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IMPULSIVE FRACTIONAL DIFFERENTIAL INCLUSIONS INVOLVING THE CAPUTO FRACTIONAL DERIVATIVE [1]

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Abstract

In this paper, we establish sufficient conditions for the existence of solutions for a class of initial value problem for impulsive fractional differential inclusions involving the Caputo fractional derivative. Both cases of convex and nonconvex valued right-hand side are considered. The topological structure of the set of solutions is also considered.

Mathematics Subject Classification: 26A33, 34A37

Key Words and Phrases: initial value problem, fractional differential inclusions, impulses, Caputo fractional derivative, fractional integral, selection, existence, fixed point theorem

1. Introduction

This paper deals with the existence and uniqueness of solutions for the initial value problems (IVP for short), for fractional order differential inclusions

$$^{c}D^{\alpha}y(t) \in F(t,y), \text{ for a.e., } t \in J = [0,T],\ t \neq t_k,\ k = 1,\ldots,m,\ 0 < \alpha \leq 1, \tag{1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)),\ k = 1,\ldots,m, \tag{2}$$

[1]This work was completed when the second author was visiting the Dept. of Mathematics of University of Marrakech. It is a pleasure for him to express his gratitude for the provided support and the warm hospitality.

$$y(0) = y_0, \tag{3}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative, $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, $I_k : \mathbb{R} \to \mathbb{R}$, $k = 1, \ldots, m$ and $y_0 \in \mathbb{R}$, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^-} y(t_k + h)$ represent the right and left limits of $y(t)$ at $t = t_k$, $k = 1, \ldots, m$. The differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [19, 29, 30, 33, 41, 42, 46]). There has been a significant development in theory of fractional calculus and fractional ordinary and partial differential equations in recent years; see e.g. the monographs of Kilbas et al [36], Kiryakova [37], Miller and Ross [43], Samko et al [51] and the papers of Belarbi et al [4, 5], Benchohra et al [6, 8, 9], Delbosco and Rodino [18], Diethelm et al [19, 20, 21], El-Sayed [22, 23, 24], Furati and Tatar [27, 28], Kilbas and Marzan [35], Lakshmikantham and Devi [40], Mainardi [41], Momani and Hadid [44], Momani *et al* [45], Ouahab [47], Podlubny et al [50], Yu and Gao [54] and Zhang [55], and the references therein.

The applied problems require definitions of fractional derivatives allowing the utilization of physically interpretable initial conditions, containing $y(0)$, $y'(0)$, etc., and the same requirements for the boundary conditions. Caputo's fractional derivative satisfies these demands. For more details on the geometric and physical interpretation for fractional derivatives of both the Riemann-Liouville and Caputo types, see [32, 49].

The web site *http://people.tuke.sk/igor.podlubny/*, authored by Igor Podlubny contains more information on fractional calculus and its applications, and hence it is very useful for those that are interested in this field.

The impulsive differential equations (for $\alpha \in \mathbb{N}$) have become important in recent years as mathematical models of phenomena in both physical and social sciences. There has been a significant development in impulsive theory, especially in the area of impulsive differential equations with fixed moments; see for instance the monographs by Bainov and Simeonov [3], Benchohra *et al* [7], Lakshmikantham *et al* [39], and Samoilenko and Perestyuk [52] and the references therein. In [10], Benchohra and Slimani have initiated the study of fractional differential equations with impulses. To the best knowledge of the authors, no papers exist in the literature devoted to differential inclusion with fractional order and impulses. Thus the results of the present paper initiate this study.

This paper is organized as follows. In Section 2 we introduce some preliminary results needed in the following sections. In Section 3 we present an existence result for the problem (1)-(3), when the right hand side is convex valued using the nonlinear alternative of Leray-Schauder type. In Section 4 two results are given for nonconvex valued right hand side. The first one is based upon a fixed point theorem for contraction multivalued maps due to Covitz and Nadler, and the second one on the nonlinear alternative of Leray Schauder type [31] for single-valued maps, combined with a selection theorem due to Bressan-Colombo [14] for lower semicontinuous multivalued maps with decomposable values. The topological structure of the solutions set is also considered in Section 5. An indication to nonlocal problems is presented in Section 6 and an example is presented in the last section. These results extend to the multivalued case some results from the above cited literature, and constitute a contribution to this emerging field of research.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that are used in the remainder of this paper. Let $[a,b]$ be a compact interval, $C([a,b],\mathbb{R})$ be the Banach space of all continuous functions from $[a,b]$ into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : a \le t \le b\},$$

and we denote by $L^1([a,b],\mathbb{R})$ the Banach space of functions $y : [a,b] \longrightarrow \mathbb{R}$ that are Lebesgue integrable with norm

$$\|y\|_{L^1} = \int_a^b |y(t)|dt.$$

$AC([a,b],\mathbb{R})$ is the space of functions $y : [a,b] \to \mathbb{R}$, which are absolutely continuous. Let $(X, \|\cdot\|)$ be a Banach space. Let $P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$, $P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$, $P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$ and $P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact and convex}\}$. A multivalued map $G : X \to P(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in X for all $B \in P_b(X)$ (i.e. $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that

$G(N_0) \subseteq N$. G is said to be completely continuous if $G(\mathcal{B})$ is relatively compact for every $\mathcal{B} \in P_b(X)$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \longrightarrow x_*$, $y_n \longrightarrow y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$. A multivalued map $G : J \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable.

