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EXISTENCE AND ASYMPTOTIC STABILITY OF SOLUTIONS OF A PERTURBED QUADRATIC FRACTIONAL INTEGRAL EQUATION *

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Abstract

We study the solvability of a perturbed quadratic integral equation of fractional order with linear modification of the argument. This equation is considered in the Banach space of real functions which are defined, bounded and continuous on an unbounded interval. Moreover, we will obtain some asymptotic characterization of solutions. Finally, we give an example to illustrate our abstract results.

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Key Words and Phrases: quadratic integral equation, fractional integral, existence, asymptotic behaviour, measure of noncompactness, Darbo fixed point theorem

1. Introduction

In this paper, we are interested in the existence and the asymptotic behaviour of solutions to the perturbed quadratic fractional integral equation

$$x(t) = g(t, x(t)) + \frac{f(t, x(t))}{\Gamma(\beta)} \int_0^t \frac{v(t, s, x(s))}{(t-s)^{1-\beta}}\, ds, \tag{1.1}$$

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where $t \in R_+ = [0, +\infty)$ and $0 < \beta < 1$. Throughout, $g : R_+ \times R_+ \to R$, $f : R_+ \times R \to R$ and $v : R_+ \times R_+ \times R \times R \to R$ are functions which satisfy special assumptions that will be given in detail in Section 3. Let us recall that the functions $f = f(t, x)$ and $g = g(t, x)$ involved in Eq. (1.1) generate superposition operators F and G, respectively, defined by

$$(Fx)(t) = f(t, x(t)) \text{ and } (Gx)(t) = g(t, x(t)), \tag{1.2}$$

where $x = x(t)$ is an arbitrary function defined on R_+; see [1].

We remark that:

- If $g(t, y) = p(t)$ in Eq. (1.1), then we have an equation studied by Banaś and O'Regan in [12].
- If $g(t, y) = a(t)$ and $v(t, s, x) = u(s, x)$ in Eq. (1.1), then we have an equation studied by Banaś and Rzepka in [11].
- If $g(t, y) = a(t)$, $f(t, y) = y$ and $v(t, s, x) = u(s, x)$ in Eq. (1.1), then we have an equation studied by Darwish in [23].

Consider the limit case where $\beta = 1$. Let $g(t, x) = h(t)$, $f(t, x) = -x$, and $v(t, s, x) = k(t, s)x$. Then Eq. (1.1) takes the form

$$x(t) + x(t) \int_0^t k(t, s)\, x(s)\, ds = h(t),\ t \in [0, 1]. \tag{1.3}$$

Eq. (1.3) is the nonlinear particle transport equation when removal effects are dominant, where t is the particle speed, the known term $h(t)$ is the intensity of the external source, and the unknown function $x(t)$ is related to the particle distribution function $y(t)$ by

$$x(t) = Q(t)\, y(t),$$

where Q is the positive macroscopic removal collision frequency of the host medium. Finally, the kernel $k(t, s)$ is given by

$$k(t, s) = \frac{1}{2tQ(t)Q(s)} \int_{|t-s|}^{t+s} v\, q(v)\, dv,$$

where q is the macroscopic removal collision frequency of the particles amongst themselves; see [15, 16, 17, 42]. On the other hand, Eq. (1.3) is a generalization of the Chandrasekhar $H-$equation in transport theory, in which t ranges from 0 to 1, $h(t) = 1$, x must be identified with the $H-$function, and

$$k(t, s) = -\frac{t\phi(s)}{t + s}$$

for a nonnegative characteristic function ϕ; see [22, 34, 36, 42].

Moreover, quadratic integral equations have numerous other useful applications in describing events and problems in the real world. For example, quadratic integral equations are often applicable in the kinetic theory of gases, in the theory of neutron transport, and in traffic theory; see [15, 16, 17, 29, 32, 34].

In the last 35 years or so, many authors have studied the existence of solutions for several classes of nonlinear quadratic integral equations with nonsingular kernels. For example, see the papers by Argyros [2], Banaś *et al.* [5, 7, 10], Banaś and Martinon [9], Benchohra and Darwish [14], Caballero *et al.* [18, 19, 20, 21], Darwish [25], Hu and Yan [33], Leggett [36], Liu and Kang [37], Stuart [41] and Spiga *et al.* [42].

More recently, following the appearance of the paper [23], there has been significant interest in the study of the existence of solutions for singular quadratic integral equations or fractional quadratic integral equations; see [11, 12, 13, 24, 26, 27, 28].

It is worth mentioning that up to now only the two papers by Banaś and D. O'Regan [12] and Darwish [26] have dealt with the study of quadratic integral equation of singular kernel in the space of real functions which are defined, continuous and bounded on an *unbounded interval.* The proofs in [12] and [26] depend on a suitable combination of the technique of measures of noncompactness and the Schauder fixed point principle.

The aim of this paper is to prove the existence of solutions to Eq. (1.1) in the space of real functions which are defined, continuous and bounded on an unbounded interval. Moreover, we will obtain some asymptotic characterizations of the solutions of Eq. (1.1). Our proof depends on a suitable combination of the technique of measures of noncompactness and the Darbo fixed point principle. Also, we give an example for indicating the natural realizations of our abstract theory presented in the paper.

2. Preliminaries

This section is devoted to collecting some definitions and results which will be needed further on. First we recall the definition of the Riemann-Liouville fractional integral; see [31, 35, 38, 39, 40] for more information.

