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INTEGRAL REPRESENTATIONS OF GENERALIZED MATHIEU SERIES VIA MITTAG-LEFFLER TYPE FUNCTIONS

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Abstract

The main purpose of this paper is to present a number of potentially useful integral representations for the generalized Mathieu series as well as for its alternating versions via Mittag-Leffler type functions.

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1. Introduction

The following infinite series:

$$S(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2 + r^2)^2} \quad (r \in \mathbb{R}^+) \quad (1.1)$$

is known as named after *Emile Leonard Mathieu* (1835-1890), who investigated it in his 1890 work [11] on elasticity of solid bodies.

A remarkable useful integral representation for $S(r)$ in the elegant form:

$$S(r) = \frac{1}{r} \int_0^{\infty} \frac{x \sin(xr)}{e^x - 1} dx \quad (1.2)$$

was given by Emersleben [6].

Several interesting problems and solutions dealing with integral representations and bounds for the following slight generalization of the *Mathieu series with a fractional power*:

$$\mathbb{S}_\mu(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2 + r^2)^\mu} \quad (r \in \mathbb{R}^+; \mu > 1) \quad (1.3)$$

can be found in the recent works by Diananda [4], Tomovski and Trenčevski [21] and Cerone and Lenard [3]. Motivated essentially by the works of Cerone and Lenard [3] (and Qi [17]), we defined in [18] a *family of generalized Mathieu series*:

$$\mathbb{S}_\mu^{(\alpha, \beta)}(r; \mathbf{a}) = \mathbb{S}_\mu^{(\alpha, \beta)}(r; \{a_k\}_{k=1}^\infty) = \sum_{n=1}^{\infty} \frac{2a_n^\beta}{(a_n^\alpha + r^2)^\mu} \quad (r, \alpha, \beta, \mu \in \mathbb{R}^+), \quad (1.4)$$

where $\lim_{k \rightarrow \infty} a_k = \infty$.

Comparing the definitions (1.1), (1.3) and (1.4), we see that $\mathbb{S}_2(r) = S(r)$ and $\mathbb{S}_\mu(r) = \mathbb{S}_\mu^{(2,1)}(r, \{k\}_{k=1}^\infty)$.

Furthermore, the special cases

$$\mathbb{S}_2^{(2,1)}(r, \{a_k\}_{k=1}^\infty), \quad \mathbb{S}_\mu^{(2,1)}(r, \{k^\gamma\}_{k=1}^\infty) \quad \text{and} \quad \mathbb{S}_\mu^{(\alpha, \alpha/2)}(r, \{k\}_{k=1}^\infty)$$

were investigated by Qi [17], Tomovski [20] and Cerone and Lenard [3].

In [18] the following integral representations were proposed:

$$\begin{aligned} & \mathbb{S}_\mu^{(\alpha, \beta)}(r; \{k^\gamma\}_{k=1}^\infty) \\ &= \frac{2}{\Gamma(\mu)} \int_0^\infty \frac{x^{\gamma(\mu\alpha - \beta) - 1}}{e^x - 1} {}_1\Psi_1 [(\mu, 1); (\gamma(\mu\alpha - \beta), \gamma\alpha); -r^2 x^{\gamma\alpha}] dx \end{aligned} \quad (1.5)$$

$$(r, \alpha, \beta, \gamma \in \mathbb{R}^+; \gamma(\mu\alpha - \beta) > 1);$$

$$\begin{aligned} & \mathbb{S}_\mu^{(\alpha, \beta)}(r; \{k^{q/\alpha}\}) \\ &= \frac{2}{\Gamma(q[\mu - \frac{\beta}{\alpha}])} \int_0^\infty \frac{x^{q[\mu - \frac{\beta}{\alpha}] - 1}}{e^x - 1} {}_1F_q \left(\mu; \Delta \left(q; q \left[\mu - \frac{\beta}{\alpha} \right] \right); -r^2 \left(\frac{x}{q} \right)^q \right) dx \end{aligned} \quad (1.6)$$

$$(r, \alpha, \beta \in \mathbb{R}^+; \mu - \frac{\beta}{\alpha} > q^{-1}; q \in \mathbb{N}),$$

where $\Delta(q; \lambda)$ is the q -tuple $\frac{\lambda}{q}, \frac{\lambda+1}{q}, \dots, \frac{\lambda+q-1}{q}$;