Let A be a subset of $[0,T] \times \mathbb{R}$. A is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $\mathcal{J} \times D$ where $\mathcal{J}$ is Lebesgue measurable in $[0,T]$ and D is Borel measurable in $\mathbb{R}$. A subset A of $L^1([0,T], \mathbb{R})$ is decomposable if for all $u, v \in A$ and $\mathcal{J} \subset [0,T]$ measurable, $u\chi_{\mathcal{J}} + v\chi_{[0,T]-\mathcal{J}} \in A$, where χ stands for the characteristic function.

Let $G : X \to \mathcal{P}(X)$ a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{x \in X : G(x) \cap B \neq \emptyset\}$ is open for any open set B in X. For more details on multivalued maps, see the books of Aubin and Cellina [1], Aubin and Frankowska [2], Deimling [17] and Hu and Papageorgiou [34].

Definition 2.1. A multivalued map $F : [a,b] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t \longmapsto F(t,u)$ is measurable for each $u \in \mathbb{R}$;

(ii) $u \longmapsto F(t,u)$ is upper semicontinuous for almost all $t \in [a,b]$.

For each $y \in C([a,b], \mathbb{R})$, define the set of selections of F by

$$S_{F,y} = \{v \in L^1([a,b], \mathbb{R}) : v(t) \in F(t, y(t)) \ a.e. \ t \in [a,b]\}.$$

Let (X,d) be a metric space induced from the normed space $(X, |\cdot|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\},$$

where $d(A,b) = \inf_{a\in A} d(a,b)$, $d(a,B) = \inf_{b\in B} d(a,b)$. Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space (see [38]).

Definition 2.2. A multivalued operator $N : X \to P_{cl}(X)$ is called

a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \le \gamma d(x, y), \quad \text{for each } x,\ y \in X,$$

b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

The following lemma will be used in the sequel.

Lemma 2.3. *Let (X, d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

Definition 2.4. ([36, 48]). The fractional (arbitrary) order integral of the function $h \in L^1([a, b], \mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds,$$

where Γ is the gamma function. When $a = 0$, we write $I^\alpha h(t) = h(t) * \varphi_\alpha(t)$, where $\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, and $\varphi_\alpha(t) = 0$ for $t \le 0$, and $\varphi_\alpha \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function.

Definition 2.5. ([36, 48]). For a function h given on the interval $[a, b]$, the αth Riemann-Liouville fractional-order derivative of h, is defined by

$$(D_{a+}^\alpha h)(t) \quad = \quad \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} h(s)ds.$$

Here $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Definition 2.6. ([36]). For a function h given on the interval $[a, b]$, the Caputo fractional-order derivative of h, is defined by

$$({}^cD_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)ds,$$

where $n = [\alpha] + 1$.

3. The convex case

In this section, we are concerned with the existence of solutions for the problem (1)-(3) when the right hand side has convex values. Initially, we assume that F is a compact and convex valued multivalued map.

Consider the following space

$PC(J,\mathbb{R}) = \{y : J \to \mathbb{R} : y \in C((t_k, t_{k+1}], \mathbb{R}),\ k = 0, \dots, m+1,$

and there exist $y(t_k^-)$ and $y(t_k^+)$, $k = 1, \dots, m$, with $y(t_k^-) = y(t_k)\}$.

$C(J,\mathbb{R})$ is a Banach space with norm

$$\|y\|_{PC} = \sup_{t\in J} |y(t)|.$$

Set $J' := [0,T] \backslash \{t_1, \dots, t_m\}$.

Definition 3.1. A function $y \in PC(J,\mathbb{R}) \bigcap \bigcup_{k=0}^{k=m} AC((t_k, t_{k+1}), \mathbb{R})$ with its α-derivative exists on J' is said to be a solution of (1)–(3) if there exists a function $v \in L^1([0,T],\mathbb{R})$such that $v(t) \in F(t,y)$ a.e. $t \in J$ satisfies the differential equation ${}^cD^\alpha y(t) = v(t)$ on J', and the conditions

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)),\ k = 1, \dots, m,$$

and

$$y(0) = y_0$$

are satisfied.

For the existence of solutions for the problem (1)–(3), we need the following auxiliary lemmas:

Lemma 3.2. ([55]) *Let $\alpha > 0$, then the differential equation*

$${}^cD^\alpha h(t) = 0$$

has solutions $h(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1}$, $c_i \in \mathbb{R}$, $i = 0, 1, ..., n-1$, $n = [\alpha] + 1$.

Lemma 3.3. ([55]) *Let $\alpha > 0$, then*

$$I^{\alpha c} D^\alpha h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, ..., n-1$, $n = [\alpha] + 1$.

As a consequence of Lemma 3.2 and Lemma 3.3, we have the following result which is useful in what follows.

