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SPECTRUM OF FUNCTIONS FOR THE DUNKL TRANSFORM ON $\mathbb{R}^d$

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Abstract

In this paper, we establish real Paley-Wiener theorems for the Dunkl transform on $\mathbb{R}^d$. More precisely, we characterize the functions in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ and in $L^2_k(\mathbb{R}^d)$ whose Dunkl transform has bounded, unbounded, convex and nonconvex support.

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1. Introduction

In the last few years there has been a great interest to the study of the spectrum of functions, i.e. the support of the transform of these functions relatively to certain integral transforms. These results have been called in some papers "real Paley-Wiener theorems". See [13] for an overview references and details for this question.

In this paper we consider the Dunkl operators $T_j, j = 1, ..., d$, which are the differential-difference operators introduced by C.F. Dunkl in [6]. These operators are very important in pure mathematics and in physics. They provide useful tool in the study of special functions with root systems.

In [7] (see also [8]), C.F. Dunkl has studied a Fourier transform $\mathcal{F}_D$, called Dunkl transform defined for a regular function f by

$$\forall\, x \in I\!R^d, \;\; \mathcal{F}_D f(x) = \int_{I\!R^d} K(-ix, y) f(y)\omega_k(y)dy,$$

where $K(-ix, y)$ represents the Dunkl kernel and ω_k a weight function.

The purpose of this paper is to prove real Paley-Wiener theorems on the Schwartz space $\mathcal{S}(I\!R^d)$ and on $L^2_k(I\!R^d)$. More precisely, we consider first the Paley-Wiener spaces associated with the Dunkl operators:

$$PW^2_k(I\!R^d) = \{f \in \mathcal{E}(I\!R^d)/\forall\, n \in I\!N, \;\; \triangle^n_k f \in L^2_k(I\!R^d) \text{ and } R^{\triangle_k}_f < +\infty\},$$

$PW_k(I\!R^d)$ being the space of $f \in \mathcal{E}(I\!R^d)$ such that for all $n, m \in I\!N$, $(1 + ||x||)^m \triangle^n_k f$ belongs to $L^2_k(I\!R^d)$ and $R^{\triangle_k}_f < +\infty$, where $\mathcal{E}(I\!R^d)$ is the space of C^∞-functions on $I\!R^d$, $\triangle_k = \sum_{j=1}^{d} T^2_j$ the Dunkl-Laplacian operator, $L^2_k(I\!R^d)$ the space of square integrable functions with respect to the measure $\omega_k(x)dx$, $||.||_{k,2}$ its norm and $R^{\triangle_k}_f = \lim_{n\to\infty} ||\triangle^n_k f||^{\frac{1}{2n}}_{k,2}$.

We establish that $\mathcal{F}_D$ is a bijection from $PW^2_k(I\!R^d)$ onto $L^2_{k,c}(I\!R^d)$ (the space of functions in $L^2_k(I\!R^d)$ with compact support), and from $PW_k(I\!R^d)$ onto $D(I\!R^d)$ (the space of C^∞-functions on $I\!R^d$ with compact support).

Next, we characterize the $L^2_k(U)$-functions by their Dunkl transform, where U is respectively a disk, a symmetric body, a nonconvex and an unbounded domain in $I\!R^d$. These results are the real Paley-Wiener theorems for square integrable functions with respect to the measure $\omega_k(x)dx$.

We generalize also a theorem of H.H. Bang [3] by characterizing the support of the Dunkl transform of functions in $\mathcal{S}(I\!R^d)$ by an L^p growth condition. More precisely, these real Paley-Wiener theorems can be stated as follows:

• The Dunkl transform $\mathcal{F}_D(f)$ of $f \in \mathcal{S}(I\!R^d)$ vanishes outside a polynomial domain $U_P = \{x \in I\!R^d, \; P(x) \leq 1\}$, with P a non constant polynomial, if and only if

$$\overline{\lim}_{n\to+\infty} ||P^n(iT)f||^{\frac{1}{n}}_{k,p} \leq 1, \; 1 \leq p \leq \infty,$$

with $T = (T_1, ..., T_d)$ and $||.||_{k,p}$ is the norm of the space $L^p_k(I\!R^d)$ of p^{th} integrable functions on $I\!R^d$ with respect to the measure $\omega_k(x)dx$.

