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COMPARATIVE ANALYSIS OF VISCOELASTIC MODELS INVOLVING FRACTIONAL DERIVATIVES OF DIFFERENT ORDERS *

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Dedicated to the bright memory of
Academician Yuriy N. Rabotnov

Abstract

In this paper, a comparative analysis of the models involving fractional derivatives of different orders is given. Such models of viscoelastic materials are widely used in various problems of mechanics and rheology. Rabotnov's hereditarily elastic rheological model is considered in detail. It is shown that this model is equivalent to the rheological model involving fractional derivatives in the stress and strain with the orders proportional to a certain positive value less than unit. In the scientific literature such a model is referred to as Koeller's model. Inversion of Rabotnov's model developed by himself based on algebra of operators results in similar rheological dependences. Inversion of Koeller's model carried out using Miller's theorem coincides inherently with Rabotnov's inversion procedure.

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1. Introduction

Many researchers in the sixties years, when constructing the theory of viscoelasticity, started from the following differential law:

$$\sum_{i=0}^{n} a_i \frac{d^i\varepsilon}{dt^i} = \sum_{j=0}^{m} b_j \frac{d^j\sigma}{dt^j}. \tag{1}$$

Here ε and σ are the strain and stress, respectively, a_i and b_j $(i, j = 1, 2, ..., n)$ are some known values, and this results in the hereditary law with the kernel in the form of a sum of exponents

$$\varepsilon = J_0 \left[\sigma + \sum_{i=1}^{n} g_i \ni_1^* (-\tau_i^{-1})\sigma \right] \tag{2}$$

with the operator

$$\ni_1^* (-\tau_i^{-1})\sigma = \int_0^t \exp(-t'/\tau_i)\sigma(t - t')dt'. \tag{3}$$

This point of view is developed in detail in [2], where the general analysis of the relationships of the type (1) is given, and the method of obtaining integral dependencies (2) and (3) is considered.

In their investigations modern researchers use the following rheological equation proposed by Bagley [1] in his PhD thesis:

$$\varepsilon + \sum_{i=1}^{n} a_i D^{\alpha_i}\varepsilon = J_0 \left(\sigma + \sum_{j=1}^{m} b_j D^{\beta_j}\sigma \right), \tag{4}$$

where α_i $(i = 1, 2, ..., n)$ and β_j $(j = 1, 2, ..., m)$ $(0 < \alpha_i, \beta_j < 1)$ are the orders of Riemann-Liouville fractional derivatives $D^{\alpha_i}\varepsilon$ and $D^{\beta_j}\sigma$, defined as well known in the fractional calculus (see [2], [17]),

$$D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha}} d\tau \qquad (0 < \alpha < 1).$$

One more example of the rheological model involving sums of fractional derivatives is the model suggested by Padovan and Sawicki [8]

$$\varepsilon + \sum_{l=1}^{N_\varepsilon} \mu_{\varepsilon_l} D^{b_l}\varepsilon = \sigma + \sum_{l=1}^{N_\sigma} \mu_{\sigma_l} D^{a_l}\sigma, \tag{5}$$

where $b_l = \frac{2l}{N_\varepsilon}$, $l \in [1, N_\varepsilon]$ and $a_l = \frac{2l}{N_\sigma}$, $l \in [1, N_\sigma]$. Examples for using the model (5) in dynamic problems of viscoelasticity can be found in [9] and [18].

In this paper, the authors do not pursue the goal to present the detailed historical aspects of the application of fractional calculus for solving the problems of mechanics, since this has been attained in their comprehensive state-of-the-art articles [15] and [16]. Here, we want to limit ourselves by highlighting the ideological inheritance of Academician Yu.N. Rabotnov devoted to *the fractional calculus models involving more than two different fractional parameters.*

