mathcal{S}_{*q}(\mathbb{R}_q)$ (resp. $\mathcal{D}_{*q}(\mathbb{R}_q)$) and we have $C_{\alpha,g} = \frac{1}{q^{4\alpha+2}} \int_0^\infty a^3 \mid \mathcal{F}_{\alpha,q}(f)(a) \mid^2 d_q a.$

Example: Consider the functions $G(x;q^2) = A_\alpha e_{q^2}^{-\frac{q^{-(2\alpha+1)}}{(1+q)^2}x^2}$ and $g = \Delta_{\alpha,q} G(.;q^2)$, where $A_\alpha = c_{\alpha,q} \int_0^\infty x^{2\alpha+1} e_{q^2}^{-x^2} d_q x$ and $e_{q^2}^{x} = \frac{1}{(x;q^2)_\infty}$. We have $G(.;q^2) \in \mathcal{S}_{*q}(\mathbb{R}_q)$ and with the use of the relation [6], $\mathcal{F}_{\alpha,q}(G(.;q^2))(x) = q^{4\alpha+2} e_{q^2}^{-x^2}$, we get $0 < \int_0^\infty |\mathcal{F}_{\alpha,q}(g)|^2(a) \frac{d_q a}{a} = \frac{q^{4\alpha}}{(1+q)}$. So, g is a q-wavelet associated with the q-Bessel operator.

Proposition 2. *Let $g \neq 0$ be a function in $L^2_{q,\alpha}(\mathbb{R}_{q,+})$ satisfying:*
1) $\mathcal{F}_{\alpha,q}(g)$ is continuous at 0.
2) $\exists \beta > 0$ such that $\mathcal{F}_{\alpha,q}(g)(x) - \mathcal{F}_{\alpha,q}(g)(0) = O(x^\beta)$, as $x \to 0$.
Then, (13) is equivalent to

$$\mathcal{F}_{\alpha,q}(g)(0) = 0. \tag{14}$$

P r o o f. • If $\mathcal{F}_{\alpha,q}(g)(0) \neq 0$, then from the condition 1) there exist $p_0 \in \mathbb{N}$ and $M > 0$, such that for all $n \geq p_0$, $\quad \mid \mathcal{F}_{\alpha,q}(g)(q^n) \mid \geq M$. Then, the q-integral in (13) would be equal to ∞.

• Conversely, we suppose that $\mathcal{F}_{\alpha,q}(g)(0) = 0$.

As $g \neq 0$, we deduce from Theorem 3, that the first inequality in (13) is satisfied.

On the other hand, from the condition 2), there exist $n_0 \in \mathbb{N}$ and $\epsilon > 0$, such that for all $n \geq n_0$, $\mid \mathcal{F}_{\alpha,q}(g)(q^n) \mid \leq \epsilon q^{n\beta}$. Then using the definition of the q-integral and Theorem 3, we obtain

$$\int_0^\infty \mid \mathcal{F}_{\alpha,q}(g)(a) \mid^2 \frac{d_q a}{a} \quad = \quad (1-q) \sum_{n=-\infty}^{\infty} \mid \mathcal{F}_{\alpha,q}(g)(q^n) \mid^2$$

$$\leq \frac{\|\mathcal{F}_{\alpha,q}(g)\|^2_{2,\alpha,q}}{q^{(2\alpha+2)n_0}} + \frac{1-q}{1-q^{2\beta}}\epsilon^2.$$

This proves the second inequality of (13). ■

REMARK 2. For $g \in L^1_{q,\alpha}(\mathbb{R}_{q,+})$, the continuity assumption in the previous proposition holds. Then (14) can be written as $\int_0^\infty g(x)x^{2\alpha+1}d_qx = 0$, which proves that g changes the sign on $\mathbb{R}_{q,+}$ and tends to 0 at ∞.

THEOREM 4. *For $a \in \mathbb{R}_{q,+}$ and $g \in L^2_{q,\alpha}(\mathbb{R}_{q,+})$ (resp. $\mathcal{S}_{*q}(\mathbb{R}_q)$), the function $g_a : x \mapsto \frac{1}{a^{2\alpha+2}}g\left(\frac{x}{a}\right)$ belongs to $L^2_{q,\alpha}(\mathbb{R}_{q,+})$ (resp. $\mathcal{S}_{*q}(\mathbb{R}_q)$) and we have*

$$\|g_a\|_{2,\alpha,q} = \frac{1}{a^{\alpha+1}}\|g\|_{2,\alpha,q}, \tag{15}$$

$$\mathcal{F}_{\alpha,q}(g_a)(\lambda) = \mathcal{F}_{\alpha,q}(g)(a\lambda), \quad \lambda \in \mathbb{R}_{q,+}. \tag{16}$$