• A function $f \in \mathcal{S}(I\!R^d)$ is the Dunkl transform of a function vanishing in some ball with radius r centered at the origin, if and only if

$$\lim_{n\to\infty} || \sum_{m=0}^{\infty} \frac{(n\triangle_k)^m f}{m!} ||^{\frac{1}{n}}_{k,p} \leq \exp(-r^2), \; 1 \leq p \leq \infty.$$

This paper is arranged as follows:

In Section 2 we recall the main results about the harmonic analysis associated to the Dunkl operators. Section 3 is devoted to study the functions such that the supports of their Dunkl transform are compact, and to establish the real Paley-Wiener theorems for $\mathcal{F}_D$ on the Schawrz space $\mathcal{S}(\mathbb{R}^d)$. In Section 4 we characterize the functions in $\mathcal{S}(\mathbb{R}^d)$ such that their Dunkl transforms vanish outside a polynomial domain. In Section 5 we give a necessary and sufficient condition for functions in $L^2_k(\mathbb{R}^d)$ such that their Dunkl transforms vanish in a disk. We study in Section 6 the functions such that their Dunkl transforms satisfy the symmetric body property, and we derive a real Paley-Wiener type theorem for these functions.

2. Harmonic analysis associated to the Dunkl operators

In the first two subsections we collect some notations and results on Dunkl operators, the Dunkl kernel and the Dunkl intertwining operators (see [6],[7],[8]).

2.1. Reflection groups, root system and multiplicity functions

We consider $\mathbb{R}^d$ with the Euclidean scalar product $\langle .,. \rangle$ and $||x|| = \sqrt{\langle x, x\rangle}$. On $\mathbb{C}^d$, $||.||$ denotes also the standard Hermitian norm while for all $z = (z_1, ..., z_d)$, $w = (w_1, ..., w_d) \in \mathbb{C}^d$, $\langle z, w\rangle = \sum_{j=1}^{d} z_j \overline{w}_j$.

For $\alpha \in \mathbb{R}^d \backslash \{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to α, i.e. $\sigma_\alpha(x) = x - 2\frac{\langle \alpha, x\rangle}{||\alpha||^2}\alpha$.

A finite set $R \subset \mathbb{R}^d\backslash\{0\}$ is called a root system, if $R \cap \mathbb{R}.\alpha = \{\alpha, -\alpha\}$ and $\sigma_\alpha R = R$ for all $\alpha \in R$. For a given root system R the reflection $\sigma_\alpha, \alpha \in R$, generate a finite group $W \subset O(d)$, the reflection group associated with R. We denote by $|W|$ its cardinality. All reflections in W correspond to suitable pairs of roots. For a given $\beta \in \mathbb{R}\backslash \cup_{\alpha\in R} H_\alpha$, we fix the positive subsystem $R_+ = \{\alpha \in R \,/ \langle \alpha, \beta\rangle > 0\}$, then for each $\alpha \in R$, either $\alpha \in R_+$ or $-\alpha \in R_+$.

A function $k : R \longrightarrow \mathbb{C}$ on a root system R is called a multiplicity function, if it is invariant under the action of the associated reflection group W. If one regards k as a function on the corresponding reflections, this means that k is constant on the conjugacy classes of reflections in W. For abbreviation, we introduce the index $\gamma = \gamma(k) = \sum_{\alpha \in R_+} k(\alpha)$, and the weight function $\omega_k(x) = \prod_{\alpha \in R_+} |\langle \alpha, x\rangle|^{2k(\alpha)}$, which is W-invariant and homogeneous of degree 2γ.

We introduce the Mehta-type constant

$$c_k = \left(\int_{I\!R^d} \exp(-||x||^2)\omega_k(x)\, dx\right)^{-1}. \tag{1}$$

Remark. For $d = 1$ and $W = \mathbf{Z}_2$, the multiplicity function k is a single parameter denoted $\gamma > 0$ and we have for all $x \in I\!R$, $\omega_k(x) = |x|^{2\gamma}$.

2.2. Dunkl operators. The Dunkl kernel and the Dunkl intertwining operator

Notations. We denote by
- $C^p(I\!R^d)$ (resp. $C^p_c(I\!R^d)$) the space of functions of class C^p on $I\!R^d$ (resp. with compact support).
- $\mathcal{E}(I\!R^d)$ the space of C^∞-functions on $I\!R^d$.
- $\mathcal{S}(I\!R^d)$ the space of C^∞-functions on $I\!R^d$ which are rapidly decreasing as their derivatives.
- $D(I\!R^d)$ the space of C^∞-functions on $I\!R^d$ which are of compact support.