Indeed, it is not well known that the rheological equation of the type

$$\sum_{i=0}^{n} a_i D^{i\alpha}\varepsilon = \sum_{j=0}^{n} b_j D^{j\alpha}\sigma, \tag{6}$$

combining in some sense Eqs. (1) and (4), was proposed by Rabotnov [12] in 1966 in the equivalent form

$$\varepsilon = J_\infty \left[\sigma + \sum_{i=1}^{n} g_i \ni_\alpha^* (-\tau_i^{-\alpha})\sigma\right] \tag{7}$$

with the operator

$$\ni_\alpha^* (-\tau_i^{-\alpha})\sigma = \int_0^t \ni_\alpha (-t'/\tau_i)\sigma(t-t')dt'. \tag{8}$$

The function $\ni_\alpha (-1, t'/\tau_i)$, which appears in Eq. (8),

$$\ni_\alpha (-1, t'/\tau_i) = \frac{t^{\alpha-1}}{\tau_i^\alpha} \sum_{n=1}^{\infty} (-1)^n \frac{(t/\tau_i)^{\alpha n}}{\Gamma[\alpha(n+1)]}, \tag{9}$$

where $\Gamma[\alpha(n+1)]$ is the gamma-function, was called by Rabotnov in his pioneering paper [11] published in 1948 as *the fractional exponential function*, since for $\alpha = 1$ it is reduced to the ordinary exponential function $\exp(-t/\tau_i)$. We shall follow the terminology by Rabotnov.

Further, in 1966 Rabotnov was the first to show in his book [12] the connection between the fractional exponential function (9) and the Mittag-Leffler function (see equation (30.3) in [12]). This relationship is presented below in Eq. (21). Moreover, in 1969 Rabotnov with his pupils [14] tabulated the fractional exponential function ant its integral, which is expressed in terms of the Mittag-Leffler function. Subsequently, some part of the tables [14] were included in [13] and published in its English edition.

It is interesting, as a fact, that the model (5) combines the features of the models (4) and (6). But despite the model (4), its fractional parameters appearing in the left and right sides are proportional to N_ε^{-1} and N_σ^{-1}, respectively; and unlike the model (6), the magnitudes of all its fractional parameters fall within the range from 0 to 2.

2. Equivalence of the models (6) and (7)

In order to establish the equivalence of the model (6) with $m = n$ and the model (7), let us apply Laplace transform to Eq. (6). Expressing transform $\bar{\varepsilon}$ in terms of transform $\bar{\sigma}$ yields

$$\overline{\varepsilon} = \frac{\sum\limits_{j=0}^{n} b_j p^{j\alpha}}{\sum\limits_{i=0}^{n} a_i p^{i\alpha}}\, \overline{\sigma}, \tag{10}$$

with p denoting the Laplace transform variable.

Suppose that equations

$$\sum_{j=0}^{n} b_j Z^j = 0, \tag{11a}$$

$$\sum_{i=0}^{n} a_i Z^i = 0 \tag{11b}$$

possess real and different roots, which we denote, respectively, as $\gamma_j = -t_j^{-\alpha}$ $(j = 1, ..., n)$ and $\beta_i = -\widetilde{t}_i^{-\alpha}$ $(i = 1, ..., n)$. Then expanding the fraction involved in (10) into simple fractions, we obtain

$$\frac{\sum\limits_{j=0}^{n} b_j p^{j\alpha}}{\sum\limits_{i=0}^{n} a_i p^{i\alpha}} = \frac{b_n}{a_n} \frac{\prod\limits_{j=1}^{n} (p^\alpha + t_j^{-\alpha})}{\prod\limits_{i=1}^{n} (p^\alpha + \widetilde{t}_i^{-\alpha})} = J_\infty \left(1 + \sum_{i=1}^{n} \frac{g_i}{1 + (p\tau_i)^\alpha} \right), \tag{12}$$

where J_∞ is compliance,

$$g_i = t_i^\alpha \, \frac{f_{n-1}(-t_i^{-\alpha})}{\prod\limits_{\substack{k=1 \\ (k\neq i)}}^{n} (\widetilde{t}_k^{-\alpha} - \widetilde{t}_i^{-\alpha})}, \quad f_{n-1}(p) = \sum_{i=0}^{n-1} \left(\frac{b_i}{b_n} - \frac{a_i}{a_n} \right) p^{i\alpha}, \quad J_\infty = \frac{b_n}{a_n}.$$

In view of (12), relationship (10) can be then represented in the form

$$\overline{\varepsilon} = J_\infty \left(1 + \sum_{i=1}^{n} \frac{g_i}{1 + (p\tau_i)^\alpha} \right) \overline{\sigma}. \tag{13}$$

Considering that Laplace transform of the fractional exponent $\ni_\alpha (-t/\tau_i)$

$$\bar{\ni}_\alpha(p) = [1 + (p\tau_i)^\alpha]^{-1} \tag{14}$$

and going in Eq. (13) from image to pre-image, we are led to relationship (7).