P r o o f. The change of variable $u = \dfrac{x}{a}$ gives the result. ■

THEOREM 5. *Let g be a q-wavelet associated with the q-Bessel operator in $L^2_{q,\alpha}(\mathbb{R}_{q,+})$ (resp. $\mathcal{S}_{*q}(\mathbb{R}_q)$). Then for all $a \in \mathbb{R}_{q,+}$ and $b \in \widetilde{\mathbb{R}}_{q,+}$, the function*

$$g_{(a,b),\alpha}(x) = \sqrt{a}\; T^\alpha_{q,b}(g_a), \tag{17}$$

is a q-wavelet associated with the q-Bessel operator in $L^2_{q,\alpha}(\mathbb{R}_{q,+})$ (resp. $\mathcal{S}_{*q}(\mathbb{R}_q)$) and we have

$$C_{\alpha,g_{(a,b),\alpha}} = a\int_0^\infty \left(j^{(3)}_\alpha\left(\frac{xb}{a};q^2\right)\right)^2 \mid \mathcal{F}_{\alpha,q}(g)(x)\mid^2 \frac{d_qx}{x}. \tag{18}$$

P r o o f. As g_a is in $L^2_{q,\alpha}(\mathbb{R}_{q,+})$ (resp. $\mathcal{S}_{*q}(\mathbb{R}_q)$), Remark 1 shows that the relation (17) defines an element of $L^2_{q,\alpha}(\mathbb{R}_{q,+})$ (resp. $\mathcal{S}_{*q}(\mathbb{R}_q)$). Furthermore, from the relations (16) and (7), we have for all $\lambda \in \mathbb{R}_{q,+}$,

$$\mathcal{F}_{\alpha,q}(g_{(a,b),\alpha})(\lambda) = \sqrt{a}\, j^{(3)}_\alpha(b\lambda;q^2)\mathcal{F}_{\alpha,q}(g)(a\lambda).$$

This relation implies (18).

On the other hand, as $g \neq 0$, we deduce from (18) and Theorem 3 that $C_{\alpha,g_{(a,b),\alpha}} \neq 0$. Moreover, from the relation (13) and the fact that $\sup_{x\in\mathbb{R}_q}|j^{(3)}_\alpha(x;q^2)| \leq \frac{1}{(q;q^2)^2_\infty}$, we deduce that $C_{\alpha,g_{(a,b),\alpha}} \leq \dfrac{a}{(q;q^2)^4_\infty}C_{\alpha,g}$, which gives the result. ■

PROPOSITION 3. *Let g be a q-wavelet associated with the q-Bessel operator in $L^2_{q,\alpha}(\mathbb{R}_{q,+})$. Then the mapping $F : (a,b) \mapsto g_{(a,b),\alpha}$ is continuous from $\mathbb{R}_{q,+} \times \widetilde{\mathbb{R}}_{q,+}$ into $L^2_{q,\alpha}(\mathbb{R}_{q,+})$.*

P r o o f. It is clear that F is a mapping from $\mathbb{R}_{q,+} \times \widetilde{\mathbb{R}}_{q,+}$ into $L^2_{q,\alpha}(\mathbb{R}_{q,+})$ and it is continuous at all $(a,b) \in \mathbb{R}_{q,+} \times \mathbb{R}_{q,+}$. The properties of the generalized q-Bessel translation, Theorem 3 and the Lebesgue theorem prove that F is continuous at all points $(a,0)$, $a \in \mathbb{R}_{q,+}$. ■

DEFINITION 2. Let g be a q-wavelet associated with the q-Bessel operator in $L^2_{q,\alpha}(\mathbb{R}_{q,+})$. We define the continuous q-wavelet transform associated with the q-Bessel operator, for $f \in L^2_{q,\alpha}(\mathbb{R}_{q,+})$, by

$$\Psi^\alpha_{q,g}(f)(a,b) = c_{\alpha,q} \int_0^\infty f(x)\overline{g_{(a,b),\alpha}}(x) x^{2\alpha+1} d_q x, \;\; a \in \mathbb{R}_{q,+}, \;\; b \in \widetilde{\mathbb{R}}_{q,+}. \quad (19)$$

REMARK 3. The relation (19) can also be written in the form

$$\Psi^\alpha_{q,g}(f)(a,b) = \sqrt{a} f *_B \overline{g_a}(b) = \sqrt{a} q^{-4\alpha-2} \mathcal{F}_{\alpha,q}\left[\mathcal{F}_{\alpha,q}(f).\mathcal{F}_{\alpha,q}(\overline{g_a})\right](b).$$

We give some properties of $\Psi^\alpha_{q,g}$ in the following proposition.