LEMMA 3.4. *Let* $0 < \alpha \leq 1$ *and let* $\rho \in PC(J, \mathbb{R})$. *A function* y *is a solution of the fractional integral equation*

$$y(t) = \begin{cases} y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \rho(s) ds & \text{if } t \in [0, t_1], \\ y_0 + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} \rho(s) ds \\ + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} \rho(s) ds \\ + \sum_{i=1}^{k} I_i(y(t_i^-)), & \text{if } t \in (t_k, t_{k+1}],\ k = 1, \dots, m, \end{cases} \tag{4}$$

if and only if y *is a solution of the fractional IVP*

$$^cD^\alpha y(t) = \rho(t), \text{ for each, } t \in J', \tag{5}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)),\ k = 1, \dots, m, \tag{6}$$

$$y(0) = y_0. \tag{7}$$

P r o o f. Assume y satisfies (5)-(7). If $t \in [0, t_1]$, then

$$^cD^\alpha y(t) = \rho(t).$$

Lemma 3.3 implies

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \rho(s) ds.$$

If $t \in (t_1, t_2]$, then Lemma 3.3 implies

$$\begin{aligned} y(t) &= y(t_1^+) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} \rho(s) ds \\ &= \Delta y|_{t=t_1} + y(t_1^-) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} \rho(s) ds \\ &= I_1(y(t_1^-)) + y_0 + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} \rho(s) ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} \rho(s) ds. \end{aligned}$$

If $t \in (t_2, t_3]$, then again Lemma 3.3 we get

$$\begin{aligned} y(t) &= y(t_2^+) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1} \rho(s) ds \\ &= \Delta y|_{t=t_2} + y(t_2^-) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1} \rho(s) ds \end{aligned}$$

$$= I_2(y(t_2^-)) + I_1(y(t_1^-)) + y_0 + \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\rho(s)ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\rho(s)ds + \frac{1}{\Gamma(\alpha)}\int_{t_2}^{t}(t-s)^{\alpha-1}\rho(s)ds.$$

If $t \in (t_k, t_{k+1}]$, then again from Lemma 3.3 we get (4).

Conversely, assume that y satisfies the impulsive fractional integral equation (4). If $t \in [0, t_1]$ then $y(0) = y_0$ and using the fact that ${}^cD^\alpha$ is the left inverse of I^α we get

$$^cD^\alpha y(t) = \rho(t), \text{ for each } t \in [0, t_1].$$

If $t \in [t_k, t_{k+1}),\ k = 1, \ldots, m$ and using the fact that ${}^cD^\alpha C = 0$, where C is a constant, we get

$$^cD^\alpha y(t) = \rho(t), \text{ for each } t \in [t_k, t_{k+1}).$$

Also, we can easily show that

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)),\ k = 1, \ldots, m.$$

■

Theorem 3.5. *Assume the following hypotheses hold:*

(H1) $F : J \times \mathbb{R} \longrightarrow \mathcal{P}_{cp,c}(\mathbb{R})$ *is a Carathéodory multi-valued map;*

(H2) There exist $p \in C(J, \mathbb{R}^+)$ *and* $\psi : [0, \infty) \to (0, \infty)$ *continuous and nondecreasing such that*

$$\|F(t,u)\|_{\mathcal{P}} = \sup\{|v| : v \in F(t,u)\} \le p(t)\psi(|u|) \text{ for } t \in J \text{ and each } u \in \mathbb{R};$$

(H3) There exists $\psi^* : [0, \infty) \to (0, \infty)$ *continuous and nondecreasing such that*

$$|I_k(u)| \le \psi^*(|u|) \text{ for each } u \in \mathbb{R};$$

(H4) There exists an number $\overline{M} > 0$ *such that*

$$\frac{\overline{M}}{|y_0| + \psi(\overline{M})\frac{T^\alpha p^0(m+1)}{\Gamma(\alpha+1)} + m\psi^*(\overline{M})} > 1,$$

where $p^0 = \sup\{p(t) :\ t \in J\}$*;*

(H5) There exists $l \in L^1(J, \mathbb{R}^+)$ such that

$$H_d(F(t,u), F(t,\overline{u})) \leq l(t)|u - \overline{u}| \quad \textit{for every } u, \overline{u} \in \mathbb{R},$$

and

$$d(0, F(t,0)) \leq l(t), \ \text{a.e. } t \in J.$$

Then the IVP (1)-(3) has at least one solution on J.

P r o o f. Transform the problem (1)–(3) into a fixed point problem. Consider the multivalued operator N: $PC(J,\mathbb{R}) \to \mathcal{P}(PC(J,\mathbb{R}))$ defined by

$$N(y) = \Big\{ h \in PC(J,\mathbb{R}) :$$

$$h(t) = y_0 + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} v(s) ds$$

$$+\frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} v(s) ds + \sum_{0<t_k<t} I_k(y(t_k^-)), \quad v \in S_{F,y} \Big\}.$$

Clearly, from Lemma 3.4, the fixed points of N are solutions to (1)–(3). We shall show that N satisfies the assumptions of the nonlinear alternative of Leray-Schauder type [31]. The proof will be given in several steps.