We provide these spaces with the classical topology .
- $\mathcal{E}'(I\!R^d)$ the space of distributions on $I\!R^d$ with compact support. It is the topological dual of $\mathcal{E}(I\!R^d)$.
- $\mathcal{S}'(I\!R^d)$ the space of tempered distributions on $I\!R^d$. It is the topological dual of $\mathcal{S}(I\!R^d)$.

The Dunkl operators $T_j,\ j = 1, ..., d$, on $I\!R^d$ associated with the finite reflection group W and the multiplicity function k are given by

$$T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in R_+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle}, \quad f \in C^1(I\!R^d). \tag{2}$$

In the case $k = 0$, the T_j, $j = 1, ..., d$, reduce to the corresponding partial derivatives. In this paper, we will assume throughout that $k \geq 0$ and $\gamma > 0$.

The Dunkl Laplacian $\triangle_k$ on $I\!R^d$ is defined by

$$\triangle_k f = \sum_{j=1}^{d} T_j^2 f = \triangle f + 2 \sum_{\alpha \in R_+} k_\alpha \delta_\alpha(f), \quad f \in C^2(I\!R^d), \tag{3}$$

where $\triangle = \sum_{j=1}^{d} \partial_j^2$ is the Laplacian on $I\!R^d$ and

$$\delta_\alpha(f)(x) = \frac{\langle \nabla f(x), \alpha\rangle}{\langle \alpha, x\rangle} - \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle^2},$$

with ∇f the gradient of f.

For f in $C_c^1(\mathbb{R}^d)$ and $g \in C^1(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} T_j f(x) g(x) \omega_k(x)\, dx = -\int_{\mathbb{R}^d} f(x) T_j g(x) \omega_k(x)\, dx, \quad j = 1, ..., d. \qquad (4)$$

For $y \in \mathbb{R}^d$, the system

$$\begin{cases} T_j u(x,y) &= \quad y_j u(x,y), \quad j = 1, ..., d, \\ \\ u(0,y) &= \quad 1, \qquad\qquad \text{for all} \;\; y \in \mathbb{R}^d. \end{cases}$$

admits a unique analytic solution on $\mathbb{R}^d$, denoted by $K(x,y)$ and called Dunkl kernel. This kernel has a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$.

EXAMPLE. If $d = 1$ and $W = Z_2$, the Dunkl kernel is given by

$$K(z,w) = j_{\gamma-\frac{1}{2}}(izw) + \frac{zw}{2\gamma+1} j_{\gamma+\frac{1}{2}}(izw), \quad z, \; w \in \mathbb{C}, \qquad (5)$$

where for $\alpha \geq \frac{-1}{2}$, j_α is the normalized Bessel function of index α defined by

$$j_\alpha(z) = 2^\alpha \Gamma(\alpha+1) \frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{2n}}{n! \Gamma(\alpha+1+n)}, \qquad (6)$$

where J_α is the Bessel function of first kind and index α.

The Dunkl kernel possesses many properties, in particular, the following proposition gives some of them:

PROPOSITION 2.1. *i) For all $z, w \in \mathbb{C}^d$ we have*

$$K(z,w) = K(w,z); \quad K(z,0) = 1; \quad K(\lambda z, w) = K(z, \lambda w), \quad \textit{for all} \;\; \lambda \in \mathbb{C}.$$

ii) For all $\nu \in \mathbb{N}^d, x \in \mathbb{R}^d$ and $z \in \mathbb{C}^d$, we have

$$|D_z^\nu K(x,z)| \leq ||x||^{|\nu|} \exp(||x|| ||Re z||), \qquad (7)$$

$$\textit{and for all}\; x, y \in \mathbb{R}^d : \; |K(ix,y)| \leq 1, \qquad (8)$$

with $D_z^\nu = \frac{\partial^\nu}{\partial z_1^{\nu_1} ... \partial z_d^{\nu_d}}$ and $|\nu| = \nu_1 + ... + \nu_d$.