PROPOSITION 4. *Let g be a q-wavelet associated with the q-Bessel operator in $L^2_{q,\alpha}(\mathbb{R}_{q,+})$ and $f \in L^2_{q,\alpha}(\mathbb{R}_{q,+})$, then, for all $a \in \mathbb{R}_{q,+}$, the function $b \mapsto \Psi^\alpha_{q,g}(f)(a,b)$ is continuous on $\widetilde{\mathbb{R}}_{q,+}$, $\lim_{b \to \infty} \Psi^\alpha_{q,g}(f)(a,b) = 0$ and*

$$\forall b \in \widetilde{\mathbb{R}}_{q,+}, \quad | \Psi^\alpha_{q,g}(f)(a,b) | \leq \frac{c_{\alpha,q}}{(q;q^2)^2_\infty a^{\alpha+1/2}} \|f\|_{2,\alpha,q} \|g\|_{2,\alpha,q}. \quad (20)$$

Additionally, if f and g are in $\mathcal{S}_{*q}(\mathbb{R}_q)$, then for all $a \in \mathbb{R}_{q,+}$, the function $b \mapsto \Psi^\alpha_{q,g}(f)(a,b)$ is in $\mathcal{S}_{*q}(\mathbb{R}_q)$.

P r o o f. The result follows from the properties of the generalized q-Bessel translation, the properties of the normalized q-Bessel function, the properties of the q-Bessel convolution product and Theorem 3. ■

THEOREM 6. *Let $g \in L^2_{q,\alpha}(\mathbb{R}_{q,+})$ be a q-wavelet associated with the operator $\Delta_{\alpha,q}$.*

i) **Plancheral formula for** $\Psi^\alpha_{q,g}$: *For $f \in L^2_{q,\alpha}(\mathbb{R}_{q,+})$, we have*

$$\frac{1}{C_{\alpha,g}} \int_0^\infty \int_0^\infty | \Psi^\alpha_{q,g}(f)(a,b) |^2 \, b^{2\alpha+1} \frac{d_q b d_q a}{a^2} = \|f\|^2_{2,\alpha,q}. \quad (21)$$

ii) **Parseval formula for** $\Psi^\alpha_{q,g}$: *For $f_1, f_2 \in L^2_{q,\alpha}(\mathbb{R}_{q,+})$, we have*

$$\int_0^\infty f_1(x)\overline{f}_2(x)x^{2\alpha+1}d_qx = \frac{1}{C_{\alpha,g}}\int_0^\infty\int_0^\infty \Psi_{q,g}^\alpha(f_1)(a,b)\overline{\Psi_{q,g}^\alpha}(f_2)(a,b)b^{2\alpha+1}\frac{d_qad_qb}{a^2}. \tag{22}$$

P r o o f. i) By using Fubini's theorem, Theorem 3, Remark 3 and the relations (16) and (11), we get the relation (21).

ii) The result is easily deduced from (21). ∎

Theorem 7. *Let g be a q-wavelet associated with the q-Bessel operator in $L^2_{q,\alpha}(\mathbb{R}_{q,+})$, then for all $f \in L^2_{q,\alpha}(\mathbb{R}_{q,+})$, we have*

$$f(x) = \frac{c_{\alpha,q}}{C_{\alpha,g}}\int_0^\infty \int_0^\infty \Psi_{q,g}^\alpha(f)(a,b)g_{(a,b),\alpha}(x)b^{2\alpha+1}\frac{d_qad_qb}{a^2}, \quad x \in \mathbb{R}_{q,+}. \tag{23}$$

P r o o f. For $f \in L^2_{q,\alpha}(\mathbb{R}_{q,+})$ and $x \in \mathbb{R}_{q,+}$, take in (22) $f_1 = f$ and $f_2 = \delta_x$. The result follows then from the definitions of $\Psi_{q,g}^\alpha$ and the q-Jackson's integral. ∎

4. Coherent states

Theorem 6 shows that the continuous q-wavelet transform associated with the q-Bessel operator $\Psi_{q,g}^\alpha$ is isometry from the Hilbert space $L^2_{q,\alpha}(\mathbb{R}_{q,+})$ into the Hilbert space $Ł^2_q(\mathbb{R}_{q,+} \times \widetilde{\mathbb{R}}_{q,+}; b^{2\alpha+1}\frac{d_qad_qb}{a^2C_{\alpha,g}})$ (the space of square integrable functions on $\mathbb{R}_{q,+} \times \widetilde{\mathbb{R}}_{q,+}$ with respect to the measure $b^{2\alpha+1}\frac{d_qad_qb}{a^2C_{\alpha,g}}$). For the characterization of the image of $\Psi_{q,g}^\alpha$, we consider the vectors $g_{(a,b),\alpha}$, $(a,b) \in \mathbb{R}_{q,+} \times \widetilde{\mathbb{R}}_{q,+}$, as a set of coherent states in the Hilbert space $L^2_{q,\alpha}(\mathbb{R}_{q,+})$ (see [11]).

