

**SMOOTHNESS PROPERTIES OF SOLUTIONS OF CAPUTO-  
TYPE FRACTIONAL DIFFERENTIAL EQUATIONS**

**Kai Diethelm**

*Dedicated to Prof. Michele Caputo  
on the occasion of his 80th birthday*

**Abstract**

We consider ordinary fractional differential equations with Caputo-type differential operators with smooth right-hand sides. In various places in the literature one can find the statement that such equations cannot have smooth solutions. We prove that this is wrong, and we give a full characterization of the situations where smooth solutions exist. The results can be extended to a class of weakly singular Volterra integral equations.

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**1. Introduction**

We consider initial value problems for fractional differential equations of the form

$$D_*^\alpha y(x) = g(x, y(x)), \quad y^{(j)}(0) = y_0^{(j)}, \quad j = 0, 1, \dots, [\alpha] - 1, \quad (1)$$

where

$$D_*^\alpha y(x) := J^{[\alpha]-\alpha} D^{[\alpha]} y(x)$$

denotes the usual Caputo-type differential operator [4]. Here, for  $m \in \mathbb{N}$ ,

$D^m$  is the classical differential operator of order  $m$ , and  $J^\beta$  is the Riemann-Liouville integral operator of order  $\beta \geq 0$  defined by  $J^0$  being the identity operator and

$$J^\beta z(x) := \frac{1}{\Gamma(\beta)} \int_0^x (x-t)^{\beta-1} z(t) dt$$

for  $\beta > 0$ . Our main interest is to give a description of the smoothness properties of the solutions of such equations, both at the point  $x = 0$  and on the interval  $(0, X)$  with some suitable  $X > 0$ , under certain smoothness assumptions on the given function  $g$ , i.e. the right-hand side of the differential equation (1).

Equations of the form (1) have proven to be an adequate and very useful way of modeling many phenomena in physics and engineering. We refer to [5, 9] and the references cited therein for a survey of the many applications. It has been shown in [5] that the initial value problem (1) is equivalent to the Volterra equation

$$y(x) = \sum_{j=0}^{[\alpha]-1} \frac{y_0^{(j)}}{j!} x^j + J^\alpha [g(\cdot, y(\cdot))](x), \quad (2)$$

if the function  $g$  is continuous, and indeed it is this fractional integral equation formulation of the problem that will be the basis of our analysis.

Section 2 will be devoted to a description of global smoothness properties of the solution such as, e.g., results concerning the differentiability of the solution on intervals like  $(0, X)$  or  $[0, X]$ . Moreover we will demonstrate that the above statement on equivalence can fail if the condition of continuity for  $g$  is relaxed.

In fact, eq. (2) is a special case of the more general weakly singular Volterra equation

$$y(x) = f(x) + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} k(x, t, y(t)) dt,$$

and it turns out that most of our methods and results can be carried over to this generalized problem. We discuss such a generalization in Section 3.

We will restrict ourselves to real-valued solutions for notational convenience, but the ideas can be generalized straightforwardly to equations for functions  $y : [0, X] \rightarrow \mathbb{R}^n$ .

The questions of existence, uniqueness and continuation of continuous solutions are not discussed here. There are a number of accounts on this topic, for example [1, 2, 3, 5, 6, 7, 10]. Local existence and uniqueness can be proved assuming restrictions on  $g$  which correspond to those found in standard text books for ordinary differential equations. In particular, we

know that a unique continuous solution  $y : [0, X] \rightarrow \mathbb{R}$  exists for some  $X > 0$  if, in a suitable domain,  $g$  is continuous and fulfils a Lipschitz condition with respect to the second variable [5]. This is sufficient for our purpose.

## 2. Global smoothness properties

As indicated above, we shall make use of the fact that the given Caputo-type initial value problem (1) is equivalent to the weakly singular Volterra equation (2) if  $g$  is continuous. Before we start the main part of our analysis we want to show the significance of the continuity assumption for  $g$ . To this end, we look at the following special case.