Step 1: *$N(y)$ is convex for each $y \in PC(J,\mathbb{R})$.*

Indeed, if h_1, h_2 belong to $N(y)$, then there exist $v_1, v_2 \in S_{F,y}$ such that for each $t \in J$ we have

$$h_i(t) = y_0 + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} v_i(s) ds$$

$$+\frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} v_i(s) ds + \sum_{0<t_k<t} I_k(y(t_k^-)), \quad i = 1, 2.$$

Let $0 \leq d \leq 1$. Then, for each $t \in J$, we have

$$(dh_1 + (1-d)h_2)(t) = \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} [dv_1(s) + (1-d)v_2(s)] ds$$

$$-\frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} [dv_1(s) + (1-d)v_2(s)] ds + \sum_{0<t_k<t} I_k(y(t_k^-)).$$

Since $S_{F,y}$ is convex (because F has convex values), we have

$$dh_1 + (1-d)h_2 \in N(y).$$

Step 2: *N maps bounded sets into bounded sets in $PC(J,\mathbb{R})$.*

Let $B_{\eta^*}=\{y\in PC(J,\mathbb{R}):\|y\|_\infty\leq\eta^*\}$ be bounded set in $PC(J,\mathbb{R})$ and $y\in B_{\eta^*}$. Then for each $h\in N(y)$ and $t\in J$, we have by (H2)-(H3)

$$\begin{aligned}|h(t)| &\leq |y_0|+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|v(s)|ds\\ &+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|v(s)|ds+\sum_{0<t_k<t}|I_k(y(t_k^-))|\\ &\leq |y_0|+\frac{mT^\alpha p^0}{\Gamma(\alpha+1)}\psi(\|y\|_\infty)+\frac{T^\alpha p^0}{\Gamma(\alpha+1)}\psi(\|y\|_\infty)+m\psi^*(\|y\|_\infty).\end{aligned}$$

Thus

$$\|h\|_\infty\leq|y_0|+\frac{mT^\alpha p^0}{\Gamma(\alpha+1)}\psi(\|y\|_\infty)+\frac{T^\alpha p^0}{\Gamma(\alpha+1)}\psi(\|y\|_\infty)+m\psi^*(\|y\|_\infty):=\ell.$$

Step 3: *N maps bounded sets into equicontinuous sets of $PC(J,\mathbb{R})$.*

Let $\tau_1,\tau_2\in J,\ \tau_1<\tau_2,\ B_{\eta^*}$ be a bounded set of $PC(J,\mathbb{R})$ as in Step 2, let $y\in B_{\eta^*}$ and $h\in N(y)$, then

$$\begin{aligned}|h(\tau_2)-h(\tau_1)| &= \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<\tau_2-\tau_1}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|v(s)|ds\\ &+\frac{1}{\Gamma(\alpha)}\int_0^{\tau_1}|(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}||v(s)|ds+\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}|(\tau_2-s)^{\alpha-1}||v(s)|ds\\ &+\sum_{0<t_k<\tau_2-\tau_1}|I_k(y(t_k^-))|\leq\frac{\psi(\|y\|_\infty)T}{\Gamma(\alpha+1)}\sum_{0<t_k<\tau_2-\tau_1}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}ds\\ &+\frac{\psi(\|y\|_\infty)p^0}{\Gamma(\alpha+1)}[2(\tau_2-\tau_1)^\alpha+\tau_2^\alpha-\tau_1^\alpha]+\sum_{0<t_k<\tau_2-\tau_1}|I_k(y(t_k^-))|.\end{aligned}$$

As $\tau_1\longrightarrow\tau_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that $N:PC(J,\mathbb{R})\longrightarrow\mathcal{P}(PC(J,\mathbb{R}))$ is completely continuous.

Step 4: *N has a closed graph.*

Let $y_n\to y_*,\ h_n\in N(y_n)$ and $h_n\to h_*$. We need to show that $h_*\in N(y_*)$. Indeed, $h_n\in N(y_n)$ means that there exists $v_n\in S_{F,y_n}$ such that, for each $t\in J$,

$$h_n(t)=y_0+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}v_n(s)ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}v_n(s)ds+\sum_{0<t_k<t}I_k(y_n(t_k^-)).$$

We must show that there exists $v_* \in S_{F,y_*}$ such that, for each $t \in J$,

$$h_*(t)=y_0+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}v_*(s)ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}v_*(s)ds+\sum_{0<t_k<t}I_k(y_*(t_k^-)).$$

Since $F(t,\cdot)$ is upper semicontinuous, then for every $\varepsilon>0$, there exist $n_0(\epsilon)\geq 0$ such that for every $n\geq n_0$, we have

$$v_n(t)\in F(t,y_n(t))\subset F(t,y_*(t))+\varepsilon B(0,1),\ \text{a.e. } t\in J.$$

Since $F(\cdot,\cdot)$ has compact values, then there exists a subsequence $v_{n_m}(\cdot)$ such that

$$v_{n_m}(\cdot)\to v_*(\cdot)\ \text{as}\ m\to\infty$$

and

$$v_*(t)\in F(t,y_*(t)),\ \text{a.e. } t\in J.$$

For every $w\in F(t,y_*(t))$, we have

$$|v_{n_m}(t)-v_*(t)|\leq|v_{n_m}(t)-w|+|w-v_*(t)|.$$

Then

$$|v_{n_m}(t)-v_*(t)|\leq d(v_{n_m}(t),F(t,y_*(t)).$$

By an analogous relation, obtained by interchanging the roles of v_{n_m} and v_*, it follows that

$$|v_{n_m}(t)-v_*(t)|\leq H_d(F(t,y_n(t)),F(t,y_*(t)))\leq l(t)\|y_n-y_*\|_\infty.$$