Definition 3. A set of coherent states in a Hilbert space $\mathcal{H}$ is a subset $\{g_l\}_{l\in\mathcal{L}}$ of $\mathcal{H}$ such that:

i) $\mathcal{L}$ is a locally compact topological space and the mapping $l \mapsto g_l$ is continuous from $\mathcal{L}$ into $\mathcal{H}$.

ii) There is a positive Borel measure dl on $\mathcal{L}$ such that, for $f \in \mathcal{H}$,

$$\| f \|^2 = \int_{\mathcal{L}} \mid (f, g_l) \mid^2 dl,$$

where $(.,.)$ and $\| . \|$ are respectively the scalar product and the norm of $\mathcal{H}$.

Let now $\mathcal{H} = L^2_{q,\alpha}(\mathbb{R}_{q,+})$, $\mathcal{L} = \mathbb{R}_{q,+} \times \widetilde{\mathbb{R}}_{q,+}$ equipped with the induced topology of $\mathbb{R}^2$. Choose a nonzero function $g \in L^2_{q,\alpha}(\mathbb{R}_{q,+})$ and let

$g_{l} = g_{(a,b),\alpha}$, $l = (a,b) \in \mathcal{L}$ be given by the relation (17). Then we have a set of coherent states. Indeed, i) of Definition 3 is satisfied, because of Proposition 3, and ii) of Definition 3 is satisfied for the measure $b^{2\alpha+1}\dfrac{d_q a d_q b}{a^2 C_{\alpha,g}}$ (see Theorem 6). By adaptation of the approach introduced by T.H. Koornwinder in [11], we obtain the following result:

THEOREM 8. *Let F be in $L^2_q(\mathbb{R}_{q,+}\times\widetilde{\mathbb{R}}_{q,+}; b^{2\alpha+1}\frac{d_q a d_q b}{a^2 C_{\alpha,g}})$. Then F belongs to $Im\Psi^{\alpha}_{q,g}$ if and only if*

$$F(a,b)=\frac{1}{C_{\alpha,g}}\int_0^\infty\int_0^\infty F(a',b')\left(\int_0^\infty g_{(a',b'),\alpha}(x)\overline{g_{(a,b),\alpha}}(x)x^{2\alpha+1}d_qx\right)(b')^{2\alpha+1}\frac{d_qa'd_qb'}{(a')^2}.$$

5. Inversion formulas for the q-Riemann-Liouville and the q-Weyl operators

In the sequel, we will use the following spaces:

- $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q) = \left\{ f \in \mathcal{S}_{*q}(\mathbb{R}_q) : \int_0^\infty f(x)\, x^{2k+2\alpha+1} d_q x = 0,\ k = 0,1,... \right\}.$
- $\mathcal{S}^0_{*q}(\mathbb{R}_q) = \left\{ f \in \mathcal{S}_{*q}(\mathbb{R}_q) : D^{2k}_q f(0) = 0,\ k = 0,1,... \right\}.$

The q-Riemann-Liouville transform $R_{\alpha,q}$ is defined on $\mathcal{D}_{*q}(\mathbb{R}_q)$ by (see [6])

$$R_{\alpha,q}(f)(x) = \frac{(1+q)\Gamma_{q^2}(\alpha+1)}{\Gamma_{q^2}\left(\frac{1}{2}\right)\Gamma_{q^2}\left(\alpha+\frac{1}{2}\right)}\int_0^1 \frac{\left(t^2q^2;q^2\right)_\infty}{\left(t^2q^{2\alpha+1};q^2\right)_\infty} f(xt)d_qt. \tag{24}$$

The q-Weyl transform is defined on $\mathcal{D}_{*q}(\mathbb{R}_q)$ by (see [6])

$$W_{\alpha,q}(f)(x)=\frac{q(1+q^{-1})^{-\alpha+\frac{1}{2}}\Gamma_{q^2}(\alpha+1)}{\Gamma^2_{q^2}(\alpha+\frac{1}{2})}\int_{qx}^\infty \frac{(x^2/t^2q^2;q^2)_\infty}{(q^{2\alpha+1}x^2/t^2;q^2)_\infty} f(t)t^{2\alpha}d_qt. \tag{25}$$

On $\mathcal{D}_{*q}(\mathbb{R}_q)$ we have the relations (see [6])

$$\Delta_{\alpha,q}\circ R_{\alpha,q} = R_{\alpha,q}\circ\Delta_q \quad \text{and} \quad R_{\alpha,q}(f *_q g) = R_{\alpha,q}(f) *_B R_{\alpha,q}(g),$$

where "$*_q$" is the q-even convolution product associated with the operator $\Delta_q : f \mapsto D^2_q(f)(q^{-1}.)$ studied in [5].