EXAMPLE 2.1. For  $\alpha \in (1, 2)$  consider the initial value problem

$$D_*^\alpha y(x) = \frac{1}{y(x) - 1}, \quad y(0) = 1, \quad y'(0) = 0. \quad (3)$$

If we ignore the discontinuity of  $g(x, y) = (y - 1)^{-1}$  at the initial point  $(x_0, y_0) = (0, 1)$  and apply the standard theory for continuous  $g$ , then we arrive at the integral equation

$$y(x) = 1 + \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha-1} \frac{1}{y(t) - 1} dt. \quad (4)$$

Equation (4) is solved by the functions

$$y(x) = 1 \pm \frac{\sqrt{\Gamma(1 - \alpha/2)}}{\sqrt{\Gamma(1 + \alpha/2)}} x^{\alpha/2}, \quad (5)$$

which is readily verified by substitution. However, both these solutions have unbounded first derivatives at 0, and hence they do not satisfy the second initial condition of (3). Thus they cannot be considered solutions of the original initial value problem (3).

Now we come to the main results of this section that deal with the smoothness of solutions of (1) in the case of a continuous function  $g$ . The inspection of classical results concerning the asymptotic expansion of the solution of (1) at the origin [8, 11] clearly shows that the solutions will not be smooth (as functions of the independent variable  $x$  only) in general, even if  $g$  is smooth. Therefore the question arises if there are special problems with smooth solutions. A remark in [8] states that it is not possible for  $g$  and  $y$  to be analytic simultaneously. But this cannot be entirely correct, as is seen by the trivial counterexample  $g \equiv 0$  in which case  $y$  is a polynomial and hence analytic. Another simple counterexample is provided in the introduction of [11]. A small modification for arbitrary  $\alpha$  in our notation is given next.

EXAMPLE 2.2. Consider

$$D_*^\alpha y(x) = 1 - y(x), \quad y_0^{(0)} = 1, \quad y_0^{(k)} = 0 \quad (k = 1, 2, \dots, [\alpha] - 1). \quad (6)$$

The solution of this problem is  $y(x) \equiv 1$ .

In equation (6), both  $g$  and  $y$  are entire functions. It is possible to give a precise account of the situations in which simultaneous analyticity of  $g$  and the solution  $y$  can occur, and we do so in our first theorem.

THEOREM 2.1. Consider the initial value problem (1) with  $\alpha > 0$  and  $\alpha \notin \mathbb{N}$ . Let the function  $f$  be defined with the help of the given initial values as

$$f(x) := \sum_{k=0}^{[\alpha]-1} \frac{y_0^{(k)}}{k!} x^k,$$

(i.e.  $f$  is the Taylor polynomial of degree  $[\alpha] - 1$  of  $y$  at the point 0). Assume that  $g$  is analytic on  $[0, X] \times G$ , where  $G \subset \mathbb{R}$  contains the range of  $f$  on  $[0, X]$ . Then,  $y$  is analytic if and only if  $g(x, f(x)) = 0$  for all  $x \in [0, X]$ .

The condition  $g(x, f(x)) \equiv 0$  is easy to check in practice because it only involves given functions.

Theorem 2.1 shows that the occurrence of an analytic solution to an equation of the form (1) with analytic right-hand side is a rare event. Nevertheless it can be used as guideline to construct problems with smooth solutions, for example if one needs test cases for numerical algorithms.

P r o o f. We first note that the analyticity of  $g$  implies the existence of a unique solution on some interval  $[0, X]$ , with  $X > 0$ .

The direction “ $\Leftarrow$ ” can be seen in the following way. In view of (2), the condition  $g(x, f(x)) \equiv 0$  implies that a solution (and hence, by uniqueness, the solution) of the initial value problem is  $y = f$ . Since  $f$  (and hence also  $y$ ) is a polynomial, we have an analytic solution.

For the other direction, we assume  $y$  to be analytic. Then, since  $g$  is analytic at  $(0, f(0))$ , the function  $z : [0, X] \rightarrow \mathbb{R}$  with  $z(x) := g(x, y(x))$  is analytic at 0 because of  $y(0) = f(0)$ . Hence we can represent it in the form

$$z(x) = \sum_{k=0}^{\infty} z_k x^k$$

with certain coefficients  $z_k$ . It follows from (2) that

$$\sum_{k=[\alpha]}^{\infty} \frac{y_0^{(k)}}{k!} x^k = y(x) - \sum_{k=0}^{[\alpha]-1} \frac{y_0^{(k)}}{k!} x^k = J^\alpha z(x) = \sum_{k=0}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)} z_k x^{k+\alpha}. \quad (7)$$