Definition 2.1. Let $f \in L_1(a, b)$, $0 \leq a < b < \infty$, and let $\beta > 0$ be a real number. The Riemann-Liouville fractional integral of order β of the function $f(t)$ is defined by

$$I^{\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{f(s)}{(t-s)^{1-\beta}}\, ds, \ a < t < b.$$

Now, let $(E, \| \cdot \|)$ be an infinite dimensional Banach space with zero element 0. Let $B(x, r)$ denotes the closed ball centered at x with radius r. The symbol B_r stands for the ball $B(0, r)$.

If X is a subset of E, then $\overline{X}$ and $ConvX$ denote the closure and convex closure of X, respectively. Moreover, we denote by $\mathcal{M}_E$ the family of all nonempty and bounded subsets of E, and by $\mathcal{N}_E$ its subfamily consisting of all relatively compact subsets.

Next we give the concept of a measure of noncompactness [3].

Definition 2.2. A mapping $\mu : \mathcal{M}_E \to R_+ = [0, \infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

1) The family $\ker\mu = \{X \in \mathcal{M}_E : \mu(X) = 0\}$ is nonempty and $\mathrm{Ker}\mu \subset \mathcal{N}_E$.
2) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
3) $\mu(\overline{X}) = \mu(ConvX) = \mu(X)$.
4) $\mu(\lambda X + (1 - \lambda)Y) \leq \lambda\, \mu(X) + (1 - \lambda)\, \mu(Y)$ for $0 \leq \lambda \leq 1$.
5) If $X_n \in \mathcal{M}_E$, $X_n = \overline{X}_n$, $X_{n+1} \subset X_n$ for $n = 1, 2, 3, \ldots$ and $\lim\limits_{n\to\infty} \mu(X_n) = 0$, then $\cap_{n=1}^{\infty} X_n \neq \phi$.

The family $\ker\mu$ described above is called the kernel of the measure of noncompactness μ. Let us observe that the intersection set X_∞ from 5) belongs to $\ker\mu$. In fact, since $\mu(X_\infty) \leq \mu(X_n)$ for every n, then we have that $\mu(X_\infty) = 0$.

In what follows we will work in the Banach space $BC(R_+)$ consisting of all real functions defined, bounded and continuous on R_+. This space is equipped with the standard norm

$$\| x \| = \sup\{|x(t)| : t \geq 0\}.$$

Now, we recollect the construction of the measure of noncompactness in $BC(R_+)$ which will be used in the next section (see [4, 6]).

Let us fix a nonempty and bounded subset X of $BC(R_+)$ and numbers $\varepsilon > 0$ and $T > 0$. For arbitrary function $x \in X$, let us denote by $\omega^T(x, \varepsilon)$ the modulus of continuity of the function x on the interval $[0, T]$, i.e.,

$$\omega^T(x, \varepsilon) = \sup\{|x(t) - x(s)| : t,\ s \in [0, T],\ |t - s| \leq \varepsilon\}.$$

Further, let us put

$$\omega^T(X, \varepsilon) = \sup\{\omega^T(x, \varepsilon) : x \in X\},$$

$$\omega_0^T(X) = \lim_{\varepsilon \to 0} \omega^T(X, \varepsilon),$$

and

$$\omega_0^\infty(X) = \lim_{T \to \infty} \omega_0^T(X, \varepsilon).$$

Moreover, for a fixed number $t \in R_+$, let us define

$$X(t) = \{x(t) : x \in X\}$$

and

$$\mathrm{diam} X(t) = \sup\{|x(t) - y(t)| : x,\ y \in X\}.$$

Finally, let us define the function μ on the family $\mathcal{M}_{BC(R_+)}$ by

$$\mu(X) = \omega_0^\infty(X) + c(X), \tag{2.1}$$

where $c(X) = \limsup_{t \to \infty} \mathrm{diam} X(t)$. The function μ is a measure of noncompactness in the space $BC(R_+)$; see [4].

The concept of the asymptotic stability of a solution $x(t)$ of Eq. (1.1) is understood in the following sense given by Banaś and Rzepka [8].

Definition 2.3. For any $\varepsilon > 0$ there exist $T(\varepsilon) > 0$ and $r(\varepsilon) > 0$ such that, if $x,\ y \in B_r$ and $x(t),\ y(t)$ are solutions of Eq. (1.1), then $|x(t) - y(t)| \le \varepsilon$ for $t \ge T(\varepsilon)$.

We will make use of a fixed point theorem due to Darbo [30]. Before quoting this theorem, we need the following definition.

Definition 2.4. Let M be a nonempty subset of a Banach space E and let $\mathcal{P} : M \to E$ be a continuous operator which transforms bounded sets onto bounded ones. We say that $\mathcal{P}$ satisfies the Darbo condition (with a constant $k \ge 0$) with respect to a measure of noncompactness μ, if for any bounded subset X of M, we have

$$\mu(\mathcal{P}X) \le k\, \mu(X).$$

If $\mathcal{P}$ satisfies the Darbo condition with $k < 1$, then $\mathcal{P}$ is called a contraction operator with respect to μ.

Theorem 2.5. *Let Q be a nonempty, bounded, closed and convex subset of the space E and let*

$$\mathcal{P} : Q \to Q$$

be a contraction with respect to the measure of noncompactness μ. Then $\mathcal{P}$ has a fixed point in the set Q.