The q-Fourier-cosine transform $\mathcal{F}_q$ (studied in [5]) and the q-Bessel transform are linked by the following relation (see [6]):

PROPOSITION 5. *For $f \in \mathcal{S}_{*q}(\mathbb{R}_q)$, we have*

$$\mathcal{F}_{\alpha,q}(f) = \mathcal{F}_q \circ W_{\alpha,q}(f). \tag{26}$$

We state the following results, useful in the sequel.

THEOREM 9. *The q-Fourier-cosine transform $\mathcal{F}_q$ is a topological isomorphism from $\mathcal{S}_{*q,-1/2}(\mathbb{R}_q)$ into $\mathcal{S}^0_{*q}(\mathbb{R}_q)$.*

P r o o f. The result follows from the Plancheral formula of $\mathcal{F}_q$ (see [2]) and the fact that $D_q^2 \cos(x; q^2) = -\cos(qx; q^2)$. ∎

Similarly, we have the following result.

THEOREM 10. *The q-Fourier-Bessel transform $\mathcal{F}_{\alpha,q}$ is a topological isomorphism from $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$ into $\mathcal{S}^0_{*q}(\mathbb{R}_q)$.*

COROLLARY 1. *The q-Weyl transform $W_{\alpha,q}$ is a topological isomorphism from $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$ into $\mathcal{S}_{*q,-1/2}(\mathbb{R}_q)$.*

P r o o f. We deduce the result from the relation $\mathcal{F}_{\alpha,q} = \mathcal{F}_q \circ W_{\alpha,q}$ and Theorems 9 and 10. ∎

PROPOSITION 6. *For f in $\mathcal{S}_{*q,-1/2}(\mathbb{R}_q)$ (resp. $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$) and g in $\mathcal{S}_{*q}(\mathbb{R}_q)$ the function $f *_q g$ (resp. $f *_B g$) belongs to $\mathcal{S}_{*q,-1/2}(\mathbb{R}_q)$ (resp. $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$).*

P r o o f. The result follows from Theorem 9 (resp. Theorem 10) and the fact that $f *_q g = \mathcal{F}_q(\mathcal{F}_q(f).\mathcal{F}_q(g))$ (resp. $f *_B g = q^{-4\alpha-2}\mathcal{F}_{\alpha,q}(\mathcal{F}_{\alpha,q}(f).\mathcal{F}_{\alpha,q}(g))$. ∎

PROPOSITION 7. *The operator $K_{\alpha,q,1}$ defined by*

$$K_{\alpha,q,1}(f) = \frac{\Gamma_{q^2}(1/2)}{q^{3\alpha+3/2}(1+q)^{(\alpha+1/2)}\Gamma_{q^2}(\alpha+1)} \mathcal{F}_q^{-1}(|\lambda|^{2\alpha+1}\mathcal{F}_q(f))$$

is a topological isomorphism from $\mathcal{S}_{*q,-1/2}(\mathbb{R}_q)$ onto itself.

P r o o f. The multiplication operator

$$f \mapsto \frac{\Gamma_{q^2}(1/2)}{q^{3\alpha+3/2}(1+q)^{(\alpha+1/2)}\Gamma_{q^2}(\alpha+1)} |\lambda|^{2\alpha+1} f$$

is a topological isomorphism from $\mathcal{S}^0_{*q}(\mathbb{R}_q)$ into itself. The inverse is given by $f \mapsto \frac{q^{3\alpha+3/2}(1+q)^{(\alpha+1/2)}\Gamma_{q^2}(\alpha+1)}{\Gamma_{q^2}(1/2)|\lambda|^{2\alpha+1}} f$. The result follows then from Theorem 9. ∎

PROPOSITION 8. *The operator* $K_{\alpha,q,2}$ *defined by*

$$K_{\alpha,q,2}(f)(x) = \frac{\Gamma_{q^2}(1/2)}{q^{3\alpha+3/2}(1+q)^{(\alpha+1/2)}\Gamma_{q^2}(\alpha+1)} \mathcal{F}^{-1}_{\alpha,q}(|\lambda|^{2\alpha+1}\,\mathcal{F}_{\alpha,q}(f))(x)$$

is a topological isomorphism from $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$ *onto itself.*