The rheological equation (7) was suggested by Rabotnov [12, 13] for approximation of hereditary kernels determined experimentally.

In 1986 Koeller [6] presented Eq. (6) in the case $n = 2$ in the form

$$(a_0 + a_1 D^\alpha + a_2 D^{2\alpha})\varepsilon = (b_0 + b_1 D^\alpha + b_2 D^{2\alpha})\sigma, \tag{15}$$

and in the equivalent form

$$(D^\alpha + \tau_1^{-\alpha})(D^\alpha + \tau_2^{-\alpha})\sigma = (D^\alpha + t_1^{-\alpha})(D^\alpha + t_2^{-\alpha})\varepsilon, \tag{16}$$

without any reference to Rabotnov's monograph [13], although in Introduction to his previous paper [5] published in 1984, the author wrote about his familiarity with Rabotnov's theory via English translation of Rabotnov's book [13] carried out in 1980.

Further researchers, see for example Welch et al [21], using equation (15) and knowing nothing about Rabotnov's investigations, without any hesitation gave Koeller credit for its derivation, although for fairness' sake, Eqs. (6) and (15) should be called as Rabotnov-Koeller equations.

Expressing from Eq. (7) σ in terms of ε yields

$$\sigma = E_\infty \left[\varepsilon - \sum_{j=1}^{n} e_j \ \ni_\alpha^* (-t_j^{-\alpha})\varepsilon \right], \tag{17}$$

where $E_\infty = J_\infty^{-1}$, and

$$e_i = t_i^\alpha \ \frac{f_{(n-1)}(-t_j^{-\alpha})}{\prod\limits_{\substack{k=1 \\ (k\neq n)}}^{n} (t_k^{-\alpha} - t_i^{-\alpha})}.$$

3. Connection between the rheological parameters of equations (7) and (17)

If, following Rabotnov [13], we substitute the value σ from Eq. (17) into Eq. (7) and use the theorem of operators' multiplication,

$$\ni_\alpha^* (-\tau_i^{-\alpha}) \ni_\alpha^* (-\tau_j^{-\alpha}) = \frac{1}{\tau_j^{-\alpha} - \tau_i^{-\alpha}} [\ni_\alpha^* (-\tau_i^{-\alpha}) - \ni_\alpha^* (-\tau_j^{-\alpha})], \tag{18}$$

then as a result, we obtain

$$\sum_{i=1}^{n} g_i \ \ni_\alpha^* (-\tau_i^{-\alpha}) - \sum_{j=1}^{n} e_j \ \ni_\alpha^* (-t_j^{-\alpha})$$
$$- \sum_{i=1}^{n}\sum_{j=1}^{n} g_i e_j \ \frac{\ni_\alpha^* (-\tau_i^{-\alpha}) - \ni_\alpha^* (-t_j^{-\alpha})}{t_j^{-\alpha} - \tau_i^{-\alpha}} = 0,$$

whence the two series of equalities follow:

$$1 + \sum_{i=1}^{n} \frac{g_i}{\tau_i^{-\alpha} - t_j^{-\alpha}} = 0, \qquad 1 + \sum_{j=1}^{n} \frac{e_j}{\tau_i^{-\alpha} - t_j^{-\alpha}} = 0. \qquad (19a, b)$$

The first series of equalities (19a) shows that the values $t_j^{-\alpha}$ are the roots of the equation

$$1 + \sum_{i=1}^{n} \frac{g_i}{\tau_i^{-\alpha} - x} = 0.$$

The second series of equalities (19b) represents a set of n linear equations for n unknown coefficients e_j.