3. Existence theorem

In this section we will study Eq. (1.1) assuming that the following hypotheses are satisfied:

(h_1) $g : R_+ \times R \to R$ is continuous and the function $t \to g(t, 0)$ is bounded on R_+ with $g^* = \sup\{|g(t, 0)| : t \in R_+\}$. Moreover, there exists a continuous function $l(t) = l : R_+ \to R_+$ such that

$$|g(t, x) - g(t, y)| \leq l(t)|x - y|$$

for all $x,\ y \in R$ and for any $t \in R_+$.

(h_2) $f : R_+ \times R \to R$ is continuous and there exists a continuous function $m(t) = m : R_+ \to R_+$ such that

$$|f(t, x) - f(t, y)| \leq m(t)|x - y|$$

for all $x,\ y \in R$ and for any $t \in R_+$.

(h_3) $v : R_+ \times R_+ \times R \to R$ is a continuous function. Moreover, there exist a function $n(t) = n : R_+ \to R_+$ being continuous on R_+ and a function $\Phi : R_+ \to R_+$ being continuous and nondecreasing on R_+, with $\Phi(0) = 0$ and such that

$$|v(t, s, x) - v(t, s, y)| \leq n(t)\Phi(|x - y|)$$

for all $t,\ s \in R_+$ such that $t \geq s$ and for all $x,\ y \in R$.

For further purpose, let us define the function $v^* : R_+ \to R_+$ by $v^*(t) = \max\{|v(t, s, 0)| : 0 \leq s \leq t\}$.

(h_4) The functions $\phi,\ \psi,\ \xi,\ \eta : R_+ \to R_+$ defined by, respectively, $\phi(t) = m(t)\ n(t)\ t^\beta$, $\psi(t) = m(t)\ v^*(t)\ t^\beta$, $\xi = n(t)|f(t, 0)|t^\beta$ and $\eta(t) = v^*(t)|f(t, 0)|t^\beta$, are bounded on R_+, and the functions ϕ and ξ vanish at infinity, i.e. $\lim_{t\to\infty} \phi(t) = \lim_{t\to\infty} \xi(t) = 0$.

(h_5) There exists a positive solution r_0 of the inequality

$$(l^*r + g^*)\Gamma(\beta+1) + [\phi^* r\Phi(r) + \psi^* r + \xi^*\Phi(r) + \eta^*] \leq r\ \Gamma(\beta+1) \quad (3.1)$$

and $l^*\Gamma(\beta+1) + \phi^*\Phi(r_0) + \psi^* < \Gamma(\beta+1)$, where $l^* = \sup\{l(t) : t \in R_+\}$, $\phi^* = \sup\{\phi(t) : t \in R_+\}$, $\psi^* = \sup\{\psi(t) : t \in R_+\}$, $\xi^* = \sup\{\xi(t) : t \in R_+\}$ and $\eta^* = \sup\{\eta(t) : t \in R_+\}$.

Now, we are in a position to state and prove our main result.

THEOREM 3.1. *Let the hypotheses* $(h_1) - (h_5)$ *be satisfied. Then* Eq. (1.1) *has at least one solution* $x \in BC(R_+)$ *and is asymptotically stable on the interval* R_+.

P r o o f. Denote by $\mathcal{F}$ the operator associated with the right-hand side of equation (1.1), i.e., equation (1.1) takes the form

$$x = \mathcal{F}x,$$

where

$$(\mathcal{F}\, x)(t) = (Gx)(t) + (Fx)(t) \cdot (\mathcal{V}\, x)(t), \tag{3.2}$$

and

$$(\mathcal{V}\, x)(t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{v(t,s,x(s))}{(t-s)^{1-\beta}}\, ds. \tag{3.3}$$

Solving Eq.(1.1) is equivalent to finding a fixed point of the operator $\mathcal{F}$ defined on the space $BC(R_+)$.

We claim that for any function $x \in BC(R_+)$ the operator $\mathcal{F}$ is continuous on R_+. To establish this claim, it suffices to show that, if $x \in BC(R_+)$ then $\mathcal{V}x$ is continuous function on R_+, thanks to (h_1) and (h_2). So, let us take an arbitrary $X \in BC(R_+)$ and fix $\varepsilon > 0$ and $T > 0$. Assume that $t_1,\ t_2 \in R_+$ are such that $|t_2 - t_1| \le \varepsilon$. Without loss of generality we can assume that $t_2 > t_1$. Then we get

$$\begin{aligned}
&|(\mathcal{V}x)(t_2) - (\mathcal{V}x)(t_1)| \\
&= \left| \frac{1}{\Gamma(\beta)} \int_0^{t_2} \frac{v(t_2,s,x(s))}{(t_2-s)^{1-\beta}}\, ds \frac{1}{\Gamma(\beta)} \int_0^{t_1} \frac{v(t_1,s,x(s))}{(t_1-s)^{1-\beta}}\, ds \right| \\
&\le \left| \frac{1}{\Gamma(\beta)} \int_0^{t_2} \frac{v(t_2,s,x(s))}{(t_2-s)^{1-\beta}}\, ds - \frac{1}{\Gamma(\beta)} \int_0^{t_1} \frac{v(t_2,s,x(s))}{(t_2-s)^{1-\beta}}\, ds \right| \\
&\quad + \left| \frac{1}{\Gamma(\beta)} \int_0^{t_1} \frac{v(t_2,s,x(s))}{(t_2-s)^{1-\beta}}\, ds - \frac{1}{\Gamma(\beta)} \int_0^{t_1} \frac{v(t_1,s,x(s))}{(t_2-s)^{1-\beta}}\, ds \right| \\
&\quad + \left| \frac{1}{\Gamma(\beta)} \int_0^{t_1} \frac{v(t_1,s,x(s))}{(t_2-s)^{1-\beta}}\, ds - \frac{1}{\Gamma(\beta)} \int_0^{t_1} \frac{v(t_1,s,x(s))}{(t_1-s)^{1-\beta}}\, ds \right| \\
&\le \frac{1}{\Gamma(\beta)} \int_{t_1}^{t_2} \frac{|v(t_2,s,x(s))|}{(t_2-s)^{1-\beta}}\, ds \\
&\quad + \frac{1}{\Gamma(\beta)} \int_0^{t_1} \frac{|v(t_2,s,x(s)) - v(t_1,s,x(s))|}{(t_2-s)^{1-\beta}}\, ds \\
&\quad + \frac{1}{\Gamma(\beta)} \int_0^{t_1} |v(t_1,s,x(s))|\, [(t_1-s)^{\beta-1} - (t_2-s)^{\beta-1}] ds.
\end{aligned}$$