Then

$$|h_n(t)-h_*(t)|\leq\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|v_{n_m}(s)-v_*(s)|ds$$
$$-\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|v_{n_m}(s)-v_*(s)|$$
$$+\sum_{0<t_k<t}|I_k(y_{n_m}(t_k^-))-I_k(y_*(t_k^-))|\leq\frac{mT^\alpha}{\Gamma(\alpha+1)}\int_0^T l(s)ds\|y_{n_m}-y_*\|_\infty$$

$$+\frac{T^{\alpha}}{\Gamma(\alpha+1)}\int_0^T l(s)ds\|y_{n_m}-y_*\|_\infty+\sum_{0<t_k<t}|I_k(y_{n_m}(t_k^-))-I_k(y_*(t_k^-))|.$$

Hence,

$$\|h_{n_m}-h_*\|_\infty\le\frac{mT^{\alpha}}{\Gamma(\alpha+1)}\int_0^T l(s)ds\|y_{n_m}-y_*\|_\infty$$

$$+\frac{T^{\alpha}}{\Gamma(\alpha+1)}\int_0^T l(s)ds\|y_{n_m}-y_*\|_\infty+\sum_{k=1}^{m}|I_k(y_{n_m}(t_k^-))-I_k(y_*(t_k^-))|\to 0 \text{ as } m\to\infty.$$

Step 5: *A priori bounds on solutions.*

Let $y\in PC(J,\mathbb{R})$ be such that $y\in\lambda N(y)$ for $\lambda\in[0,1]$. Then, there exists $v\in S_{F,y}$ such that, for each $t\in J$,

$$|y(t)|\le|y_0|+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}p(s)\psi(|y(s)|)ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}p(s)\psi(|y(s)|)ds+\sum_{0<t_k<t}\psi^*(|y(s)|)$$

$$\le|y_0|+\psi(\|y\|_\infty)\frac{mT^{\alpha}p^0}{\Gamma(\alpha+1)}+\psi(\|y\|_\infty)\frac{T^{\alpha}p^0}{\Gamma(\alpha+1)}+m\psi^*(\|y\|_\infty).$$

Thus

$$\frac{\|y\|_\infty}{|y_0|+\psi(\|y\|_\infty)\frac{T^{\alpha}p^0(m+1)}{\Gamma(\alpha+1)}+m\psi^*(\|y\|_\infty)}\le 1.$$

Then by condition (H4), there exists M such that $\|y\|_\infty\ne M$.

Let

$$U=\{y\in PC(J,\mathbb{R}):\|y\|_\infty<M\}.$$

The operator $N:\overline{U}\to\mathcal{P}(PC(J,\mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of U, there is no $y\in\partial U$ such that $y\in\lambda N(y)$ for some $\lambda\in(0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [31], we deduce that N has a fixed point y in $\overline{U}$ which is a solution of the problem (1)–(3). This completes the proof. ■

4. The nonconvex case

This section is devoted to the existence of solutions for the problem (1)-(3) with a nonconvex valued right hand side. Our first result is based on the fixed point theorem for contraction multivalued map given by Covitz and Nadler [16], and the second one on a selection theorem due to Bressan-Colombo [14] for lower semicontionus operators with decomposable values combined with the nonlinear Leray-Schauder alternative.

THEOREM 4.1. *Assume (H5) and the following hypotheses hold:*

(H6) $F : J \times \mathbb{R} \longrightarrow P_{cp}(\mathbb{R})$ *has the property that* $F(\cdot, u) : J \to P_{cp}(\mathbb{R})$ *is measurable, convex valued and integrably bounded for each* $u \in \mathbb{R}$*;*

(H7) There exists a constant $l^* > 0$ *such that*

$$|I_k(u) - I_k(\overline{u})| \leq l^*|u - \overline{u}|, \text{ for each } u,\ \overline{u} \in \mathbb{R}, \text{ and } k = 1, \dots, m.$$

Let $l = \sup\{l(t) :\ t \in J\}$*. If*

$$\left[\frac{T^\alpha l(m+1)}{\Gamma(\alpha+1)} + ml^*\right] < 1, \tag{8}$$

then the IVP (1)-(3) has a solution on J*.*

P r o o f. For each $y \in PC(J, \mathbb{R})$, the set $S_{F,y}$ is nonempty since by (H6), F has a measurable selection (see [15], Theorem III.6). We shall show that N satisfies the assumptions of Lemma 2.3. The proof will be given in two steps.

Step 1: $N(y) \in P_{cl}(PC(J, \mathbb{R}))$ *for each* $y \in PC(J, \mathbb{R})$.