The left-hand side of (7) clearly is an analytic function at the point 0, so its right-hand side must be analytic there too. Since we assumed  $\alpha \notin \mathbb{N}$ , this is true if and only if  $z_k = 0$  for all  $k$ , which is equivalent to saying that  $0 = z(x) = g(x, y(x))$  for all  $x$ . Thus the integral equation form (2) of our initial value problem reduces to

$$y(x) = f(x) + J^\alpha z(x) = f(x),$$

and hence we have, for all  $x$ ,

$$0 = g(x, y(x)) = g(x, f(x)).$$

■

An inspection of the proof of Theorem 2.1 immediately reveals another important property:

**COROLLARY 2.2.** *Assume the hypotheses of Theorem 2.1. If the given initial value problem has an analytic solution  $y$  then  $y = f$ , i.e.  $y$  is the polynomial from the kernel of the Caputo differential operator that fits the initial conditions.*

Another direct consequence of these results is the following statement.

**COROLLARY 2.3.** *Consider the initial value problem (1) with  $\alpha > 0$  and  $\alpha \notin \mathbb{N}$ . If the solution  $y$  of this problem is analytic but not a polynomial then the function  $g$  is not analytic.*

If we do not want to require the given function  $g$  to be analytic then we can still prove some useful results about the differentiability properties of the solution of the initial value problem (1) on the interval where the solution exists.

**THEOREM 2.4.** *Consider the initial value problem (1) with  $\alpha > 0$  and  $g$  being continuous and satisfying a Lipschitz condition with respect to its second variable. Then, the solution  $y$  satisfies  $y \in C^{[\alpha]-1}[0, X]$ .*

**P r o o f.** In view of (2), the function  $y$  satisfies the Volterra equation

$$y(x) = p(x) + J^\alpha [g(\cdot, y(\cdot))](x)$$

with  $p$  being some polynomial whose precise form is not of interest at the moment. Now let  $k \in \{0, 1, 2, \dots, [\alpha] - 1\}$  (this implies  $k < \alpha$ ) and differentiate this equation  $k$  times:

$$\begin{aligned} D^k y(x) &= D^k p(x) + D^k J^\alpha [g(\cdot, y(\cdot))](x) \\ &= D^k p(x) + D^k J^k J^{\alpha-k} [g(\cdot, y(\cdot))](x) \\ &= D^k p(x) + J^{\alpha-k} [g(\cdot, y(\cdot))](x) \end{aligned}$$

in view of the semigroup property of fractional integration and the classical fundamental theorem of calculus. Now recall that  $y$  is continuous; thus the argument of the integral operator  $J^{\alpha-k}$  is a continuous function. Hence, in view of the polynomial structure of  $p$  and the well known mapping properties of  $J^{\alpha-k}$ , the function on the right-hand side of the equation is continuous, and so the function on the left, viz.  $D^k y$ , must be continuous too. ■

**THEOREM 2.5.** *Assume the hypotheses of Theorem 2.4. Moreover let  $\alpha > 1$ ,  $\alpha \notin \mathbb{N}$  and  $g \in C^1(G)$  with  $G = [y_0^{(0)} - K, y_0^{(0)} + K] \times \mathbb{R}$ . Then  $y \in C^{[\alpha]}(0, X)$ . Furthermore,  $y \in C^{[\alpha]}[0, X]$  if and only if  $g(0, y_0^{(0)}) = 0$ .*

**REMARK 2.1.** Since the function  $g$  and the initial value  $y_0^{(0)}$  are given, it is easy to check whether the condition  $g(0, y_0^{(0)}) = 0$  is fulfilled or not.

**REMARK 2.2.** In the case of integer-order differential equations we know that smoothness of the given function  $g$  implies smoothness of the solution  $y$  on the closed interval  $[0, X]$ ; however in the fractional setting this holds only under certain additional conditions. The precise behaviour of the solution at the starting point  $x = 0$  is a very interesting question in its own right. Some results in this respect are known [8, 11], but the investigations done in these papers contain a few small errors and are by no means complete. We intend to discuss this matter in detail in a forthcoming paper.