Therefore, if

$$\omega_d^T(v,\varepsilon) = \sup\{|v(t_2,s,y) - v(t_1,s,z)| : \ s,\ t_1,\ t_2 \in [0,T],\\ t_1 \geq s,\ t_2 \geq s,\ |t_2 - t_1| \leq \varepsilon, \text{ and } y,\ z \in [-d,d]\},$$

then we obtain

$$\begin{aligned}
&|(\mathcal{V}x)(t_2) - (\mathcal{V}x)(t_1)| \\
&\leq \frac{1}{\Gamma(\beta)} \int_{t_1}^{t_2} \frac{|v(t_2,s,x(s)) - v(t_2,s,0)| + |v(t_2,s,0)|}{(t_2-s)^{1-\beta}}\, ds \\
&\quad + \frac{1}{\Gamma(\beta)} \int_0^{t_1} \frac{\omega_{\|x\|}^T(v,\varepsilon)}{(t_2-s)^{1-\beta}}\, ds \\
&\quad + \frac{1}{\Gamma(\beta)} \int_0^{t_1} [|v(t_1,s,x(s)) - v(t_1,s,0)| + |v(t_1,s,0)|] \\
&\quad \times [(t_1-s)^{\beta-1} - (t_2-s)^{\beta-1}] ds \\
&\leq \frac{1}{\Gamma(\beta)} \int_{t_1}^{t_2} \frac{n(t_2)\Phi(|x(s)|) + v^*(t_2)}{(t_2-s)^{1-\beta}}\, ds + \frac{\omega_{\|x\|}^T(v,\varepsilon)}{\Gamma(\beta+1)}[t_2^\beta - (t_2-t_1)^\beta] \\
&\quad + \frac{1}{\Gamma(\beta)} \int_0^{t_1} [n(t_1)\Phi(|x(s)|) + v^*(t_1)]\ [(t_1-s)^{\beta-1} - (t_2-s)^{\beta-1}] ds \\
&\leq \frac{n(t_2)\Phi(\|x\|) + v^*(t_2)}{\Gamma(\beta+1)}(t_2-t_1)^\beta + \frac{\omega_{\|x\|}^T(v,\varepsilon)}{\Gamma(\beta+1)} t_1^\beta \\
&\quad + \frac{n(t_1)\Phi(\|x\|) + v^*(t_1)}{\Gamma(\beta+1)}[t_1^\beta - t_2^\beta + (t_2-t_1)^\beta].
\end{aligned}$$

Thus

$$\omega^T(\mathcal{V}x,\varepsilon) \leq \frac{2\varepsilon^\beta[\hat{n}(T)\Phi(\|x\|) + \hat{v}(T)] + T^\beta \omega_{\|x\|}^T(v,\varepsilon)}{\Gamma(\beta+1)}, \tag{3.4}$$

where we denoted

$$\hat{n}(T) = \max\{n(t) : t \in [0,T]\}$$

and

$$\hat{v}(T) = \max\{v^*(t) : t \in [0,T]\}.$$

In view of the uniform continuity of the function v on $[0,T] \times [0,T] \times [-\|x\|, \|x\|]$, we have that $\omega_{\|x\|}^T(v,\varepsilon) \to 0$ as $\varepsilon \to 0$. From the above inequality we infer that the function $\mathcal{V}x$ is continuous on the interval $[0,T]$ for any $T > 0$. This yields the continuity of $\mathcal{V}x$ on R_+, and consequently, the function $\mathcal{F}x$ is continuous on R_+.

Now, we show that $\mathcal{F}x$ is bounded on R_+. Indeed, in view of our hypotheses for arbitrary $x \in BC(R_+)$ and for a fixed $t \in R_+$, we have

$$\begin{aligned}
|(\mathcal{F}x)(t)| &\le \left| g(t,x(t)) + \frac{f(t,x(t))}{\Gamma(\beta)} \int_0^t \frac{v(t,s,x(s))}{(t-s)^{1-\beta}}\, ds \right| \\
&\le |g(t,x(t)) - g(t,0)| + |g(t,0)| \\
&\quad + \frac{1}{\Gamma(\beta)} [|f(t,x(t)) - f(t,0)| + |f(t,0)|] \\
&\qquad \times \int_0^t \frac{|v(t,s,x(s)) - v(t,s,0)| + |v(t,s,0)|}{(t-s)^{1-\beta}}\, ds \\
&\le l(t)\|x\| + |g(t,0)| \\
&\quad + \frac{m(t)\|x\| + |f(t,0)|}{\Gamma(\beta)} \int_0^t \frac{n(t)\Phi(|x(s)|) + v^*(t)}{(t-s)^{1-\beta}}\, ds \\
&\le l(t)\|x\| + |g(t,0)| \\
&\quad + \frac{m(t)\|x\| + |f(t,0)|}{\Gamma(\beta+1)} [n(t)\ \Phi(\|x\|) + v^*(t)]\ t^\beta \\
&\le l^*\|x\| + g^* + \frac{1}{\Gamma(\beta+1)} [\phi(t)\|x\|\Phi(\|x\|) \\
&\quad + \psi(t)\|x\| + \xi(t)\Phi(\|x\|) + \eta(t)].
\end{aligned}$$