Suppose that $g_i > 0$, $\tau_i^{-\alpha} > 0$ and $\tau_k^{-\alpha} > \tau_{k-1}^{-\alpha} > 0$, and consider the following two functions

$$F_1(x) = 1 + \sum_{i=1}^{n} \frac{g_i}{\tau_i^{-\alpha} - x}, \qquad F_2(y) = 1 + \sum_{j=1}^{n} \frac{e_j}{y - t_j^{-\alpha}}\,. \qquad (20a, b)$$

A reference to formulas (20) shows that the function $F_1(x)$ possesses zeros $F_1(\gamma_j) = 0$ located between its poles $F_1(\beta_i) = \pm\infty$, that is

$$\beta_k < \gamma_k < \beta_{k+1}, \qquad \beta_n < \gamma_n,$$

therefore $F_1(0) = 1$, and $F_1(\infty) = F_1(-\infty) = 1$. As for the function $F_2(y)$, $0 < F_2(0) < 1$, $F_2(\pm\infty) = 1$, and then the values $e_j > 0$ $(j = 1, ..., n)$.

Let us put $\sigma = \sigma_0 =$const in (7), and take into account Eq. (30.3) from [12], i.e.,

$$\ni_\alpha^* (-t/\tau_i^{-1}) \cdot 1 = \{1 - E_\alpha [-(t/\tau_j)^\alpha]\}\, \tau_i^\alpha, \qquad (21)$$

where

$$E_\alpha [-(t/\tau_j)^\alpha] = \sum_{n=0}^{\infty} \frac{(-1)^n (t/\tau_i)^{\alpha n}}{\Gamma(1 + \alpha n)}$$

is the Mittag-Leffler function. Then going to limit as $t \to \infty$ and considering that $E_\alpha(-\infty) = 0$, we find

$$\varepsilon(\infty) = J_\infty \left[1 + \sum_{i=1}^{n} g_i \tau_i^\alpha\right] \sigma_0 = F_1(0) J_\infty \sigma_0. \qquad (22a)$$

Similarly, from (17) it follows that if $\varepsilon = \varepsilon_0$ =const, then

$$\sigma(\infty) = E_\infty \left[1 - \sum_{j=1}^{n} e_j t_j^\alpha \right] \varepsilon_0 = F_2(0) E_\infty \varepsilon_0. \qquad (22b)$$

Since $F_1(0) > 1$, and $\varepsilon(0) = J_\infty \sigma_0$, then Eq. (22a) describes finite creep; as $0 < F_2(0) < 1$, and $\sigma_0 = E_\infty \varepsilon(0)$, then Eq. (22b) describes incomplete relaxation.

If $\beta_1 = 0$, this means that one term in the sum (7) represents the operator of fractional integration I^α; the product of this operator with a constant increases without bounds as time goes on. Formally putting $\tau_1^{-\alpha} = 0$ in (22a), we obtain $F_1(0) = \infty$, and $F_2(0) = 0$. In other words, if the creep is unbounded, then the relaxation is full. The creep and relaxation functions for the models (7) and (17) can be written, respectively, as

$$J(t) = J_\infty \left\{ 1 + \sum_{i=1}^{n} g_i \left[1 - E_\alpha \left(-(t/\tau_i)^\alpha \right) \right] \right\}, \qquad (23a)$$

$$G(t) = E_\infty \left\{ 1 - \sum_{j=1}^{n} e_i \left[1 - E_\alpha \left(-(t/t_j)^\alpha \right) \right] \right\}. \qquad (23b)$$

Formulas (23) for $n = 2$ were used in [21] for describing experimental data obtained by Gottenberg and Christensen [4] and for illustrating advantages of the fractional derivative models over the viscoelastic models with integer order derivatives (1).

An immediate generalization of the Rabotnov's model (17) is the model proposed by Koeller [5]

$$\sigma = E_\infty \left[\varepsilon - \sum_{j=1}^{n} e_i \ \ni_{\alpha_j}^{*} (-t_j^{-\alpha_j}) \varepsilon \right]. \qquad (24)$$

Inversion of this equation is rather difficult, but nevertheless it may be used for solving different problems of mechanics.