$$\begin{aligned} \mathbb{S}_{\mu+1}^{(\alpha, \alpha/2)}(r; \{k^{2/\alpha}\}_{k=1}^\infty) &= \mathbb{S}_{\mu+1}^{(2,1)}(r; \{k\}_{k=1}^\infty) = \mathbb{S}_{\mu+1}(r) \\ &= \frac{\sqrt{\pi}}{(2r)^{\mu-\frac{1}{2}}\Gamma(\mu+1)} \int_0^\infty \frac{x^{\mu+\frac{1}{2}}}{e^x-1} J_{\mu-\frac{1}{2}}(rx) dx, \quad (r, \mu \in \mathbb{R}^+); \end{aligned} \tag{1.7}$$

$$\begin{aligned} \mathbb{S}_\mu^{(\alpha,0)}(r; \{k^{2/\alpha}\}_{k=1}^\infty) &= \sum_{n=1}^\infty \frac{2}{(n^2+r^2)^\mu} \\ &= \frac{2\sqrt{\pi}}{(2r)^{\mu-\frac{1}{2}}\Gamma(\mu)} \int_0^\infty \frac{x^{\mu-\frac{1}{2}}}{e^x-1} J_{\mu-\frac{1}{2}}(rx) dx, \quad (r \in \mathbb{R}^+; \mu > \frac{1}{2}). \end{aligned} \tag{1.8}$$

Here ${}_p\Psi_q$ denotes the *Fox-Wright generalization of the hypergeometric ${}_pF_q$ function* with p numerator and q denominator parameters (see for example [7], v.1; [19]), p.50, eq. 1.5 (21); [10], Appendix) and J_ν – the Bessel function of first kind.

In [14] we introduced an alternating variant of $S(r)$,

$$\tilde{S}(r) = \sum_{n=1}^\infty (-1)^{n-1} \frac{2n}{(n^2+r^2)^2} \quad (r \in \mathbb{R}^+), \tag{1.9}$$

called *alternative Mathieu series*.

Using the formulas given in [12]:

$$\sum_{k=1}^\infty \mathcal{L}'(k) = - \int_0^\infty \frac{tf(t)}{e^t-1} dt, \tag{1.10}$$

$$\sum_{k=1}^\infty (-1)^{k-1} \mathcal{L}'(k) = \int_0^\infty \frac{tf(t)}{e^t+1} dt, \tag{1.11}$$

where $\mathcal{L}(k)$ is the classical Laplace transform of $f(t)$, we obtained in [14] the integral representation of $\tilde{S}(r)$:

$$\tilde{S}(r) = \frac{1}{r} \int_0^\infty \frac{x \sin(xr)}{e^x+1} dx. \tag{1.12}$$

Let $\tilde{\mathbb{S}}_\mu^{(\alpha,\beta)}(r; \mathbf{a})$ be alternating variant of (1.4). Integral representations for the alternating variant of (1.8) are given in [14] by:

$$\begin{aligned} & \tilde{S}_\mu^{(\alpha, \beta)}(r; \{k^\gamma\}_{k=1}^\infty) \\ &= \frac{2}{\Gamma(\mu)} \int_0^\infty \frac{x^{\gamma(\mu\alpha - \beta) - 1}}{e^x + 1} {}_1\Psi_1 [(\mu, 1); (\gamma(\mu\alpha - \beta), \gamma\alpha); -r^2 x^{\gamma\alpha}] dx \end{aligned} \quad (1.13)$$

($r, \alpha, \beta, \gamma \in \mathbb{R}^+$; $\gamma(\mu\alpha - \beta) > 1$);

$$\tilde{S}_{\mu+1}(r) = \frac{\sqrt{\pi}}{(2r)^{\mu - \frac{1}{2}} \Gamma(\mu + 1)} \int_0^\infty \frac{x^{\mu + \frac{1}{2}}}{e^x + 1} J_{\mu - \frac{1}{2}}(rx) dx \quad (r, \mu \in \mathbb{R}^+); \quad (1.14)$$

$$\tilde{S}_\mu^{(\alpha, 0)}(r; \{k^{2/\alpha}\}_{k=1}^\infty) = \frac{2\sqrt{\pi}}{(2r)^{\mu - \frac{1}{2}} \Gamma(\mu)} \int_0^\infty \frac{x^{\mu - \frac{1}{2}}}{e^x + 1} J_{\mu - \frac{1}{2}}(rx) dx \quad (r \in \mathbb{R}^+, \mu > \frac{1}{2}). \quad (1.15)$$

2. Mittag-Leffler type functions

The special functions of the form

$$E_\rho(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\rho k + 1)}, \quad E_{\rho, \mu}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\rho k + \mu)} \quad (2.1)$$

with complex $\rho, \mu \in \mathbb{C}$ ($Re(\rho) > 0, Re(\mu) > 0$) are known as *Mittag-Leffler (M-L) functions*.