Hence, $\mathcal{F}x$ is bounded on R_+, thanks to hypothesis (h_4). This assertion, in conjunction with the continuity of $\mathcal{F}x$ on R_+ allows us to conclude that the operator $\mathcal{F}$ maps $BC(R_+)$ into itself. Moreover, from the last estimate we have

$$\|\mathcal{F}x\| \le l^*\|x\| + g^* + \frac{1}{\Gamma(\beta+1)} [\phi^*\|x\|\Phi(\|x\|) + \psi^*\|x\| + \xi^*\Phi(\|x\|) + \eta^*].$$

Linking this estimate with hypothesis (h_5), we deduce that there exists $r_0 > 0$ such that the operator $\mathcal{F}$ transforms the ball B_{r_0} into itself.

In what follows let us take a nonempty set $X \subset B_{r_0}$. Then, for arbitrary $x,\ y \in X$ and for a fixed $t \in R_+$, we obtain

$$\begin{aligned}
&|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)| \\
&\le |g(t,x(t)) - g(t,y(t))| \\
&\quad + \left| \frac{f(t,x(t))}{\Gamma(\beta)} \int_0^t \frac{v(t,s,x(s))}{(t-s)^{1-\beta}}\, ds - \frac{f(t,y(t))}{\Gamma(\beta)} \int_0^t \frac{v(t,s,y(s))}{(t-s)^{1-\beta}}\, ds \right| \\
&\le l(t)|x(t) - y(t)| + \frac{|f(t,x(t)) - f(t,y(t))|}{\Gamma(\beta)} \int_0^t \frac{|v(t,s,x(s))|}{(t-s)^{1-\beta}}\, ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{|f(t,y(t))|}{\Gamma(\beta)}\int_0^t \frac{|v(t,s,x(s))-v(t,s,y(s))|}{(t-s)^{1-\beta}}ds\\
&\le l(t)|x(t)-y(t)|\\
&+\frac{m(t)|x(t)-y(t)|}{\Gamma(\beta)}\int_0^t \frac{|v(t,s,x(s))-v(t,s,0)|+|v(t,s,0)|}{(t-s)^{1-\beta}}\,ds\\
&+\frac{m(t)|y(t)|+|f(t,0)|}{\Gamma(\beta)}\int_0^t \frac{n(t)\ \Phi(|x(s)-y(s)|)}{(t-s)^{1-\beta}}ds\\
&\le l(t)|x(t)-y(t)|\\
&+\frac{m(t)|x(t)-y(t)|}{\Gamma(\beta)}\int_0^t \frac{n(t)\Phi(|x(s)|)+v^*(t)}{(t-s)^{1-\beta}}\,ds\\
&+\frac{m(t)|y(t)|+|f(t,0)|}{\Gamma(\beta)}\int_0^t \frac{n(t)\Phi(|x(s)|+|y(s)|)}{(t-s)^{1-\beta}}ds\\
&\le l(t)|x(t)-y(t)|\\
&+\frac{m(t)n(t)(|x(t)|+|y(t)|)}{\Gamma(\beta)}\int_0^t \frac{\Phi(|x(s)|)}{(t-s)^{1-\beta}}\,ds\\
&+\frac{m(t)v^*(t)|x(t)-y(t)|}{\Gamma(\beta)}\int_0^t \frac{ds}{(t-s)^{1-\beta}}\\
&+\frac{m(t)n(t)|y(t)|}{\Gamma(\beta)}\int_0^t \frac{\Phi(|x(s)|+|y(s)|)}{(t-s)^{1-\beta}}ds\\
&+\frac{n(t)|f(t,0)|}{\Gamma(\beta)}\int_0^t \frac{\Phi(|x(s)|+|y(s)|)}{(t-s)^{1-\beta}}ds\\
&\le l(t)\mathrm{diam}X(t)+\frac{2m(t)n(t)r_0\Phi(r_0)}{\Gamma(\beta)}\int_0^t \frac{ds}{(t-s)^{1-\beta}}\\
&+\frac{m(t)v^*(t)\mathrm{diam}X(t)}{\Gamma(\beta)}\int_0^t \frac{ds}{(t-s)^{1-\beta}}\\
&+\frac{m(t)n(t)r_0\Phi(2r_0)}{\Gamma(\beta)}\int_0^t \frac{ds}{(t-s)^{1-\beta}}\\
&+\frac{n(t)|f(t,0)|\Phi(2r_0)}{\Gamma(\beta)}\int_0^t \frac{ds}{(t-s)^{1-\beta}}\\
&\le \frac{2\phi(t)r_0\Phi(r_0)}{\Gamma(\beta+1)}+\left(l(t)+\frac{\psi(t)}{\Gamma(\beta+1)}\right)\mathrm{diam}X(t)\\
&\quad+\frac{\phi(t)r_0\Phi(2r_0)}{\Gamma(\beta+1)}+\frac{\xi(t)\Phi(2r_0)}{\Gamma(\beta+1)}.
\end{aligned}
$$