Inversion of formulas (7) into (17), or (17) into (7), is given in Rabotnov's books [12] and [13] published in 1966 and 1977, respectively, and translated in English in 1969 and 1980, respectively. However, references to these monographs in articles devoted to the applications of fractional calculus to viscoelasticity are few and far between. Moreover, a reader can find citation of [12] and/or [13] more frequently in papers by Western researchers than in those by Russian scientists. To the authors deep regret, Russian young researchers know not much about outstanding investigation of our distinguished Academician. And hereafter is one striking example.

4. Application of Rabotnov's equations for solving some fractional derivative equations

Using the fractional Green's function (see [7]), Surguladze [19] constructs the solution of the following set of equations:

$$Q(D^{\nu})f(t) = g(t), \qquad \nu = \frac{1}{q}, \tag{25}$$

$$D^{k\nu}f(0) = 0 \qquad (0 \leq k \leq N-1)$$

$$Q(D^{\nu}) = a_0 D^{n\nu} + a_1 D^{(n-1)\nu} + ... + a_n D^0, \tag{26}$$

where $g(t)$ is the known function, N is the smallest integer which is larger or equal to $n\nu$, and q is a positive integer.

However, the solution of Eq. (25) with the operator (26) was obtained by Rabotnov ([12]) 35 years ago, without using Green's function. Moreover, Rabotnov [12] solved a more general problem, since he considered sums of operators in both sides of Eq. (25).

Indeed, Eq. (7) is equivalent to the equation

$$\sigma = Q(D^{\nu})\varepsilon. \tag{27}$$

In order to show this, it is sufficient to apply Laplace transformation to Eq. (27). Then

$$\overline{\sigma} = Q(p^{\nu})\overline{\varepsilon}, \tag{28a}$$

or

$$\overline{\varepsilon} = \overline{\sigma}\,\frac{1}{a_0 p^{n\nu} + a_1 p^{(n-1)\nu} + ... + a_n}\,. \tag{28b}$$

Denoting the roots of the equation

$$a_0 x^n + a_1 x^{(n-1)} + ... + a_n = 0$$

by $\alpha_i = -\tau_i^{-\nu}$, we rewrite (28b) in the form

$$\overline{\varepsilon} = \overline{\sigma}\,\frac{J_{\infty}}{\prod\limits_{i=1}^{n}(1 + p^{\nu}\tau_i^{\nu})}\,. \tag{29}$$

Expanding then the right part of expression (29) in terms of simple fractions, we obtain, taking into account (13), the relationship (7).

Inversion of the relationship (7) is just the formula (17), while the parameters involved therein are found from (19a,b). It is not clear why would the author of [19] and [20] carrying out the inversion of long time known hereditarily elastic relationships has not used Rabotnov's method, which is more effective than that of Miller and Ross [7].

5. Conclusion

Appearance of viscoelastic models involving fractional derivatives of different orders often is related to the names of Bagley [1] and Koeller [6]. However, due to our knowledge, the first such model appeared in 1966 in Russia and was suggested by Academician Yu.N. Rabotnov in the equivalent operator form, see [12]. Could Rabotnov write his model in terms of fractional derivatives? Of course he could! As far back as in 1948, in his classical papers [10] and [11] he wrote about this matter. However, he considered fractional derivatives as some mathematical abstraction without any physical meaning and demonstrativeness and thus, suggested to use the hereditary mechanics models instead of the operators of fractional calculus.

In this paper, an attempt has been made to improve a historical unfairness, as well as to show the importance of Rabotnov's ideas, which have a lead of several decades in the development of viscoelasticity. For this purpose, a comparative analysis of *the models involving fractional derivatives of different orders* is given. Such models of viscoelastic materials are widely used now in various problems of mechanics and rheology. Rabotnov's hereditarily elastic rheological model is considered in detail. For this model it is shown that it is equivalent to the rheological model involving fractional derivatives in the stress and strain with the orders proportional to a certain positive value less than unit.

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