The former was introduced by Mittag-Leffler [13] and named after him in [7], vol. 3; and the latter is due to Agarwal [1] and also studied in details by Dzrbashjan (see e.g. [5]) who denoted it as $\Phi_{1/\rho}(z; \mu)$.

Prabhakar [16] introduced the M-L type function $E_{\rho, \mu}^\delta(z)$ of the form

$$E_{\rho, \mu}^\delta(z) = \sum_{k=0}^\infty \frac{(\delta)_k}{\Gamma(\rho k + \mu)} \frac{z^k}{k!} \quad (\rho, \mu, \gamma \in \mathbb{C}, Re(\rho) > 0), \quad (2.2)$$

where $(\delta)_k$ is the Pochhammer symbol:

$$(\delta)_0 = 1, \quad (\delta)_k = \delta(\delta + 1) \dots (\delta + k - 1) \quad (k = 1, 2, \dots).$$

For $\delta = 1$, we have

$$E_{\rho, \mu}^1(z) = E_{\rho, \mu}(z), \quad E_{\rho, 1}^1(z) = E_\rho(z). \quad (2.3)$$

The function $E_{\rho, \mu}^\delta$ can be represented via the Fox-Wright Psi-function ${}_1\Psi_1$ and the generalized hypergeometric function ${}_1F_m$ as follows:

$$E_{\rho,\mu}^\delta(z) = \frac{1}{\Gamma(\delta)} {}_1\Psi_1[(\delta, 1), (\mu, \rho); z], \tag{2.4}$$

$$E_{m,\nu}^\delta(z) = \frac{1}{\Gamma(\delta)} {}_1F_m(\delta; \Delta(m; \nu); \frac{z}{m^m}) \quad (m \in \mathbb{N}, \delta, \nu \in \mathbb{C}). \tag{2.5}$$

Let $m > 1$ be an integer, $\rho_1, \rho_2, \dots, \rho_m > 0$ and $\mu_1, \mu_2, \dots, \mu_m$ be arbitrary real numbers. By means of the "multi-indices" $(\rho_i), (\mu_i)$, Kiryakova [9] introduced the so-called *multi-index* (m -tuple, multiple) *M-L functions*:

$$E_{(1/\rho_i), (\mu_i)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu_1 + k/\rho_1)\Gamma(\mu_2 + k/\rho_2) \dots \Gamma(\mu_m + k/\rho_m)}. \tag{2.6}$$

About generalized fractional calculus differential and integral operators and Laplace type integral transform, closely related to the functions $E_{(1/\rho_i), (\mu_i)}$, see also Al-Mussalam, Kiryakova and Vu Kim Tuan [2].

The special case $m = 2$ of (2.6) was introduced by Dzrbashjan [5]. He denoted it by

$$E_{(1/\rho_1, 1/\rho_2), (\mu_1, \mu_2)} = \Phi_{\rho_1, \rho_2}(z; \mu_1, \mu_2) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu_1 + k/\rho_1)\Gamma(\mu_2 + k/\rho_2)}. \tag{2.7}$$

We mention here some special cases of (2.6), presented by Kiryakova in [9]:

$$E_{1/\rho, \mu}(z) = E_{(1/\rho, 0), (\mu, 1)}(z) = \Phi_{p, \infty}(z; \mu, 1), \tag{2.8}$$

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu E_{(1, 1), (\nu+1, 1)}\left(-\frac{z^2}{4}\right) = \left(\frac{z}{2}\right)^\nu \Phi_{1, 1}\left(-\frac{z^2}{4}; 1, \nu + 1\right), \tag{2.9}$$

$$\begin{aligned} E_{(1, 1, \dots, 1), (\mu_{i+1})}(z) &= {}_1\Psi_m \left[\begin{matrix} (1, 1) \\ (\mu_i, 1)_1^m \end{matrix} \middle| z \right] \\ &= \left(\prod_{i=1}^m \Gamma(\mu_i) \right)^{-1} \cdot {}_1F_m(1; \mu_1, \mu_2, \dots, \mu_m; z). \end{aligned} \tag{2.10}$$

