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ON q -LAPLACE TRANSFORMS OF THE q -BESSEL FUNCTIONS

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Abstract

The present paper deals with the evaluation of the q -Laplace transforms of a product of basic analogues of the Bessel functions. As applications, several useful special cases have been deduced.

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Key Words and Phrases: q -Laplace transforms, basic hypergeometric functions, basic analogue of the Bessel function

1. Introduction

Recently, Yadav and Purohit [12]-[14] evaluated the q -Laplace images of a number of q -polynomials and generalized basic hypergeometric functions of one and more variables, including the basic analogue of Fox's H -function (due to Saxena, Modi and Kalla [9]), and Purohit, Yadav and Vyas [7] obtained q -Laplace transform of a basic analogue of the I -function (due to Saxena and Kumar [8]).

Hahn [6] defined the q -analogues of the well-known classical Laplace transform

$$\phi(s) = \int_0^{\infty} e^{-st} f(t) dt, \quad (1.1)$$

by means of the following q -integrals:

$${}_qL_s \{f(t)\} = \frac{1}{(1-q)} \int_0^{s^{-1}} E_q(qst) f(t) d(t; q), \quad (1.2)$$

and

$${}_qL_s \{f(t)\} = \frac{1}{(1-q)} \int_0^\infty e_q(-st) f(t) d(t; q), \quad (1.3)$$

where the q -exponential series (analogues of the classical exponential function) are defined by

$$e_q(t) = \sum_{n=0}^{\infty} \frac{t^n}{(q; q)_n}, \quad (1.4)$$

and

$$E_q(t) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2} t^n}{(q; q)_n}. \quad (1.5)$$

The basic integration (cf. Gasper and Rahman [4]) is defined by

$$\int_0^x f(t) d(t; q) = x(1-q) \sum_{k=0}^{\infty} q^k f(xq^k). \quad (1.6)$$

By virtue of (1.6), the operator (1.2) can be expressed as

$$\phi(s) \equiv {}_qL_s \{f(t)\} = \frac{(q; q)_\infty}{s} \sum_{j=0}^{\infty} \frac{q^j f(s^{-1}q^j)}{(q; q)_j}. \quad (1.7)$$

The correspondence defined by operators (1.2) and (1.7) shall be denoted symbolically by

$$f(t) \supset_q \phi(s),$$

where the function $f(t)$ is called the original function, and $\phi(s)$ is named as the q -Laplace transform, or q -image of the original function $f(t)$.

For real or complex a and $|q| < 1$, the q -shifted factorial is defined as:

$$(a; q)_n = \begin{cases} 1 & ; \text{if } n = 0 \\ (1-a)(1-aq) \cdots (1-aq^{n-1}) & ; \text{if } n \in N, \end{cases} \quad (1.8)$$

also

$$(x-y)_\nu = x^\nu \prod_{n=0}^{\infty} \left[\frac{1 - (y/x)q^n}{1 - (y/x)q^{\nu+n}} \right], \quad (1.9)$$

and

$$\Gamma_q(a) = \frac{(q; q)_\infty}{(q^a; q)_\infty (1-q)^{a-1}} = \frac{(q; q)_{a-1}}{(1-q)^{a-1}}, \quad (1.10)$$

where $a \neq 0, -1, -2, \dots$.

The generalized basic hypergeometric series ${}_r\Phi_s(\cdot)$ (cf. Slater [10]), is:

$${}_r\Phi_s \left[\begin{matrix} a_1, \dots, a_r; \\ b_1, \dots, b_s; \end{matrix} \middle| q, x \right] = \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_r; q)_n x^n}{(q, b_1, \dots, b_s; q)_n}, \quad (1.11)$$

where for the convergence of the series (1.11), we require $|q| < 1$ and $|x| < 1$ if $r = s + 1$.

The q -analogue of the Bessel function (cf. Gasper and Rahman [4]) is defined as

$$J_\nu(x; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \left(\frac{x}{2}\right)^\nu {}_2\Phi_1 \left[\begin{matrix} 0, 0; \\ q^{\nu+1}; \end{matrix} \middle| q, -x^2/4 \right]. \quad (1.12)$$

Abdi [1] has investigated the fundamental properties of the q -Laplace transforms and established several theorems related to q -images of basic functions.

The main motivation for this paper is to evaluate the q -Laplace transform of product of basic analogues of the Bessel functions. Interesting special cases of the main result are also discussed.