2.3. The Dunkl transform

NOTATIONS. We denote by $L_k^p(\mathbb{R}^d)$ the space of measurable functions on $\mathbb{R}^d$ such that

$$||f||_{k,p} = (\int_{I\!R^d} |f(x)|^p \omega_k(x)\, dx)^{\frac{1}{p}} < +\infty, \quad \text{if } 1 \leq p < +\infty,$$

$$||f||_{k,\infty} = ess\ sup_{x\in I\!R^d} |f(x)| < +\infty.$$

The Dunkl transform of a function f in $D(I\!R^d)$ is given by

$$\forall y \in I\!R^d, \quad \mathcal{F}_D(f)(y) = \int_{I\!R^d} f(x) K(-iy, x)\omega_k(x) dx. \tag{9}$$

Further, we give some properties of this transform (see [7],[8]):

i) For all f in $L^1_k(I\!R^d)$ we have

$$||\mathcal{F}_D(f)||_{k,\infty} \leq ||f||_{k,1}. \tag{10}$$

ii) For all f in $\mathcal{S}(I\!R^d)$ we have

$$\forall y \in I\!R^d, \quad \mathcal{F}_D(T_j f)(y) = iy_j \mathcal{F}_D(f)(y) \quad , j = 1, ..., d. \tag{11}$$

iii) For all f in $L^1_k(I\!R^d)$ such that $\mathcal{F}_D(f)$ is in $L^1_k(I\!R^d)$, we have the inversion formula

$$f(y) = \frac{c_k^2}{4^{\gamma+\frac{d}{2}}} \int_{I\!R^d} \mathcal{F}_D(f)(x) K(ix, y)\omega_k(x)\, dx, \quad a.e. \tag{12}$$

THEOREM 2.2. *The Dunkl transform $\mathcal{F}_D$ is a topological isomorphism:*
i) From $\mathcal{S}(I\!R^d)$ onto itself.
ii) From $D(I\!R^d)$ onto $H(\mathbb{C}^d)$ (the space of entire functions on $\mathbb{C}^d$, rapidly decreasing and of exponential type.)
The inverse transform $\mathcal{F}_D^{-1}$ is given by

$$\forall y \in I\!R^d, \quad \mathcal{F}_D^{-1}(f)(y) = \frac{c_k^2}{4^{\gamma+\frac{d}{2}}} \mathcal{F}_D(f)(-y), \quad f \in S(I\!R^d). \tag{13}$$

THEOREM 2.3. *i) Plancherel formula for $\mathcal{F}_D$: For all f in $\mathcal{S}(I\!R^d)$ we have*

$$\int_{I\!R^d} |f(x)|^2 \omega_k(x)\, dx = \frac{c_k^2}{4^{\gamma+\frac{d}{2}}} \int_{I\!R^d} |\mathcal{F}_D(f)(\xi)|^2 \omega_k(\xi)\, d\xi. \tag{14}$$

ii) Plancherel theorem for $\mathcal{F}_D$: The renormalized Dunkl transform $f \to 2^{-(\gamma+\frac{d}{2})} c_k \mathcal{F}_D(f)$ can be uniquely extended to an isometric isomorphism on $L^2_k(I\!R^d)$.

PROPOSITION 2.4. *Let $1 \leq p \leq 2$. The Dunkl transform $\mathcal{F}_D$ can be extended to a continuous mapping from $L^p_k(\mathbb{R}^d)$ into $L^q_k(\mathbb{R}^d)$, with q the conjugate component of p.*

DEFINITION 2.5. i) The Dunkl transform of a distribution τ in $\mathcal{S}'(\mathbb{R}^d)$ is defined by

$$< \mathcal{F}_D(\tau), \phi >=< \tau, \mathcal{F}_D(\phi) >, \quad \phi \in \mathcal{S}(\mathbb{R}^d). \tag{15}$$

ii) We define the Dunkl transform of a distribution τ in $\mathcal{E}'(\mathbb{R}^d)$ by

$$\forall\, y \in \mathbb{R}^d,\ \mathcal{F}_D(\tau)(y) = \langle \tau_x, K(-ix, y) \rangle. \tag{16}$$

THEOREM 2.6. *The Dunkl transform $\mathcal{F}_D$ is a topological isomorphism:*
i) From $\mathcal{S}'(\mathbb{R}^d)$ onto itself.
ii) From $\mathcal{E}'(\mathbb{R}^d)$ onto $\mathcal{H}(\mathbb{C}^d)$(the space of entire functions on $\mathbb{C}^d$, slowly increasing and of exponential type.)