Indeed,let $(h_n)_{n\geq 0} \in N(y)$ such that $h_n \longrightarrow \tilde{h}$ in $PC(J, \mathbb{R})$. Then, $\tilde{h} \in PC(J, \mathbb{R})$ and there exists $v_n \in S_{F,y}$ such that, for each $t \in J$,

$$h_n(t) = y_0 + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} v_n(s)ds$$
$$+\frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} v_n(s)ds + \sum_{0<t_k<t} I_k(y(t_k^-)).$$

Using the fact that F has compact values and from (H5), we may pass to a subsequence if necessary to get that v_n converges weakly to v in $L^1_w(J, \mathbb{R})$ (the space endowed with the weak topology). An application of Mazur's theorem ([53]) implies that v_n converges strongly to v and hence $v \in S_{F,y}$. Then, for each $t \in J$,

$$h_n(t) \longrightarrow \tilde{h}(t) = y_0 + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} v(s)ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}v(s)ds+\sum_{0<t_k<t}I_k(y(t_k^-)),$$

So, $\tilde{h}\in N(y)$.

Step 2: *There exists* $\gamma<1$ *such that*

$$H_d(N(y),N(\overline{y}))\leq\gamma\|y-\overline{y}\|_\infty \text{ for each } y,\overline{y}\in PC(J,\mathbb{R}).$$

Let $y,\overline{y}\in PC(J,\mathbb{R})$ and $h_1\in N(y)$. Then, there exists $v_1(t)\in F(t,y(t))$ such that for each $t\in J$,

$$h_1(t)=y_0+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}v_1(s)ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}v_1(s)ds+\sum_{0<t_k<t}I_k(y(t_k^-)).$$

From (H5) it follows that

$$H_d(F(t,y(t)),F(t,\overline{y}(t)))\leq l(t)|y(t)-\overline{y}(t)|.$$

Hence, there exists $w\in F(t,\overline{y}(t))$ such that

$$|v_1(t)-w|\leq l(t)|y(t)-\overline{y}(t)|,\ t\in J.$$

Consider $U:J\to\mathcal{P}(\mathbb{R})$ given by

$$U(t)=\{w\in\mathbb{R}:|v_1(t)-w|\leq l(t)|y(t)-\overline{y}(t)|\}.$$

Since the multivalued operator $V(t)=U(t)\cap F(t,\overline{y}(t))$ is measurable (see Proposition III.4 in [15]), there exists a function $v_2(t)$ which is a measurable selection for V. So, $v_2(t)\in F(t,\overline{y}(t))$, and for each $t\in J$,

$$|v_1(t)-v_2(t)|\leq l(t)|y(t)-\overline{y}(t)|.$$

Let us define for each $t\in J$

$$h_2(t)=y_0+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}v_2(s)ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}v_2(s)ds+\sum_{0<t_k<t}I_k(\overline{y}(t_k^-)).$$

Then for $t\in J$,

$$|h_1(t)-h_2(t)|\leq\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|v_1(s))-v_2(s)|ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|v_1(s)-v_2(s)|ds$$

$$+\sum_{0<t_k<t}|I_k(y(t_k^-))-I_k(\overline{y}(t_k^-))|\le\frac{l}{\Gamma(\alpha)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|y(s)-\overline{y}(s)|ds$$

$$+\frac{l}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|y(s)-\overline{y}(s)|ds+\sum_{k=1}^{m}l^*|y(t_k^-)-\overline{y}(t_k^-)|$$

$$\le\frac{mlT^\alpha}{\Gamma(\alpha+1)}\|y-\overline{y}\|_\infty+\frac{T^\alpha l}{\Gamma(\alpha+1)}\|y-\overline{y}\|_\infty+ml^*\|y-\overline{y}\|_\infty.$$

Thus,

$$\|h_1-h_2\|_\infty\le\left[\frac{T^\alpha l(m+1)}{\Gamma(\alpha+1)}+ml^*\right]\|y-\overline{y}\|_\infty.$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, it follows that

$$H_d(N(y),N(\overline{y}))\le\left[\frac{T^\alpha l(m+1)}{\Gamma(\alpha+1)}+ml^*\right]\|y-\overline{y}\|_\infty.$$

So by (8), N is a contraction and thus, by Lemma 2.3, N has a fixed point y which is solution to (1)–(3). The proof is complete. ■

Now we present a result for the problem (1)-(3) in the spirit of the nonlinear alternative of Leray Schauder type [31] for single-valued maps, combined with a selection theorem due to Bressan-Colombo for lower semicontinuous multivalued maps with decomposable values. Details on multivalued maps with decomposable values and their properties can be found in the recent book by Fryszkowski [26].

Definition 4.2. Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,T],\mathbb{R}))$ be a multivalued operator. We say N has property (BC), if

1) N is lower semi-continuous (l.s.c.);
2) N has nonempty closed and decomposable values.

Let $F:[0,T]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Assign to F the multivalued operator

$$\mathcal{F}:PC([0,T],\mathbb{R})\to\mathcal{P}(L^1([0,T],\mathbb{R}))$$

by letting

$$\mathcal{F}(y)=\{w\in L^1([0,T],\mathbb{R}):w(t)\in F(t,y(t))\text{ for a.e. }t\in[0,T]\}.$$

The operator $\mathcal{F}$ is called the Niemytzki operator associated to F.

DEFINITION 4.3. Let $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type), if its associated Niemytzki operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Next we state a selection theorem due to Bressan and Colombo.