P r o o f. From the relation $\mathcal{F}_{\alpha,q} = \mathcal{F}_q \circ W_{\alpha,q}$ and the definition of $K_{\alpha,q,1}$, we have for all $f \in \mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$

$$K_{\alpha,q,2} = W^{-1}_{\alpha,q} \circ K_{\alpha,q,1} \circ W_{\alpha,q}. \tag{27}$$

We deduce the result from Proposition 7 and Corollary 1. ∎

PROPOSITION 9. *i) For all* $f \in \mathcal{S}_{*q,-1/2}(\mathbb{R}_q)$ *and* $g \in \mathcal{S}_{*q}(\mathbb{R}_q)$, *we have*

$$K_{\alpha,q,1}(f *_q g) = K_{\alpha,q,1}(f) *_q g.$$

ii) For all $f \in \mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$ *and* $g \in \mathcal{S}_{*q}(\mathbb{R}_q)$, *we have*

$$K_{\alpha,q,2}(f *_B g) = K_{\alpha,q,2}(f) *_B g.$$

P r o o f. The result follows from the properties of the q-convolution products and the definitions of $K_{\alpha,q,1}$ and $K_{\alpha,q,2}$. ∎

THEOREM 11. *For all* $f \in \mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$, *we have the following inversion formulas for the operator* $R_{\alpha,q}$

$$f = R_{\alpha,q} \circ K_{\alpha,q,1} \circ W_{\alpha,q}(f), \tag{28}$$

$$f = R_{\alpha,q} \circ W_{\alpha,q} \circ K_{\alpha,q,2}(f). \tag{29}$$

P r o o f. Using the properties of the operator $R_{\alpha,q}$, studied in [6], Theorem 3 and Proposition 5, we obtain for $x \in \widetilde{\mathbb{R}}_{q,+}$,

$$\begin{aligned} q^{4\alpha+2} f(x) &= c_{\alpha,q} \int_0^{\infty} \mathcal{F}_{\alpha,q}(f)(\lambda) j^{(3)}_{\alpha}(\lambda x; q^2) \lambda^{2\alpha+1} d_q\lambda \\ &= R_{\alpha,q}\left[c_{\alpha,q} \int_0^{\infty} \mathcal{F}_{\alpha,q}(f)(\lambda) \cos(\lambda .; q^2) \lambda^{2\alpha+1} d_q\lambda\right](x) \\ &= R_{\alpha,q}\left\{\frac{c_{\alpha,q}}{c_{-1/2,q}} \mathcal{F}^{-1}_q\left[\lambda^{2\alpha+1}\mathcal{F}_q \circ W_{\alpha,q}(f)\right]\right\}(x) \\ &= q^{4\alpha+2} R_{\alpha,q} \circ K_{\alpha,q,1} \circ W_{\alpha,q}(f)(x). \end{aligned}$$

We deduce the second from the first relation and the the relation (27). ∎

Corollary 2. *The operator* $R_{\alpha,q}$ *is a topological isomorphism from* $\mathcal{S}_{*q,-1/2}(\mathbb{R}_q)$ *into* $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$.

P r o o f. We deduce the result from Proposition 7, Corollary 1 and relation (28). ■

Similarly, we have the following result.

Theorem 12. *For all* $f \in \mathcal{S}_{*q,-1/2}(\mathbb{R}_q)$, *we have the following inversion formulas for the operator* $W_{\alpha,q}$

$$f = W_{\alpha,q} \circ R_{\alpha,q} \circ K_{\alpha,q,1}(f), \tag{30}$$

$$f = W_{\alpha,q} \circ K_{\alpha,q,2} \circ R_{\alpha,q}(f). \tag{31}$$

P r o o f. For $f \in \mathcal{S}_{*q,-1/2}(\mathbb{R}_q)$, Corollary 1 (resp. Corollary 2) implies that $W^{-1}_{\alpha,q}(f)$ (resp. $R_{\alpha,q}(f)$) belongs to $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$. Then by writing the relation (28) (resp. (29)) for $W^{-1}_{\alpha,q}(f)$ (resp. $R_{\alpha,q}(f)$), we obtain the result. ■

Corollary 3. i) *For all* $f, g \in \mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$, *we have*

$$W_{\alpha,q}\left(f *_B g\right) = W_{\alpha,q}(f) *_q W_{\alpha,q}(g). \tag{32}$$

ii) *For all* $f, g \in \mathcal{S}_{*q,-1/2}(\mathbb{R}_q)$ *we have*

$$R_{\alpha,q}\left(f *_q g\right) = R_{\alpha,q}\left(f\right) *_B W^{-1}_{\alpha,q}(g). \tag{33}$$