2. q -Laplace image of product of q -Bessel functions

THEOREM. Consider a $t^{\nu-1}$ -weighted product of n different q -Bessel functions $J_{2\mu_j}(2\sqrt{a_j t}; q)$, $j = 1, \dots, n$. Then, their q -Laplace transform is given by

$${}_qL_s \{t^{\nu-1} J_{2\mu_1}(2\sqrt{a_1 t}; q) \cdots J_{2\mu_n}(2\sqrt{a_n t}; q)\} = \frac{\Gamma_q(\nu+M)(1-q)^{\nu-M-1} a_1^{\mu_1} \cdots a_n^{\mu_n}}{s^{\nu+M} \Gamma_q(2\mu_1+1) \cdots \Gamma_q(2\mu_n+1)} \\ \times \Psi_2^{(n)} \left(q^{\nu+M}; q^{2\mu_1+1}, \dots, q^{2\mu_n+1}; q; \frac{-a_1}{s}, \dots, \frac{-a_n}{s} \right), \quad (2.1)$$

where $M = \mu_1 + \cdots + \mu_n$, $Re(\nu + M) > 0$ and $Re(s) > 0$.

P r o o f. To prove (2.1), we put

$$f(t) = t^{\nu-1} J_{2\mu_1}(2\sqrt{a_1 t}; q) \cdots J_{2\mu_n}(2\sqrt{a_n t}; q)$$

into definition (1.7) and use (1.12), to obtain

$${}_qL_s \{t^{\nu-1} J_{2\mu_1}(2\sqrt{a_1 t}; q) \cdots J_{2\mu_n}(2\sqrt{a_n t}; q)\} = \frac{(q; q)_\infty}{s} \sum_{j=0}^{\infty} \frac{q^j}{(q; q)_j} \left(\frac{q^j}{s}\right)^{\nu-1}$$

$$\begin{aligned} & \times \frac{(q^{2\mu_1+1}; q)_\infty}{(q; q)_\infty} \left(\frac{a_1 q^j}{s}\right)^{\mu_1} \cdots \frac{(q^{2\mu_n+1}; q)_\infty}{(q; q)_\infty} \left(\frac{a_n q^j}{s}\right)^{\mu_n} \\ & \times \sum_{m_1, \dots, m_n=0}^{\infty} \frac{(-a_1 q^j/s)^{m_1}}{(q^{2\mu_1+1}; q)_{m_1} (q; q)_{m_1}} \cdots \frac{(-a_n q^j/s)^{m_n}}{(q^{2\mu_n+1}; q)_{m_n} (q; q)_{m_n}}. \end{aligned} \quad (2.2)$$

Interchanging the order of summations in the right-hand side of the above equation (2.2), this yields

$$\begin{aligned} & \frac{(q^{2\mu_1+1}; q)_\infty}{(q; q)_\infty} \cdots \frac{(q^{2\mu_n+1}; q)_\infty}{(q; q)_\infty} \frac{(q; q)_\infty a_1^{\mu_1} \cdots a_n^{\mu_n}}{s^{\nu+\mu_1+\cdots+\mu_n}} \\ & \times \sum_{m_1, \dots, m_n=0}^{\infty} \frac{(-a_1/s)^{m_1}}{(q^{2\mu_1+1}; q)_{m_1} (q; q)_{m_1}} \cdots \frac{(-a_n/s)^{m_n}}{(q^{2\mu_n+1}; q)_{m_n} (q; q)_{m_n}} \\ & \times \sum_{j=0}^{\infty} \frac{q^{j(\nu+\mu_1+\cdots+\mu_n+m_1+\cdots+m_n)}}{(q; q)_j}. \end{aligned}$$

Using equation (1.10), and then summing the inner ${}_0\Phi_0(\cdot)$ -series with the help of the formula

$${}_0\Phi_0(-; -; q; t) = \frac{1}{(t; q)_\infty}, \quad (2.3)$$

the above expression yields:

$$\begin{aligned} & \frac{(q; q)_\infty (1-q)^{-2(\mu_1+\cdots+\mu_n)} a_1^{\mu_1} \cdots a_n^{\mu_n}}{s^{\nu+\mu_1+\cdots+\mu_n} \Gamma_q(2\mu_1+1) \cdots \Gamma_q(2\mu_n+1)} \\ & \times \sum_{m_1, \dots, m_n=0}^{\infty} \frac{(-a_1/s)^{m_1}}{(q^{2\mu_1+1}; q)_{m_1} (q; q)_{m_1}} \cdots \\ & \times \frac{(-a_n/s)^{m_n}}{(q^{2\mu_n+1}; q)_{m_n} (q; q)_{m_n}} \frac{1}{(q^{\nu+\mu_1+\cdots+\mu_n+m_1+\cdots+m_n}; q)_\infty} \\ & = \frac{(q; q)_\infty (1-q)^{-2(\mu_1+\cdots+\mu_n)} a_1^{\mu_1} \cdots a_n^{\mu_n}}{s^{\nu+\mu_1+\cdots+\mu_n} \Gamma_q(2\mu_1+1) \cdots \Gamma_q(2\mu_n+1) (q^{\nu+\mu_1+\cdots+\mu_n}; q)_\infty} \\ & \times \sum_{m_1, \dots, m_n=0}^{\infty} \frac{(q^{\nu+\mu_1+\cdots+\mu_n}; q)_{m_1+\cdots+m_n}}{(q^{2\mu_1+1}; q)_{m_1} \cdots (q^{2\mu_n+1}; q)_{m_n}} \frac{(-a_1/s)^{m_1}}{(q; q)_{m_1}} \cdots \frac{(-a_n/s)^{m_n}}{(q; q)_{m_n}}. \end{aligned}$$