Using the relations (2.4) and (2.5) we give here some integral representations for the Mathieu type series $\mathbb{S}_\mu^{(\alpha, \beta)}(r, \mathbf{a})$ and $\tilde{\mathbb{S}}_\mu^{(\alpha, \beta)}(r, \mathbf{a})$ via (M-L) type functions:

$$\mathbb{S}_\mu^{(\alpha, \beta)}(r; \{k^\gamma\}_{k=1}^\infty) = 2 \int_0^\infty \frac{x^{\gamma(\mu\alpha - \beta) - 1}}{e^x - 1} E_{\gamma\alpha, \gamma(\mu\alpha - \beta)}^\mu(-r^2 x^{\gamma\alpha}) dx; \tag{2.11}$$

$$\begin{aligned} \tilde{\mathbb{S}}_{\mu}^{(\alpha,\beta)}(r; \{k^{\gamma}\}_{k=1}^{\infty}) &= 2 \int_0^{\infty} \frac{x^{\gamma(\mu\alpha-\beta)-1}}{e^x + 1} E_{\gamma\alpha, \gamma(\mu\alpha-\beta)}^{\mu}(-r^2 x^{\gamma\alpha}) dx \\ &\quad (r, \alpha, \beta, \gamma \in \mathbb{R}^+; \quad \gamma(\mu\alpha - \beta) > 1); \end{aligned} \quad (2.12)$$

$$\begin{aligned} \mathbb{S}_{\mu}^{(\alpha,\beta)}(r; \{k^{q/\alpha}\}_{k=1}^{\infty}) &= 2 \int_0^{\infty} \frac{x^{q[\mu-\beta/\alpha]-1}}{e^x - 1} E_{q, q[\mu-\beta/\alpha]}^{\mu}(-r^2 x^q) dx \\ &\quad (r, \alpha, \beta \in \mathbb{R}^+; \quad \mu - \beta/\alpha > q^{-1}; \quad q \in \mathbb{N}). \end{aligned} \quad (2.13)$$

From the relation (see [16])

$$\begin{aligned} \mathcal{L} \left[t^{\mu-1} E_{\rho, \mu}^{\delta}(wt^{\rho}) \right] (s) &= s^{-\mu} (1 - ws^{-\rho})^{-\delta} \\ s &\in \mathbb{C}, \quad \text{Res} > 0, \quad |w/s^{\rho}| < 1, \end{aligned} \quad (2.14)$$

and (1.11), upon setting $f(t) = t^{q[\mu-\beta/\alpha]-2} E_{q, q[\mu-\beta/\alpha]}^{\mu}(-r^2 t^q)$, it easily follows the integral representation

$$\tilde{\mathbb{S}}_{\mu}^{(\alpha,\beta)}(r; \{k^{q/\alpha}\}_{k=1}^{\infty}) = 2 \int_0^{\infty} \frac{x^{q[\mu-\beta/\alpha]-1}}{e^x + 1} E_{q, q[\mu-\beta/\alpha]}^{\mu}(-r^2 x^q) dx. \quad (2.15)$$

Next, using the relation (2.7), we obtain also the integral representations:

$$\begin{aligned} \mathbb{S}_{\mu+1}^{(\alpha,\alpha/2)}(r; \{k^{2/\alpha}\}_{k=1}^{\infty}) &= \mathbb{S}_{\mu+1}^{(2,1)}(r; \{k\}_{k=1}^{\infty}) = \mathbb{S}_{\mu+1}(r) \\ &= \frac{\sqrt{\pi}}{2^{2\mu}\Gamma(\mu+1)} \int_0^{\infty} \frac{x^{2\mu}}{e^x - 1} E_{(1,1), (\mu+\frac{1}{2}, 1)} \left(-\frac{r^2 x^2}{4} \right) dx; \end{aligned} \quad (2.16)$$

$$\begin{aligned} \tilde{\mathbb{S}}_{\mu+1}^{(\alpha,\alpha/2)}(r; \{k^{2/\alpha}\}_{k=1}^{\infty}) &= \frac{\sqrt{\pi}}{2^{2\mu}\Gamma(\mu+1)} \int_0^{\infty} \frac{x^{2\mu}}{e^x + 1} E_{(1,1), (\mu+\frac{1}{2}, 1)} \left(-\frac{r^2 x^2}{4} \right) dx \\ &\quad (r, \mu \in \mathbb{R}^+); \end{aligned} \quad (2.17)$$