Hence, we can easily deduce the following inequality

$$\begin{aligned}\operatorname{diam}(\mathcal{F}X)(t) &\le \frac{2\phi(t)r_0\Phi(r_0)}{\Gamma(\beta+1)} + \left(l(t) + \frac{\psi(t)}{\Gamma(\beta+1)}\right)\operatorname{diam}X(t)\\ &\quad + \frac{\Phi(2r_0)}{\Gamma(\beta+1)}(\phi(t)r_0 + \xi(t)).\end{aligned}$$

Now, taking into account hypothesis (h_4), we obtain

$$c(\mathcal{F}X) \le k\, c(X), \tag{3.5}$$

where we denoted $k = l^* + \frac{\phi^*\Phi(r_0)+\psi^*}{\Gamma(\beta+1)} \ge l^* + \frac{\psi^*}{\Gamma(\beta+1)}$. Obviously, in view of hypothesis (h_5), we have that $k < 1$.

In what follows, let us take arbitrary numbers $\varepsilon > 0$ and $T > 0$. Choose a function $x \in X$ and take $t_1,\ t_2 \in [0,T]$ such that $|t_2 - t_1| \le \varepsilon$. Without loss of generality we can assume that $t_2 > t_1$. Then, taking into account our hypotheses and (3.4), we have

$$\begin{aligned}
&|(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)|\\
&\le |g(t_2,x(t_2)) - g(t_1,x(t_1))| + |(Fx)(t_2)(\mathcal{V}x)(t_2) - (Fx)(t_1)(\mathcal{V}x)(t_2)|\\
&\qquad + |(Fx)(t_1)(\mathcal{V}x)(t_2) - (Fx)(t_1)(\mathcal{V}x)(t_1)|\\
&\le |g(t_2,x(t_2)) - g(t_2,x(t_1))| + |g(t_2,x(t_1)) - g(t_1,x(t_1))|\\
&\quad + \frac{|f(t_2,x(t_2)) - f(t_1,x(t_1))|}{\Gamma(\beta)}\\
&\quad \times \int_0^{t_2} \frac{|v(t_2,s,x(s)) - v(t_2,s,0)| + |v(t_2,s,0)|}{(t_2-s)^{1-\beta}}\, ds\\
&\quad + \frac{|f(t_1,x(t_1)) - f(t_1,0)| + |f(t_1,0)|}{\Gamma(\beta+1)}\\
&\quad \times \left\{2\varepsilon^\beta[\hat{n}(T)\Phi(\|x\|) + \hat{v}(T)] + T^\beta\omega^T_{\|x\|}(v,\varepsilon)\right\}\\
&\le l(t_2)|x(t_2) - x(t_1)| + \omega_g^T(\varepsilon)\\
&\quad + \frac{m(t_2)|x(t_2) - x(t_1)| + \omega_f^T(\varepsilon)}{\Gamma(\beta)} \int_0^{t_2} \frac{n(t_2)\Phi(|x(s)|) + v^*(t_2)}{(t_2-s)^{1-\beta}}\, ds\\
&\quad + \frac{m(t_1)|x(t_1)| + |f(t_1,0)|}{\Gamma(\beta+1)} \left\{2\varepsilon^\beta[\hat{n}(T)\Phi(\|x\|) + \hat{v}(T)] + T^\beta\omega^T_{\|x\|}(v,\varepsilon)\right\}\\
&\le l(t_2)\omega^T(x,\varepsilon) + \omega_g^T(\varepsilon)\\
&\quad + \frac{\phi(t_2)\Phi(r_0) + \psi(t_2)}{\Gamma(\beta+1)}\omega^T(x,\varepsilon) + \frac{n(t_2)\Phi(r_0) + v^*(t_2)}{\Gamma(\beta+1)} t_2^\beta\, \omega_f^T(\varepsilon)
\end{aligned}$$

$$+\frac{\hat{m}(T)r_0+\hat{f}(T)}{\Gamma(\beta+1)}\left\{2\varepsilon^\beta[\hat{n}(T)\Phi(r_0)+\hat{v}(T)]+T^\beta\omega^T_{r_0}(v,\varepsilon)\right\}$$
$$\leq \omega^T_g(\varepsilon)+\left(l^*+\frac{\phi^*\Phi(r_0)+\psi^*}{\Gamma(\beta+1)}\right)\omega^T(x,\varepsilon)+\frac{\hat{n}(T)\Phi(r_0)+\hat{v}(T)}{\Gamma(\beta+1)}T^\beta\omega^T_f(\varepsilon)$$
$$+\frac{\hat{m}(T)r_0+\hat{f}(T)}{\Gamma(\beta+1)}\left\{2\varepsilon^\beta[\hat{n}(T)\Phi(r_0)+\hat{v}(T)]+T^\beta\omega^T_{r_0}(v,\varepsilon)\right\},$$

where we denoted

$$\begin{aligned}\omega^T_h(\varepsilon) &= \sup\{|h(t_2,x(t_1))-h(t_1,x(t_1))| : t_1,\ t_2\in[0,T],\\ &\qquad |t_2-t_1|\leq\varepsilon,\ x\in[-r_0,r_0]\},\end{aligned}$$