Further simplifications lead to

$$\frac{\Gamma_q(\nu+M)(1-q)^{\nu-M-1} a_1^{\mu_1} \cdots a_n^{\mu_n}}{s^{\nu+M} \Gamma_q(2\mu_1+1) \cdots \Gamma_q(2\mu_n+1)}$$

$$\times \Psi_2^{(n)} \left(q^{\nu+M}; q^{2\mu_1+1}, \dots, q^{2\mu_n+1}; q; \frac{-a_1}{s}, \dots, \frac{-a_n}{s} \right), \quad (2.4)$$

where $M = \mu_1 + \dots + \mu_n$ and $\Psi_2^{(n)}(\cdot)$ denotes the confluent q -hypergeometric function defined as

$$\begin{aligned} & \Psi_2^{(n)}(a; c_1, \dots, c_n; q; x_1, \dots, x_n) \\ &= \sum_{m_1, \dots, m_n=0}^{\infty} \frac{(a; q)_{m_1+\dots+m_n}}{(c_1; q)_{m_1} \dots (c_n; q)_{m_n}} \frac{x_1^{m_1}}{(q; q)_{m_1}} \dots \frac{x_n^{m_n}}{(q; q)_{m_n}}. \end{aligned} \quad (2.5)$$

This completes the proof of (2.1). ■

3. Special cases

In this section we evaluate the q -Laplace transforms involving the q -Bessel functions as applications of our main result (2.1). First, we put $n = 2$ in (2.1) and this yields a q -Laplace image of a product of two q -Bessel functions, namely:

$$\begin{aligned} {}_qL_s \left\{ t^{\nu-1} J_{2\mu_1}(2\sqrt{a_1t}; q) J_{2\mu_2}(2\sqrt{a_2t}; q) \right\} &= \frac{\Gamma_q(\nu + M)(1 - q)^{\nu - M - 1} a_1^{\mu_1} a_2^{\mu_2}}{s^{\nu + M} \Gamma_q(2\mu_1 + 1) \Gamma_q(2\mu_2 + 1)} \\ &\times \Psi_2 \left(q^{\nu+M}; q^{2\mu_1+1}, q^{2\mu_2+1}; q; \frac{-a_1}{s}, \frac{-a_2}{s} \right), \end{aligned} \quad (3.1)$$

where $M = \mu_1 + \mu_2$, $Re(\nu + M) > 0$ and $Re(s) > 0$.

If we put $n = 1$, $\mu_1 = \nu$, $\nu = \mu$ and $a_1 = a$ in (2.1), we obtain a Laplace q -image of basic analogue of the Bessel function as below:

$${}_qL_s \left\{ t^{\mu-1} J_{2\nu}(2\sqrt{at}; q) \right\} = \frac{\Gamma_q(\mu + \nu)(1 - q)^{\mu - \nu - 1} a^\nu}{s^{\mu + \nu} \Gamma_q(2\nu + 1)} {}_1\Phi_1 \left[\begin{matrix} q^{\mu + \nu}; \\ q^{2\nu + 1}; \end{matrix} \quad q, -a/s \right], \quad (3.2)$$

for $Re(\mu + \nu) > 0$ and $Re(s) > 0$.

Replacing μ and ν by $\frac{\nu}{2} + 1$ and $\frac{\nu}{2}$ respectively in (3.2), yields

$${}_qL_s \left\{ t^{\nu/2} J_\nu(2\sqrt{at}; q) \right\} = a^{\nu/2} s^{-\nu-1} e_q(-a/s), \quad Re(s) > 0. \quad (3.3)$$

Next, we put $\nu = 1$ in (3.3) and get

$${}_qL_s \left\{ t^{1/2} J_1(2\sqrt{at}; q) \right\} = a^{1/2} s^{-2} e_q(-a/s), \quad Re(s) > 0. \quad (3.4)$$

Similarly, for $\nu = 0$ the equation (3.3) reduces to

$${}_qL_s \left\{ J_0(2\sqrt{at}; q) \right\} = s^{-1} e_q(-a/s), \quad \text{Re}(s) > 0. \quad (3.5)$$