THEOREM 4.4. ([14]) *Let Y be separable metric space and let $N : Y \to \mathcal{P}(L^1([0,T],\mathbb{R}))$ be a multivalued operator which has property (BC). Then N has a continuous selection, i.e. there exists a continuous function (single-valued) $\tilde{g} : Y \to L^1([0,1],\mathbb{R})$ such that $\tilde{g}(y) \in N(y)$ for every $y \in Y$.*

Let us introduce the following hypotheses:

(H8) $F : [0,T] \times \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$ is a nonempty compact valued multivalued map such that:
a) $(t,u) \mapsto F(t,u)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable;
b) $y \mapsto F(t,y)$ is lower semi-continuous for a.e. $t \in [0,T]$;

(H9) for each $q > 0$, there exists a function $h_q \in L^1([0,T],\mathbb{R}^+)$ such that $\|F(t,u)\|_{\mathcal{P}} \le h_q(t)$ for a.e. $t \in [0,T]$ and for $u \in \mathbb{R}$ with $|u| \le q$.

The following lemma is crucial in the proof of our main theorem:

LEMMA 4.5. ([25]) *Let $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty, compact values. Assume that $(H8)$ and $(H9)$ hold. Then F is of l.s.c. type.*

THEOREM 4.6. *Suppose that hypotheses (H2)-(H4), (H8), (H9) are satisfied. Then the problem (1)–(3) has at least one solution.*

P r o o f. (H8) and (H9) imply by Lemma 4.5 that F is of lower semi-continuous type. Then from Theorem 4.4 there exists a continuous function $f : PC([0,T],\mathbb{R}) \to L^1([0,T],\mathbb{R})$ such that $f(y) \in \mathcal{F}(y)$ for all $y \in PC([0,T],\mathbb{R})$. Consider the following problem

$$ {}^cD^\alpha y(t) \in f(y)(t) \text{ a.e., } t \in J = [0,T],\ t \neq t_k,\ k = 1,\dots,m,\ 0 < \alpha \le 1, \tag{9}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)),\ k = 1,\dots,m, \tag{10}$$

$$y(0) = y_0. \tag{11}$$

REMARK 4.7. If $y \in PC(J,\mathbb{R}) \bigcap \bigcup_{k=0}^{k=m} AC((t_k,t_{k+1}),\mathbb{R})$ is a solution of the problem (9)–(11), then y is a solution to the problem (1)-(3).

Problem (9)-(11) is the reformulated as a fixed point problem for the operator $N_1 : PC([0,T,\mathbb{R}) \to PC([0,T],\mathbb{R})$ defined by:

$$N_1(y)(t) = y_0 + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} f(y(s))ds$$
$$+\frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f(y)(s)ds + \sum_{0<t_k<t} I_k(y(t_k^-)).$$

Using (H2)-(H4) we can easily show (using similar argument as in Theorem 3.5 that the operator N_1 satisfies all conditions in the Leray-Schauder alternative.

5. Topological structure of the solutions set

In this section, we present a result on the topological structure of the set of solutions of (1)–(3).

THEOREM 5.1. *Assume that (H1) and the following hypotheses hold:*

(H10) There exists $p_1 \in C(J,\mathbb{R})$ such that

$$\|F(t,u)\|_{\mathcal{P}} \leq p_1(t) \text{ for } t \in J \text{ and each } u \in \mathbb{R};$$

(H11) There exists $d \in \mathbb{R}_+^*$ such that

$$|I_k(u)| \leq d \text{ for each } u \in \mathbb{R}.$$

Then the solution set of (1)-(3) in not empty and compact in $PC(J,\mathbb{R})$.

P r o o f. Let

$$S = \{y \in PC(J,\mathbb{R}) : y \text{ is solution of } (1)-(3)\}.$$

From Theorem 3.5, $S \neq \emptyset$. Now, we prove that S is compact. Let $(y_n)_{n\in\mathbb{N}} \in S$, then there exists $v_n \in S_{F,y_n}$ and $t \in J$ such that

$$y_n(t) = y_0 + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} v_n(s)ds$$
$$+\frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} v_n(s)ds + \sum_{0<t_k<t} I_k(y_n(t_k^-)).$$

From (H1), (H10) and (H11) we can prove that there exists $M_1 > 0$ such that

$$\|y_n\|_\infty \leq M_1, \quad \text{for every } n \geq 1.$$

As in Step 3 in Theorem 3.5, we can easily show using (H1), (H9) and (H10) that the set $\{y_n : n \geq 1\}$ is equicontinuous in $PC(J,\mathbb{R})$, hence by Arzéla-Ascoli Theorem we can conclude that, there exists a subsequence (denoted again by $\{y_n\}$) of $\{y_n\}$ such that y_n converges to y in $PC(J,\mathbb{R})$. We shall show that there exist $v(.) \in F(., y(.))$ and $t \in J$ such that

$$\begin{aligned} y(t) &= y_0 + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} v(s)ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} v(s)ds + \sum_{0<t_k<t} I_k(y(t_k^-)). \end{aligned}$$

Since $F(t,.)$ is upper semicontinuous, then for every $\varepsilon > 0$, there exists $n_0(\epsilon) \geq 0$ such that for every $n \geq n_0$, we have

$$v_n(t) \in F(t, y_n(t)) \subset F(t, y(t)) + \varepsilon B(0,1), \text{ a.e. } t \in J.$$