Let τ be in $\mathcal{S}'(\mathbb{R}^d)$. We define the distribution $T_j\tau$, $j = 1, ..., d$, by

$$< T_j\tau, \psi >= - < \tau, T_j\psi >, \ \text{ for all } \psi \in \mathcal{S}(\mathbb{R}^d). \tag{17}$$

This distribution satisfies the following properties:

$$\mathcal{F}_D(T_j\tau) = iy_j\mathcal{F}_D(\tau), \quad j = 1, ..., d. \tag{18}$$

$$\mathcal{F}_D(\triangle_k\tau) = -||y||^2\mathcal{F}_D(\tau). \tag{19}$$

We consider f in $L^2_k(\mathbb{R}^d)$. The distribution $T_{f\omega_k}$ given by the function $f\omega_k$ belongs to $\mathcal{S}'(\mathbb{R}^d)$ and the relation (19) is written in this case in the form

$$\mathcal{F}_D(\triangle_k T_{f\omega_k}) = -||y||^2\mathcal{F}_D(T_{f\omega_k}). \tag{20}$$

NOTATIONS. We denote by
- $L^2_{k,c}(\mathbb{R}^d)$ the space of functions in $L^2_k(\mathbb{R}^d)$ with compact support.
- $\mathcal{H}_{L^2_k}(\mathbb{C}^d)$ the space of entire functions f on $\mathbb{C}^d$ of exponential type such that $f_{|\mathbb{R}^d}$ belongs to $L^2_k(\mathbb{R}^d)$.

THEOREM 2.7. *The Dunkl transform $\mathcal{F}_D$ is bijective from $L^2_{k,c}(\mathbb{R}^d)$ onto $\mathcal{H}_{L^2_k}(\mathbb{C}^d)$.*

P r o o f. i) We consider the function f on $\mathbb{C}^d$ given by

$$\forall\, z \in \mathbb{C}^d,\ f(z) = \int_{\mathbb{R}^d} g(x)K(-ix, z)\omega_k(x)dx, \tag{21}$$

with $g \in L^2_{k,c}(I\!R^d)$.

By derivation under the integral sign and by using the inequality (7), we deduce that the function f is entire on $\mathbb{C}^d$ and of exponential type.

On the other hand, the relation (21) can also be written in the form

$$\forall\, y \in I\!R^d,\ f(y) = \mathcal{F}_D(g)(y).$$

Thus from Theorem 2.3, the function $f_{|I\!R^d}$ belongs to $L^2_k(I\!R^d)$. Thus $f \in \mathcal{H}_{L^2_k}(\mathbb{C}^d)$.

ii) Reciprocally, let ψ be in $\mathcal{H}_{L^2_k}(\mathbb{C}^d)$. From Theorem 2.6 ii), there exists S belonging to $\mathcal{E}'(I\!R^d)$ with support in the ball $B(o,a)$ of center o and radius a, such that

$$\forall\, y \in I\!R^d,\ \psi(y) = \langle S_x, K(-ix,y)\rangle. \tag{22}$$

On the other hand as $\psi_{|I\!R^d}$ belongs to $L^2_k(I\!R^d)$, then from Theorem 2.3 there exists $h \in L^2_k(I\!R^d)$ such that

$$\psi_{|I\!R^d} = \mathcal{F}_D(h). \tag{23}$$

Thus from (22), for all $\varphi \in D(I\!R^d)$ we have

$$\int_{I\!R^d} \psi(y)\overline{\mathcal{F}_D(\varphi)(y)}\omega_k(y)dy \;=\; \langle S_x, \int_{I\!R^d} K(-ix,y)\overline{\mathcal{F}_D(\varphi)(y)}\omega_k(y)dy\rangle.$$

Thus using (13) we deduce that

$$\int_{I\!R^d} \psi(y)\overline{\mathcal{F}_D(\varphi)(y)}\omega_k(y)dy = \frac{4^{\gamma+\frac{d}{2}}}{c_k^2}\langle S, \varphi\rangle. \tag{24}$$

On the other hand, (23) implies

$$\int_{I\!R^d} \psi(y)\overline{\mathcal{F}_D(\varphi)(y)}\omega_k(y)dy = \int_{I\!R^d} \mathcal{F}_D(h)(y)\overline{\mathcal{F}_D(\varphi)(y)}\omega_k(y)dy.$$

But from Theorem 2.3 we deduce that

$$\int_{I\!R^d} \mathcal{F}_D(h)(y)\overline{\mathcal{F}_D(\varphi)(y)}\omega_k(y)dy = \frac{4^{\gamma+\frac{d}{2}}}{c_k^2}\int_{I\!R^d} h(y)\varphi(y)\omega_k(y)dy = \frac{4^{\gamma+\frac{d}{2}}}{c_k^2}\langle T_{h\omega_k}, \varphi\rangle. \tag{25}$$

Thus the relations (24),(25) imply

$$S = T_{h\omega_k}.$$

This relation shows that the support of h is compact. Then $h \in L^2_{k,c}(I\!R^d)$. ∎

3. Functions with compact spectrum

We begin this section by the following definition.