$$\begin{aligned} \mathbb{S}_{\mu}^{(\alpha,0)}(r; \{k^{2/\alpha}\}_{k=1}^{\infty}) &= \frac{\sqrt{\pi}}{2^{2\mu-1}\Gamma(\mu)} \int_0^{\infty} \frac{x^{2\mu-1}}{e^x - 1} E_{(1,1), (\mu+\frac{1}{2}, 1)} \left(-\frac{r^2 x^2}{4} \right) dx; \end{aligned} \quad (2.18)$$

$$\begin{aligned} \tilde{\mathbb{S}}_{\mu}^{(\alpha,0)}(r; \{k^{2/\alpha}\}_{k=1}^{\infty}) &= \frac{\sqrt{\pi}}{2^{2\mu-1}\Gamma(\mu)} \int_0^{\infty} \frac{x^{2\mu-1}}{e^x + 1} E_{(1,1), (\mu+\frac{1}{2}, 1)} \left(-\frac{r^2 x^2}{4} \right) dx \\ &\quad (r, \in \mathbb{R}^+, \quad \mu > \frac{1}{2}). \end{aligned} \quad (2.19)$$

3. Mathieu-type series whose terms contain multi-index M-L functions

In the recent articles ([14], [15], [22]) Pogany, Srivastava and Tomovski considered special kind of Mathieu-type series and their alternating variants whose terms contain the Gauss hypergeometric function ${}_2F_1$, generalized hypergeometric ${}_pF_q$ and the Meijer G functions. The derived results concern, among others, closed integral form expressions for the considered series and bilateral bounding inequalities. Here we are interested in giving the integral expression for the Mathieu-type series (and its alternating variants) whose terms contain the multi-index M-L functions from [9] and [2].

DEFINITION. By a Fox's H -function we mean a generalized hypergeometric function, defined by means of the Mellin-Barnes-type contour integral

$$H_{p,q}^{m,n} = \left[\sigma \left| \begin{matrix} (a_k, A_k)_1^p \\ (b_k, B_k)_1^q \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_{\mathcal{L}'} \frac{\prod_{k=1}^m \Gamma(b_k - B_k s) \prod_{j=1}^n (1 - a_j + s A_j)}{\prod_{k=m+1}^q \Gamma(1 - b_k + s B_k) \prod_{j=n+1}^p (a_j - s A_j)} \sigma^s ds, \tag{3.1}$$

where \mathcal{L}' is a suitable contour in \mathbb{C} , the orders (m, n, p, q) are integers, $0 \leq m \leq q, 0 \leq n \leq p$ and the parameters $a_j \in \mathbb{R}, A_j > 0, j = 1, 2, \dots, p, b_k \in \mathbb{R}, B_k > 0, k = 1, 2, \dots, q$ are such that $A_j(b_k + l) \neq B_k(a_j - l' - 1), l, l' = 0, 1, 2, \dots$

For various type of contours and conditions for existence and analyticity of function (3.1) in disks $\subset \mathbb{C}$ whose radii are $\rho = \prod_{j=1}^p A_j^{-A_j} \prod_{k=1}^q B_k^{B_k} > 0$, one can see, for example [10], Appendix.

We define the Mathieu-type series $\mu_{(1/\rho_i), (\mu_i)}$ and its alternating variant $\tilde{\mu}_{(1/\rho_i), (\mu_i)}$ as follows:

$$\mu_{(1/\rho_i), (\mu_i)}(\mathbf{a}; r) = \sum_{n=1}^{\infty} \frac{E_{(1/\rho_i), (\mu_i)}\left(-\frac{r^2}{a_n}\right)}{a_n^\lambda (a_n + r^2)^\eta}, \tag{3.2}$$

$$\tilde{\mu}_{(1/\rho_i), (\mu_i)}(\mathbf{a}; r) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{E_{(1/\rho_i), (\mu_i)}\left(-\frac{r^2}{a_n}\right)}{a_n^\lambda (a_n + r^2)^\eta} \quad (\lambda, \eta, r \in \mathbb{R}^+), \tag{3.3}$$

where we make the convention that the real sequence $\mathbf{a} = \{a_n\}_{n \in \mathbb{N}}$ increases and tends to ∞ .