**P r o o f.** We introduce the abbreviation  $z(t) := g(t, y(t))$  and write out the identity stated in the previous proof for  $k = [\alpha] - 1$ :

$$\begin{aligned} D^{[\alpha]-1}y(x) &= D^{[\alpha]-1}p(x) + J^{\alpha-[\alpha]+1}z(x) \\ &= D^{[\alpha]-1}p(x) + \frac{1}{\Gamma(\alpha - [\alpha] + 1)} \int_0^x (x-t)^{\alpha-[\alpha]} z(t) dt. \end{aligned}$$

We differentiate once again, recall that  $p$  is a polynomial of degree  $[\alpha] - 1$  and find

$$\begin{aligned} D^{[\alpha]}y(x) &= D^{[\alpha]}p(x) + \frac{1}{\Gamma(\alpha - [\alpha] + 1)} \frac{d}{dx} \int_0^x (x-t)^{\alpha-[\alpha]} z(t) dt \\ &= \frac{1}{\Gamma(\alpha - [\alpha] + 1)} \frac{d}{dx} \int_0^x u^{\alpha-[\alpha]} z(x-u) du \\ &= \frac{1}{\Gamma(\alpha - [\alpha] + 1)} \left( x^{\alpha-[\alpha]} z(0) + \int_0^x u^{\alpha-[\alpha]} z'(x-u) du \right) \\ &= \frac{1}{\Gamma(\alpha - [\alpha] + 1)} x^{\alpha-[\alpha]} g(0, y_0^{(0)}) + J^{\alpha-[\alpha]+1} z'(x). \quad (8) \end{aligned}$$

Since  $\alpha > 1$  we deduce that  $[\alpha] \geq 2$ , and thus Theorem 2.4 asserts that  $y \in C^1[0, X]$ . An explicit calculation gives that

$$z'(t) = \frac{\partial}{\partial x} g(t, y(t)) + \frac{\partial}{\partial y} g(t, y(t)) y'(t).$$

Hence, by our differentiability assumption on  $g$ , the function  $z'$  is continuous, and so  $J^{\alpha-[\alpha]+1} z' \in C[0, X]$  too. The fact that  $[\alpha] > \alpha$  then finally yields that the right-hand side of (8), and therefore also the left-hand side of this equation, i.e. the function  $D^{[\alpha]} y$ , is always continuous on the half-open interval  $(0, X]$  whereas it is continuous on the closed interval  $[0, X]$  if and only if  $g(0, y_0^{(0)}) = 0$ . ■

By repeating the arguments used in the proof of Theorem 2.5, it is possible to generalize this idea and to keep Remarks 2.1 and 2.2 valid:

**THEOREM 2.6.** *Assume the hypotheses of Theorem 2.4. Moreover let  $k \in \mathbb{N}$ ,  $\alpha > k$ ,  $\alpha \notin \mathbb{N}$  and  $g \in C^k(G)$ . Let  $z(t) := g(t, y(t))$ . Then,  $y \in C^{[\alpha]+k-1}(0, X]$ . Furthermore,  $y \in C^{[\alpha]+k-1}[0, X]$  if and only if  $z$  has got a  $k$ -fold zero at the origin.*

### 3. Weakly singular Volterra equations

The Volterra integral equation (2) has been a very useful tool in our analysis so far because of its equivalence with the Caputo initial value problem (1). By taking into consideration the well known definition of the Riemann-Liouville integral operator, it is evident that (2) is a special case of the more general singular Volterra equation

$$y(x) = f(x) + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} k(x, t, y(t)) dt. \quad (9)$$

In particular, eq. (2) can be obtained from (9) by choosing  $f$  to be a polynomial and  $k$  to be independent of the free variable  $x$ .

It is a natural generalization of the results developed in Section 2 to ask the following question with respect to the general equation (9): Under which conditions will the given functions  $f$  and  $k$  and the unknown solution  $y$  be smooth simultaneously? Of course a sufficient set of conditions is immediately obtained from the results of Section 2, but our goal is to give a characterization. Indeed it is possible to provide such a characterization, and it is contained in the following statement.

**THEOREM 3.1.** *Consider the integral equation (9) with  $\alpha > 0$  and  $\alpha \notin \mathbb{N}$ . Assume that  $f$  is analytic in  $[0, X]$  and that  $k$  is analytic in  $T \times G$  where  $T := \{(x, t) : x \in [0, X], 0 \leq t \leq x\}$  and  $G \subset \mathbb{R}$  contains the range of  $f$  on  $[0, X]$ . Then,  $y$  is analytic if and only if*

$$\int_0^x (x-t)^{\alpha-1} k(x, t, f(t)) dt = 0 \quad (10)$$

for all  $x \in [0, X]$ .