$$\hat{m}(T)=\max\{m(t):t\in[0,T]\},$$

and

$$\hat{f}(T)=\max\{|f(t,0)|:t\in[0,T]\}.$$

Now, keeping in mind the uniform continuity of the functions $f=f(t,x)$ and $g=g(t,x)$ on the set $[0,T]\times[r_0,r_0]$ and the uniform continuity of the function $v=v(t,s,x)$ on the set $[0,T]\times[0,T]\times[r_0,r_0]$, from the last estimate we derive the following one:

$$\omega^T_0(\mathcal{F}X)\leq k\,\omega^T_0(X).$$

Hence we have

$$\omega^\infty_0(\mathcal{F}X)\leq k\,\omega^\infty_0(X). \tag{3.6}$$

From (3.5) and (3.6) and the definition of the measure of noncompactness μ given by formula (2.1), we obtain

$$\mu(\mathcal{F}X)\leq k\,\mu(X). \tag{3.7}$$

Now, the above established inequality together with the fact that $k<1$ enables us to apply Theorem , and hence Eq. (1.1) has at least one solution $x\in BC(R_+)$.

Moreover, it is easy to see from the above arguments that any solution of Eq. (1.1) which belongs to the ball B_{r_0} is asymptotically stable. This completes the proof. ∎

4. Example

Consider the following perturbed quadratic integral equation of fractional order $\beta=\frac{1}{3}$,

$$x(t) = \frac{1}{2\pi}\arctan(t + x(t)) + \frac{t + t^2 x(t)}{5\Gamma(\frac{1}{3})}\int_0^t \frac{|x(s)|e^{-3t-s} + 1/(1 + 5t^{7/3})}{(t-s)^{2/3}}\, ds. \tag{4.1}$$

In this example, we have that $g(t, x) = \frac{4}{\pi}\arctan(t + x)$ and this function satisfies hypothesis (h_1) with $l(t) = \frac{1}{2\pi}$ and $g^* = 0.25$. Moreover, $f(t, x(t)) = (t + t^2 x(t))/5$ and satisfies hypothesis (h_2) with $m(t) = t^2/5$, and $|f(t, 0)| = f(t, 0) = t/5$. Also, $v(t, s, x) = xe^{-3t-s} + 1/(1+5t^{7/3})$ verifies hypothesis (h_3) with $n(t) = e^{-3t}$, $\Phi(x) = x$ and $v(t, s, 0) = 1/(1 + 5t^{7/3})$. Now, we have

$$\phi(t) = \frac{1}{5}t^{7/3}e^{-3t},\ \psi(t) = \frac{t^{7/3}}{5(1+5t^{7/3})},\ \xi(t) = \frac{1}{5}t^{4/3}e^{-3t},\ \eta(t) = \frac{t^{4/3}}{5(1+5t^{7/3})}.$$

It is easy to see that the functions ϕ, ψ, ξ and η are bounded on R_+ and also $\lim_{t\to\infty}\phi(t) = \lim_{t\to\infty}\xi(t) = 0$. Hence, hypothesis (h_4) is satisfied. Moreover, we have

$$\phi^* = \phi(7/6) = \frac{1}{5}(7/6)^{7/3}e^{-7/2} = 0.0277897...,$$

$$\psi^* = 0.04,$$

$$\xi^* = \xi(4/9) = \frac{1}{5}(4/9)^{4/3}e^{-4/3} = 0.0178811...$$

and

$$\eta^* = \eta((4/15)^{3/7}) = 0.0072288... .$$

In this case the inequality (3.1) has the form

$$(l^* r + g^*)\Gamma(\frac{4}{3}) + r^2\phi^* + r\psi^* + r\xi^* + \eta^* \le r\Gamma(\frac{4}{3}). \tag{4.2}$$

Let us denote by $H(r)$ the left hand side of the last inequality, i.e.,

$$H(r) = (l^* r + g^*)\Gamma(\frac{4}{3}) + r^2\phi^* + r\psi^* + r\xi^* + \eta^*.$$

For $r = 1$ we obtain

$$H(1) = (l^* + g^*)\Gamma(\frac{4}{3}) + \phi^* + \psi^* + \xi^* + \eta^* = \Gamma(\frac{4}{3})\ 0.4091549... + 0.0928997... .$$

Hence, inequality (4.2) admits $r_0 = 1$ as a positive solution since $\Gamma(\frac{4}{3}) \simeq 0.8929796$. Moreover,

$$l^*\Gamma(\frac{4}{3}) + \phi^*\phi(r_0) + \psi^* \simeq 0.2099124 < \Gamma(\frac{4}{3}).$$

Therefore, Theorem 3.1 guarantees that equation (4.1) has a solution $x = x(t)$ in the space R_+ belonging to the ball B_1. Moreover, this solution is asymptotically stable.

References

[1] J. Appell and P.P. Zabrejko, *Nonlinear Superposition Operators*, Cambridge Tracts in Mathematics, **95**, Cambridge University Press (1990).

[2] I.K. Argyros, Quadratic equations and applications to Chandrasekhar's and related equations, *Bull. Austral. Math. Soc.* **32** (1985), 275-292.

[3] J. Banaś and K. Goebel, *Measures of Noncompactness in Banach Spaces*, Lecture Notes in Pure and Applied Mathematics, **60**, Marcel Dekker, New York (1980).

[4] J. Banaś, Measures of noncompactness in the space of continuous tempered functions, *Demonstratio Math.* **14** (1981), 127-133.

[5] J. Banaś, M. Lecko and W. G. El-Sayed, Existence theorems of some quadratic integral equations, *J. Math. Anal. Appl.* **222** (1998), 276-285.