6. Inversion formulas for the q-Riemann-Liouville and the q-Weyl operators using wavelets

In this section, we assume that the reader is familiar with the notions and notations presented in [3]. In particular, we recall the following two notations:

$$H_a(f)(x) = \frac{1}{\sqrt{a}} f\left(\frac{x}{a}\right) \quad \text{and} \quad C_g = \int_0^\infty |\mathcal{F}_q(g)|^2(a)\frac{d_q a}{a}.$$

We begin by the next useful and easily verified result.

Proposition 10. *For all* $a \in \mathbb{R}_{q,+}$ *and* $g \in L^2_{q,\alpha}(\mathbb{R}_{q,+})$, *we have*

$$g_a = \frac{1}{a^{2\alpha+3/2}} H_a(g) = \frac{q^{-4\alpha-2}}{\sqrt{a}} \mathcal{F}_{\alpha,q} \circ H_{a^{-1}} \circ \mathcal{F}_{\alpha,q}(g) = \frac{1}{\sqrt{a}} W^{-1}_{\alpha,q} \circ H_a \circ W_{\alpha,q}(g). \tag{34}$$

PROPOSITION 11. *Let g be a q-wavelet associated with the q-Bessel operator in $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$. Then for all f in $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$, we have the following relation*

$$\Psi_{q,g}^{\alpha}(f)(a,.) = W_{\alpha,q}^{-1}\left[\Phi_{q,W_{\alpha,q}(g)}\left(W_{\alpha,q}(f)\right)(a,.)\right], \qquad a \in \mathbb{R}_{q,+}. \tag{35}$$

P r o o f. Let $a \in \mathbb{R}_{q,+}$, from the properties of the continuous q-wavelet transform (see [3]), and relations (32) and (34), we have

$$\begin{aligned}\Psi_{q,g}^{\alpha}(f)(a,.) &= \sqrt{a} f *_B \overline{g_a} = \sqrt{a} W_{\alpha,q}^{-1}\left[W_{\alpha,q}(f) *_q W_{\alpha,q}(\overline{g_a})\right] \\ &= W_{\alpha,q}^{-1}\left[W_{\alpha,q}(f) *_q \overline{H_a \circ W_{\alpha,q}(g)}\right] \\ &= W_{\alpha,q}^{-1}\left[\Phi_{q,W_{\alpha,q}(g)}\left(W_{\alpha,q}(f)\right)(a,.)\right].\end{aligned}$$

∎

THEOREM 13. *Let g be a q-wavelet associated with the q-Bessel operator in $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$. Then:*

1) For all f in $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$, we have for $a \in \mathbb{R}_{q,+}$, $b \in \widetilde{\mathbb{R}}_{q,+}$,

$$\Psi_{q,g}^{\alpha}(f)(a,b) = R_{\alpha,q}\left[\Phi_{q,W_{\alpha,q}(g)}\left(R_{\alpha,q}^{-1}(f)\right)(a,.)\right](b). \tag{36}$$

2) For all f in $\mathcal{S}_{*q,-1/2}(\mathbb{R}_q)$, we have for $a \in \mathbb{R}_{q,+}$, $b \in \widetilde{\mathbb{R}}_{q,+}$,

$$\Phi_{q,W_{\alpha,q}(g)}(f)(a,b) = W_{\alpha,q}\left[\Psi_{q,g}^{\alpha}\left(W_{\alpha,q}^{-1}(f)\right)(a,.)\right](b). \tag{37}$$

P r o o f. We deduce the result from Corollary 3, the properties of the continuous q-wavelet transform (see [3]) and the relation (34). ∎

PROPOSITION 12. *1) If g is a q-wavelet in $\mathcal{S}_{*q,-1/2}(\mathbb{R}_q)$, then $K_{\alpha,q,1}(g)$ is a q-wavelet in $\mathcal{S}_{*q,-1/2}(\mathbb{R}_q)$ and we have*

$$K_{\alpha,q,1} \circ H_a(g) = \frac{1}{a^{2\alpha+1}} H_a \circ K_{\alpha,q,1}(g), \qquad a \in \mathbb{R}_{q,+}. \tag{38}$$

2) If g is a q-wavelet associated with the q-Bessel operator in $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$, then $K_{\alpha,q,2}(g)$ is a q-wavelet in $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$ and we have

$$K_{\alpha,q,2}(g_a) = \frac{1}{a^{2\alpha+1}}\left(K_{\alpha,q,2}(g)\right)_a, \qquad a \in \mathbb{R}_{q,+}. \tag{39}$$