Since $F(.,.)$ has compact values, there exists subsequence $v_{n_m}(.)$ such that

$$v_{n_m}(.) \to v(.) \text{ as } m \to \infty$$

and

$$v(t) \in F(t, y(t)), \text{ a.e. } t \in J.$$

It is clear that

$$|v_{n_m}(t)| \leq p_1(t), \text{ a.e. } t \in J.$$

By Lebesgue's dominated convergence theorem, we conclude that $v \in L^1(J,\mathbb{R})$ which implies that $v \in S_{F,y}$. Thus, for $t \in J$, we have

$$\begin{aligned} y(t) &= y_0 + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} v(s)ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} v(s)ds + \sum_{0<t_k<t} I_k(y(t_k^-)). \end{aligned}$$

Then $S \in \mathcal{P}_{cp}(PC(J,\mathbb{R}))$. ∎

6. Nonlocal initial value problem

In this section we indicate without proofs a generalization of the problem (1)-(3) by considering a nonlocal initial condition instead of the classical initial condition. More precisely we consider the following nonlocal problem

$$^cD^\alpha y(t) \in F(t,y), \text{ for a.e., } t \in J = [0,T], \ t \neq t_k, \ k = 1, \dots, m, \ 0 < \alpha \leq 1, \tag{12}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \ k = 1, \dots, m, \tag{13}$$

$$y(0) + g(y) = y_0, \tag{14}$$

where ${}^cD^\alpha, F, I_k$ are as in in the previous sections and $g : PC(J, \mathbb{R}) \to \mathbb{R}$ a given function. Nonlocal conditions were initiated by Byszewski [13] when he proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems. As remarked by Byszewski [11, 12], the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena. For example, $g(y)$ may be given by

$$g(y) = \sum_{i=1}^{p} c_i y(\tau_i),$$

where $c_i,\ i = 1, \dots, p$, are given constants and $0 < \tau_1 < ... < \tau_p \leq T$. Let us introduce the following conditions on the function g.

(C1) There exists a constant $k^* > 0$ such that

$$|g(u) - g(\overline{u})| \leq k^*|u - \overline{u}|, \text{ for each } t \in J, \text{ and all } u,\ \overline{u} \in PC([0, T], \mathbb{R}).$$

(C2) There exists a constant $M_1^* > 0$ such that

$$|g(y)| \leq M_1^* \text{ for all } y \in PC([0, T], \mathbb{R}).$$

(C3) There exists $\overline{M}^* > 0$ such that

$$\frac{\overline{M}^*}{M_1^* + |y_0| + \psi(\overline{M}^*)\frac{T^\alpha p^0(m+1)}{\Gamma(\alpha+1)} + m\psi^*(\overline{M}^*)} > 1.$$

Theorem 6.1. *Assume (H1)-(H3), (H5), (C2), (C3) hold. Then the problem (12)-(14) has at least one solution.*

Theorem 6.2. *Assume (H5)-(H7), (C1) hold. If*

$$\left[k^* + \frac{T^\alpha l(m+1)}{\Gamma(\alpha+1)} + ml^*\right] < 1,$$

then the problem (12)-(14) has at least one solution.

Theorem 6.3. *Suppose that hypotheses (H2), (H3), (C3), (H8), (H9) are satisfied. Then the problem (12)–(14) has at least one solution.*

7. An example

As an application of the main results in the convex case, we consider the fractional differential inclusion

$$ {}^cD^\alpha y(t) \in F(t,y), \text{ a.e. } t \in J=[0,1], \quad t \neq \frac{1}{2}, \quad 0<\alpha\leq 1, \tag{15}$$

$$\Delta y|_{t=\frac{1}{2}} = \frac{1}{3+|y(\frac{1}{2}^-)|}, \tag{16}$$

$$y(0)=0. \tag{17}$$

Set

$$F(t,y)=\{v\in\mathbb{R} : f_1(t,y)\leq v\leq f_2(t,y)\},$$

where $f_1, f_2 : J\times\mathbb{R}\to\mathbb{R}$. We assume that for each $t\in J$, $f_1(t,\cdot)$ is lower semi-continuous (i.e, the set $\{y\in\mathbb{R} : f_1(t,y)>\mu\}$ is open for each $\mu\in\mathbb{R}$), and assume that for each $t\in J$, $f_2(t,\cdot)$ is upper semi-continuous (i.e the set $\{y\in\mathbb{R} : f_2(t,y)<\mu\}$ is open for each $\mu\in\mathbb{R}$). Assume that there are $p\in C(J,\mathbb{R}^+)$ and $\psi : [0,\infty)\to(0,\infty)$ continuous and nondecreasing such that

$$\max(|f_1(t,y)|,\ |f_2(t,y)|)\leq p(t)\psi(|y|), \quad t\in J, \text{ and all } y\in\mathbb{R}.$$

Assume there exists a constant $M>0$ such that

$$\frac{M}{\frac{2p^0\psi(M)}{\Gamma(\alpha+1)}+\frac{1}{3}}>1.$$

It is clear that F is compact and convex valued, and it is upper semi-continuous (see [17]). Since all the conditions of Theorem 3.5 are satisfied, the problem (15)-(17) has at least one solution y on J.

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