Once again we note that the condition (10) mentioned in the theorem only involves given data and thus can be checked effectively.

**P r o o f.** As in the proof of Theorem 2.1, we begin by noting that the analyticity of  $f$  and  $k$  implies the existence of a unique solution on some interval  $[0, X]$ , with  $X > 0$ .

The direction “ $\Leftarrow$ ” can be seen in the following way. The condition (10) implies that a solution (and hence, by uniqueness, *the* solution) of the Volterra equation is  $y = f$ . Since  $f$  is analytic by assumption, we have an analytic solution.

For the other direction, we cannot use the same techniques as in the proof of Theorem 2.1 in a straightforward way. Assuming  $y$  to be analytic, we rather proceed as follows. Since  $k$  is analytic at  $(0, 0, f(0))$  and  $y$  is analytic, the function  $z : T \rightarrow \mathbb{R}$  with  $z(x, t) := k(x, t, y(t))$  is analytic at  $(0, 0)$  because of  $y(0) = f(0)$ . Hence we can represent it in the form of a bivariate power series centered at  $(0, 0)$ , i.e.

$$z(x, t) = \sum_{j,k=0}^{\infty} z_{jk} x^k t^j$$

with certain coefficients  $z_{jk}$ . It follows from (9) that

$$\begin{aligned} y(x) - f(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} z(x, t) dt \\ &= \sum_{j,k=0}^{\infty} z_{jk} x^k \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^j dt \\ &= \sum_{j,k=0}^{\infty} z_{jk} \frac{\Gamma(j+1)}{\Gamma(j+1+\alpha)} x^{k+j+\alpha} = x^\alpha \sum_{j,k=0}^{\infty} z_{jk} \frac{\Gamma(j+1)}{\Gamma(j+1+\alpha)} x^{k+j} \end{aligned} \quad (11)$$

(the interchange of summation and integration is allowed because of the absolute and uniform convergence of the power series). The left-hand side of (11) clearly is an analytic function at the point 0, so its right-hand side



must be analytic there too. Since we assumed  $\alpha \notin \mathbb{N}$ , this is true if and only if the infinite series vanishes identically. According to (11) this is equivalent to the condition

$$\int_0^x (x-t)^{\alpha-1} k(x, t, y(t)) dt = 0$$

for all  $x \in [0, X]$ . Thus the integral equation (9) reduces to  $y(x) = f(x)$  which implies that we have, for all  $x$ ,

$$0 = \int_0^x (x-t)^{\alpha-1} k(x, t, y(t)) dt = \int_0^x (x-t)^{\alpha-1} k(x, t, f(t)) dt,$$

i.e. condition (10) as required.  $\blacksquare$

REMARK 3.1. We note explicitly that it is not possible to conclude from condition (10) that  $k(x, t, y(t))$  vanishes identically. As a counterexample we present the equation

$$y(x) = 1 + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \left( t - \frac{\Gamma(1+\alpha)}{\Gamma(2+\alpha)} x \right) y(t) dt.$$

This equation has an analytic and nonvanishing kernel  $k$  and an analytic (indeed, constant) function  $f$ , and it is easy to verify that its solution is given by  $y(x) = 1$ , an analytic function.

We conclude by noting that the proof of Theorem 3.1 actually gives us an analogue to Corollary 2.2.

COROLLARY 3.2. *Assume the hypotheses of Theorem 3.1. If the given weakly singular Volterra equation (9) has an analytic solution  $y$  then  $y = f$ .*

The extension of Theorems 2.4, 2.5 and 2.6 to general Volterra equations of the form (9) is possible under appropriate assumptions but is technically very cumbersome and outside of the scope of this paper.

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\* *Institut Computational Mathematics*  
*Technische Universität Braunschweig*  
*Pockelsstraße 14*  
*38106 Braunschweig – GERMANY*  
*e-mail: k.diethelm@tu-bs.de*

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*GNS Gesellschaft für Numerische Simulation mbH*  
*Am Gaußberg 2, 38114 Braunschweig – GERMANY*  
*e-mail: diethelm@gns-mbh.com*

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\* Corresponding address