[6] J. Banaś and L. Olszpwz, Measures of noncompactness related to monotonicity, *Comment. Math.* **41** (2001), 13-23.

[7] J. Banaś, J. Rocha and K. Sadarangani, Solvability of a nonlinear integral equation of Volterra type, *J. Comput. Appl. Math.* **157** (2003), 31-48.

[8] J. Banaś and B. Rzepka, On existence and asymptotic stability of solutions of a nonlinear integral equation, *J. Math. Anal. Appl.* **284** (2003), 165-173.

[9] J. Banaś and A. Martinon, Monotonic solutions of a quadratic integral equation of Volterra type, *Comput. Math. Appl.* **47** (2004), 271-279.

[10] J. Banaś, J. Caballero, J. Rocha and K. Sadarangani, Monotonic solutions of a class of quadratic integral equations of Volterra type, *Comput. Math. Appl.* **49** (2005), 943-952.

[11] J. Banaś and B. Rzepka, Monotonic solutions of a quadratic integral equation of fractional order, *J. Math. Anal. Appl.* **332** (2007), 1370-1378.

[12] J. Banaś and D. O'Regan, On existence and local attractivity of solutions of a quadratic integral equation of fractional order, *J. Math. Anal. Appl.* **345** (2008), 573-582.

[13] J. Banaś and B. Rzepka, Nondecreasing solutions of a quadratic singular Volterra integral equation, *Mathematical and Computer Modelling* (2008), doi:10.1016/j.mcm.2007.10.021.

[14] M. Benchohra and M.A. Darwish, On monotonic solutions of a quadratic integral equation of Hammerstein type, *Int. J. Appl. Math. Sci.* (to appear).

[15] V.C. Boffi and G. Spiga, An equation of Hammerstein type arising in particle transport theory, *J. Math. Phys.* **24** (1983), 1625-1629.

[16] V.C. Boffi and G. Spiga, Nonlinear removal effects in time-dependent particle transport theory, *Z. Angew. Math. Phys.* **34** (1983), 347-357.

[17] K.M. Case and P.F. Zweifel, *Linear Transport Theory*, Addison-Wesley, Reading, MA (1967).

[18] J. Caballero, D. O'Regan and K. Sadarangani, On solutions of an integral equation related to traffic flow on unbounded domains, *Arch. Math. (Basel)* **82** (2004), 551-563.

[19] J. Caballero, B. López and K. Sadarangani, On monotonic solutions of an integral equation of Volterra type with supremum, *J. Math. Anal. Appl.* **305** (2005), 304-315.

[20] J. Caballero, J. Rocha and K. Sadarangani, On monotonic solutions of an integral equation of Volterra type, *J. Comput. Appl. Math.* **174** (2005), 119-133.

[21] J. Caballero, B. López and K. Sadarangani, Existence of nondecreasing and continuous solutions for a nonlinear integral equation with supremum in the kernel, *Z. Anal. Anwend.* **26** (2007), 195-205.

[22] S. Chandrasekher, *Radiative Transfer*, Dover Publications, New York (1960).

[23] M.A. Darwish, On quadratic integral equation of fractional orders, *J. Math. Anal. Appl.* **311** (2005), 112-119.

[24] M.A. Darwish, On a singular quadratic integral equation of Volterra type with supremum, *IC/2007/071, Trieste, Italy* (2007), 1-13.

[25] M.A. Darwish, On solvability of some quadratic functional-integral equation in Banach algebra, *Commun. Appl. Anal.* **11** (2007), 441-450.

[26] M. A. Darwish, On existence and asympototic behaviour of solutions of a fractional integral equation with linear modification of the argument, *arXiv: 0805.1422v1* (2008).

[27] M.A. Darwish, On monotonic solutions of a singular quadratic integral equation with supremum, *Dynam. Syst. Appl.* 17 (2008), 539-550.

[28] M.A. Darwish, On monotonic solutions of an integral equation of Abel type. *Math. Bohemica* **133**, No 4 (2008), 407-420.

[29] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin (1985).

[30] J. Dugundji and A. Granas, *Fixed Point Theory*, Monografie Mathematyczne, PWN, Warsaw (1982).

[31] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore (2000).
[32] S. Hu, M. Khavani and W. Zhuang, Integral equations arising in the kinetic theory of gases, *Appl. Anal.* **34** (1989), 261-266.
[33] X. Hu and J. Yan, The global attractivity and asymptotic stability of solution of a nonlinear integral equation, *J. Math. Anal. Appl.* **321** (2006), 147-156.
[34] C.T. Kelly, Approximation of solutions of some quadratic integral equations in transport theory, *J. Integral Eq.* **4** (1982), 221-237.
[35] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, **204**. Elsevier Science B.V., Amsterdam (2006).
[36] R.W. Leggett, A new approach to the H-equation of Chandrasekher, *SIAM J. Math.* **7** (1976), 542-550.
[37] Z. Liu and S. Kang, Existence of monotone solutions for a nonlinear quadratic integral equation of Volterra type, *Rocky Mountain J. Math.* **37** (2007), 1971-1980.
[38] K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York (1993).
[39] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego (1999).
[40] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Science Publs., Amsterdam (1993) [Russian Edition: Nauka, Moscow (1987)].
[41] C.A. Stuart, Existence theorems for a class of nonlinear integral equations, *Math. Z.* **137** (1974), 49-66.
[42] G. Spiga, R.L. Bowden and V.C. Boffi, On the solutions of a class of nonlinear integral equations arising in transport theory, *J. Math. Phys.* **25** (1984), 3444-3450.

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