We will need the Laplace transform of $x^{\lambda-1} H_{0,m+1}^{1,0}(wx|\cdot)$. Using the definition (3.1), for real w it follows easily

$$\int_0^{\infty} e^{-Ax} x^{\lambda-1} H_{0,m+1}^{1,0} \left(wx^{\rho} \left| (0,1), (1-\mu_i, \frac{1}{\rho_i})_{i=1}^m \right. \right) dx = A^{-\lambda} E_{(1/\rho_i),(\mu_i)} \left(\frac{w}{A^{\rho}} \right),$$

$$\text{where } m > 0, \quad \operatorname{Re}\{A\}, \quad \operatorname{Re}\{\lambda\} + \rho, \quad \min_{1 \leq j \leq m} \operatorname{Re} \left\{ \frac{1-\mu_j}{1/\rho_j} \right\} > 0. \quad (3.4)$$

THEOREM. Let $\lambda > 0, \eta > 0, r > 0$ and let the real sequence \mathbf{a} increase and tend to ∞ . Then,

$$\mu_{(1/\rho_i),(\mu_i)}(\mathbf{a}; r) = \eta I(\lambda, \eta + 1) + I(\lambda + 1, \eta), \quad (3.5)$$

$$\tilde{\mu}_{(1/\rho_i),(\mu_i)}(\mathbf{a}; r) = \eta \tilde{I}(\lambda, \eta + 1) + \tilde{I}(\lambda + 1, \eta), \quad (3.6)$$

where

$$I(\lambda, \eta) = \int_{a_1}^{\infty} E_{(1/\rho_i),(\mu_i)} \left(-\frac{r^2}{x} \right) \frac{[a^{-1}(x)]}{x^{\lambda}(r^2+x)^{\eta}} dx, \quad (3.7)$$

$$\tilde{I}(\lambda, \eta) = \int_{a_1}^{\infty} E_{(1/\rho_i),(\mu_i)} \left(-\frac{r^2}{x} \right) \frac{\sin^2(\frac{\pi}{2}[a^{-1}(x)])}{x^{\lambda}(r^2+x)^{\eta}} dx, \quad (3.8)$$

and $a : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is an increasing function such that $a(x)|_{x \in \mathbb{N}} = \mathbf{a}$, $a^{-1}(x)$ denotes the inverse of $a(x)$, $[a^{-1}(x)]$ stands for the integer part of the quantity $a^{-1}(x)$.

P r o o f. Taking $\xi = a_n + r^2$ in the formula

$$\Gamma(\eta)\xi^{-\eta} = \int_0^{\infty} e^{-\xi t} t^{\eta-1} dt \quad (\operatorname{Re} \xi > 0, \operatorname{Re} \eta > 0)$$

and $A = a_n, \rho = 1, w = -r^2$ in (3.4), we get

$$\begin{aligned} \mu_{(1/\rho_i),(\mu_i)}(\mathbf{a}; r) &= \frac{1}{\Gamma(\eta)} \int_0^{\infty} \int_0^{\infty} H_{0,m+1}^{1,0} \left(-r^2 t \left| (0,1), (1-\mu_i, \frac{1}{\rho_i})_{i=1}^m \right. \right) \mathcal{D}_{\mathbf{a}}(s+t) \\ &\quad \times e^{-r^2 s} t^{\lambda-1} s^{\eta-1} ds dt, \end{aligned}$$

where

$$\mathcal{D}_{\mathbf{a}}(s+t) = \sum_{n=1}^{\infty} e^{a_n(s+t)} = (s+t) \int_{a_1}^{\infty} e^{-(s+t)x} [a^{-1}(x)] dx$$

is the integral representation of the Dirichlet series (see [8], [14], [15]). Hence,

$$\begin{aligned} \mu_{(1/\rho_i),(\mu_i)}(\mathbf{a}; r) &= \frac{1}{\Gamma(\eta)} \int_0^\infty \int_0^\infty \int_{a_1}^\infty H_{0,m+1}^{1,0} \left(-r^2 t \left| \overline{(0, 1), (1 - \mu_i, \frac{1}{\rho_i})_{i=1}^m} \right. \right) \\ &\times e^{-(r^2+x)s-tx} s^{\eta-1} t^{\lambda-1} (s+t) [a^{-1}(x)] ds dx dt = I_s + I_t, \end{aligned}$$