Again, on replacing μ and ν by $1 - \frac{\nu}{2}$ and $\frac{\nu}{2}$ respectively, (3.2) yields

$${}_qL_s \left\{ t^{-\nu/2} J_\nu(2\sqrt{at}; q) \right\} = \frac{(1-q)^{-\nu} a^{\nu/2}}{s \Gamma_q(\nu+1)} {}_1\Phi_1 \left[\begin{matrix} q; \\ q^{\nu+1}; \end{matrix} q, -a/s \right].$$

By further simplifications this reduces to

$${}_qL_s \left\{ t^{-\nu/2} J_\nu(2\sqrt{at}; q) \right\} = \frac{(-1)^\nu s^{\nu-1}}{a^{\nu/2} \Gamma_q(\nu)} e_q(-a/s) \Gamma_q(\nu, -a/s), \quad (3.6)$$

where $\Gamma_q(\alpha, x)$ denotes the q -extension of the incomplete gamma function $\gamma(\alpha, x)$ (cf. Gupta [5], eqn.(9), p.253):

$$\Gamma_q(\alpha, x) = \frac{x^\alpha (x; q)_\infty (1-q)^{1-\alpha}}{(1-q^\alpha)} {}_1\Phi_1 \left[\begin{matrix} q; \\ q^{\alpha+1}; \end{matrix} q, x \right]. \quad (3.7)$$

Further, for $\nu = 0$ and $a = 0$, we obtain a q -extension of the well-known result for the Laplace transform, namely:

$${}_qL_s \{ t^{\mu-1} \} = \frac{\Gamma_q(\mu)(1-q)^{\mu-1}}{s^\mu}, \quad \text{Re}(s) > 0. \quad (3.8)$$

Finally, it is interesting to observe that in view of the limit formulae

$$\lim_{q \rightarrow 1^-} \Gamma_q(a) = \Gamma(a) \quad \text{and} \quad \lim_{q \rightarrow 1^-} \frac{(q^a; q)_n}{(1-q)^n} = (a)_n, \quad (3.9)$$

where

$$(a)_n = a(a+1) \cdots (a+n-1), \quad (3.10)$$

the main result (2.1) and the results (3.2) to (3.6), give q -extensions of the known results mentioned in Erdélyi, Magnus, Oberhettinger and Tricomi [3] (table number (4.14), pp. 182-187), namely:

$$\begin{aligned} L \{ t^{\nu-1} J_{2\mu_1}(2\sqrt{a_1 t}) \cdots J_{2\mu_n}(2\sqrt{a_n t}) \} &= \frac{\Gamma(\nu+M) a_1^{\mu_1} \cdots a_n^{\mu_n}}{s^{\nu+M} \Gamma(2\mu_1+1) \cdots \Gamma(2\mu_n+1)} \\ &\times \Psi_2^{(n)} \left(\nu+M; 2\mu_1+1, \dots, 2\mu_n+1; \frac{-a_1}{s}, \dots, \frac{-a_n}{s} \right), \quad (3.11) \end{aligned}$$

where $M = \mu_1 + \dots + \mu_n$, $Re(\nu + M) > 0$ and $Re(s) > 0$;

$$L \left\{ t^{\mu-1} J_{2\nu}(2\sqrt{at}) \right\} = \frac{\Gamma(\mu + \nu) a^\nu}{s^{\mu+\nu} \Gamma(2\nu + 1)} {}_1F_1 \left[\begin{matrix} \mu + \nu; \\ 2\nu + 1; \end{matrix} -a/s \right], \quad (3.12)$$

where $Re(\mu + \nu) > 0$ and $Re(s) > 0$;

$$L \left\{ t^{\nu/2} J_\nu(2\sqrt{at}) \right\} = a^{\nu/2} s^{-\nu-1} e^{-a/s}, \quad Re(s) > 0; \quad (3.13)$$

$$L \left\{ t^{1/2} J_1(2\sqrt{at}) \right\} = a^{1/2} s^{-2} e^{-a/s}, \quad Re(s) > 0; \quad (3.14)$$

$$L \left\{ J_0(2\sqrt{at}) \right\} = s^{-1} e^{-a/s}, \quad Re(s) > 0; \quad (3.15)$$

$$L \left\{ t^{-\nu/2} J_\nu(2\sqrt{at}) \right\} = \frac{e^{i\pi\nu} s^{\nu-1}}{a^{\nu/2} \Gamma(\nu)} e^{-a/s} \gamma\left(\nu, \frac{e^{i\pi} a}{s}\right). \quad (3.16)$$

The results proved in this paper give some contributions to the theory of the q -series, especially q -Bessel functions, and may find applications to solutions of certain q -difference and q -integral equations associated with various q -Bessel functions. In this regard, one can refer to the work of Abdi [2].

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