P r o o f. 1) Let g be a q-wavelet in $\mathcal{S}_{*q,-1/2}(\mathbb{R}_q)$. From the definition of $K_{\alpha,q,1}$, we have for $\lambda \in \mathbb{R}_{q,+}$,

$$\mathcal{F}_q(K_{\alpha,q,1}(g))(\lambda) = \frac{\Gamma_{q^2}(1/2)}{q^{3\alpha+3/2}(1+q)^{(\alpha+1/2)}\Gamma_{q^2}(\alpha+1)}\lambda^{2\alpha+1}\mathcal{F}_q(g)(\lambda).$$

Proposition 4 of [3], implies that $K_{\alpha,q,1}(g)$ is a q-wavelet. On the other hand, using the fact $\mathcal{F}_q \circ H_a = H_{a^{-1}} \circ \mathcal{F}_q$, $a \in \mathbb{R}_{q,+}$ and the above equality, we obtain

$$\begin{aligned}&\mathcal{F}_q(H_a \circ K_{\alpha,q,1}(g))(\lambda)\\ &\quad = a^{2\alpha+1}\frac{\Gamma_{q^2}(1/2)}{q^{3\alpha+3/2}(1+q)^{(\alpha+1/2)}\Gamma_{q^2}(\alpha+1)}\lambda^{2\alpha+1}\mathcal{F}_q\left(H_a(g)\right)(\lambda),\end{aligned}$$

which gives the result.

2) The same way of 1) leads to the result. ∎

Theorem 14. *Let g be a q-wavelet associated with the q-Bessel operator in $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$. Then for $a \in \mathbb{R}_{q,+}$ and $b \in \widetilde{\mathbb{R}}_{q,+}$, we have:*

1) For all f in $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$,

$$\Psi^{\alpha}_{q,g}(f)(a,b) = \frac{1}{a^{2\alpha+1}}R_{\alpha,q}\left[\Phi_{q,K_{\alpha,q,1}\circ W_{\alpha,q}(g)}(W_{\alpha,q}(f))(a,.)\right](b). \tag{40}$$

2) For all f in $\mathcal{S}_{*q,-1/2}(\mathbb{R}_q)$,

$$\Phi_{q,W_{\alpha,q}(g)}(f)(a,b) = \frac{1}{a^{2\alpha+1}}W_{\alpha,q}\left[\Psi^{\alpha}_{q,K_{\alpha,q,2}(g)}(R_{\alpha,q}(f))(a,.)\right](b). \tag{41}$$

P r o o f. 1) Let f be in $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$, $a \in \mathbb{R}_{q,+}$ and $b \in \widetilde{\mathbb{R}}_{q,+}$. Using Corollary 3, we obtain

$$\Psi^{\alpha}_{q,g}(f)(a,b) = \sqrt{a}f *_B \overline{g_a}(b) = \sqrt{a}R_{\alpha,q}\left[W_{\alpha,q}(f) *_q R^{-1}_{\alpha,q}(\overline{g_a})\right](b).$$

So, Theorem 11, Proposition 12 and the relation (34), achieve the proof.

2) Follows from Corollary 3, Theorem 12 and Propositions 9 and 12. ∎

Theorem 15. *Let g be a q-wavelet associated with the q-Bessel operator in $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$. Then for all $x \in \mathbb{R}_{q,+}$:*

1) For all f in $\mathcal{S}_{*q,-1/2}(\mathbb{R}_q)$, we have

$$W^{-1}_{\alpha,q}(f)(x) = \frac{c_{\alpha,q}}{C_{\alpha,g}} \int_0^\infty$$
$$\times \left(\int_0^\infty R_{\alpha,q}[\Phi_{q,K_{\alpha,q,1}\circ W_{\alpha,q}(g)}(f)(a,.)](b) \times g_{(a,b),\alpha}(x) \frac{b^{2\alpha+1}}{a^{2\alpha+3}} d_q b \right) d_q a.$$

2) For all f in $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$, we have

$$R^{-1}_{\alpha,q}(f)(x) = \frac{c_{-\frac{1}{2},q}}{C_g} \int_0^\infty \left(\int_0^\infty W_{\alpha,q}\left[\Psi^\alpha_{q,K_{\alpha,q,2}(g)}(f)(a,.)\right](b) g_{a,b}(x) \frac{d_q b}{a^{2\alpha+3}} \right) d_q a.$$

P r o o f. The result follows from the previous theorem, Theorem 7 and ([3], Theorem 7). ∎

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