where

$$\begin{aligned} I_s &= \frac{1}{\Gamma(\eta)} \int_0^\infty \int_0^\infty \int_{a_1}^\infty H_{0,m+1}^{1,0} \left(-r^2 t \left| \overline{(0, 1), (1 - \mu_i, \frac{1}{\rho_i})_{i=1}^m} \right. \right) \\ &\times e^{-(r^2+x)s-tx} t^{\lambda-1} s^\eta [a^{-1}(x)] ds dx dt \\ &= \frac{1}{\Gamma(\eta)} \int_0^\infty \int_{a_1}^\infty H_{0,m+1}^{1,0} \left(-r^2 t \left| \overline{(0, 1), (1 - \mu_i, \frac{1}{\rho_i})_{i=1}^m} \right. \right) \\ &\times e^{-tx} t^{\lambda-1} [a^{-1}(x)] \left(\int_0^\infty e^{-(r^2+x)s} s^\eta ds \right) dx dt \\ &= \eta \int_{a_1}^\infty \frac{[a^{-1}(x)]}{(r^2+x)^{\eta+1}} \left(\int_0^\infty H_{0,m+1}^{1,0} \left(-r^2 t \left| \overline{(0, 1), (1 - \mu_i, \frac{1}{\rho_i})_{i=1}^m} \right. \right) t^{\lambda-1} e^{-tx} dt \right) dx \\ &= \eta \int_{a_1}^\infty \frac{[a^{-1}(x)]}{(r^2+x)^{\eta+1} x^\lambda} E_{(1/\rho_i),(\mu_i)} \left(-\frac{r^2}{x} \right) dx; \end{aligned}$$

$$\begin{aligned} I_t &= \frac{1}{\Gamma(\eta)} \int_0^\infty \int_0^\infty \int_{a_1}^\infty H_{0,m+1}^{1,0} \left(-r^2 t \left| \overline{(0, 1), (1 - \mu_i, \frac{1}{\rho_i})_{i=1}^m} \right. \right) \\ &\times e^{-(r^2+x)s-tx} s^{\eta-1} t^\lambda [a^{-1}(x)] ds dx dt \\ &= \frac{1}{\Gamma(\eta)} \int_0^\infty \int_{a_1}^\infty e^{-(r^2+x)s} s^{\eta-1} [a^{-1}(x)] \\ &\times \left(\int_0^\infty H_{0,m+1}^{1,0} \left(-r^2 t \left| \overline{(0, 1), (1 - \mu_i, \frac{1}{\rho_i})_{i=1}^m} \right. \right) t^\lambda e^{-tx} dt \right) ds dx \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\Gamma(\eta)} \int_{a_1}^{\infty} E_{(1/\rho_i), (\mu_i)} \left(-\frac{r^2}{x} \right) \frac{[a^{-1}(x)]}{x^{\lambda+1}} \left(\int_0^{\infty} e^{-(r^2+x)s} s^{\eta-1} ds \right) dx \\
&= \int_{a_1}^{\infty} E_{(1/\rho_i), (\mu_i)} \left(-\frac{r^2}{x} \right) \frac{[a^{-1}(x)]}{x^{\lambda+1} (r^2+x)^\eta} dx.
\end{aligned}$$

The derivation of (3.6) is similar to the previous proof; as the alternating Dirichlet series $\tilde{D}_{\mathbf{a}}(x)$ equals to (see [14]):

$$\tilde{D}_{\mathbf{a}}(x) = \sum_{n=1}^{\infty} (-1)^{n-1} e^{-a_n x} = x \int_{a_1}^{\infty} e^{-xt} \tilde{A}(t) dt,$$

where the so-called continuing function $\tilde{A}(t)$ has been found easily in the following manner:

$$\tilde{A}(t) = \sum_{n: a_n \leq t} (-1)^{n-1} = \frac{1 - (-1)^{[a^{-1}(t)]}}{2} = \sin^2 \left(\frac{\pi}{2} [a^{-1}(t)] \right).$$

Hence $\tilde{D}_{\mathbf{a}}(s+t) = (s+t) \int_{a_1}^{\infty} e^{-x(s+t)} \sin^2 \left(\frac{\pi}{2} [a^{-1}(x)] \right) dx$, i.e. we get (3.6). ■

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