DEFINITION 3.1. i) We define the support of $g \in L^2_k(I\!R^d)$ and we denote it by supp g, the smallest closed set such that the function g vanishes almost everywhere outside it.

ii) We denote by R_g the radius of the support of g given by

$$R_g = \sup_{\lambda \in suppg} ||\lambda||.$$

REMARK. It is clear that R_g is finite if and only if, g has compact support.

PROPOSITION 3.2. *Let g be in $L^2_k(I\!R^d)$ such that for all $n \in I\!N$, the function $||\lambda||^{2n} g(\lambda)$ belongs to $L^2_k(I\!R^d)$. Then,*

$$R_g = \lim_{n\to\infty} \left\{ \int_{I\!R^d} ||\lambda||^{4n} |g(\lambda)|^2 \omega_k(\lambda) d\lambda \right\}^{\frac{1}{4n}}. \tag{26}$$

P r o o f. We suppose that $||g||_{k,2} \neq 0$, otherwise $R_g = 0$ and formula (26) is trivial.

Assume now that g has compact support with $R_g > 0$. We have

$$\left\{ \int_{I\!R^d} ||\lambda||^{4n} |g(\lambda)|^2 \omega_k(\lambda) d\lambda \right\}^{\frac{1}{4n}} \leq \left\{ \int_{||\lambda||\leq R_g} |g(\lambda)|^2 \omega_k(\lambda) d\lambda \right\}^{\frac{1}{4n}} R_g.$$

Thus,

$$\overline{\lim_{n\to\infty}} \left\{ \int_{I\!R^d} ||\lambda||^{4n} |g(\lambda)|^2 \omega_k(\lambda) d\lambda \right\}^{\frac{1}{4n}} \leq R_g. \tag{27}$$

On the other hand, for any positive ε we have

$$\int_{R_g - \varepsilon \leq ||\lambda|| \leq R_g} |g(\lambda)|^2 \omega_k(\lambda) d\lambda > 0.$$

Hence

$$\underline{\lim}_{n\to\infty} \left\{ \int_{I\!R^d} ||\lambda||^{4n} |g(\lambda)|^2 \omega_k(\lambda) d\lambda \right\}^{\frac{1}{4n}} \geq R_g - \varepsilon. \tag{28}$$

Then we deduce (26) from (27) and (28).

We prove now the assertion in the case where g has unbounded support. Indeed, for any positive N, we have

$$\underline{\lim}_{n\to\infty} \left\{ \int_{I\!R^d} ||\lambda||^{4n} |g(\lambda)|^2 \omega_k(\lambda) d\lambda \right\}^{\frac{1}{4n}}$$

$$\geq \underline{\lim}_{n\to\infty}\left\{\int_{||\lambda||\geq N}||\lambda||^{4n}|g(\lambda)|^2\omega_k(\lambda)d\lambda\right\}^{\frac{1}{4n}}\geq N.$$

Thus,

$$\underline{\lim}_{n\to\infty}\left\{\int_{I\!R^d}||\lambda||^{4n}|g(\lambda)|^2\omega_k(\lambda)d\lambda\right\}^{\frac{1}{4n}}=\infty.$$

■

NOTATIONS. We denote by
- $L^2_{k,R}(I\!R^d) := \{g \in L^2_{k,c}(I\!R^d)/R_g = R\}$, for $R \geq 0$.
- $D_R(I\!R^d) := \{g \in D(I\!R^d)/R_g = R\}$, for $R \geq 0$.

DEFINITION 3.3. We define the Paley-Wiener spaces $PW^2_k(I\!R^d)$ and $PW^2_{k,R}(I\!R^d)$ as follows:
i) $PW^2_k(I\!R^d)$ is the space of functions $f \in \mathcal{E}(I\!R^d)$ satisfying
a) $\triangle^n_k f \in L^2_k(I\!R^d)$ for all $n \in I\!N$.
b) $R^{\triangle_k}_f := \lim\limits_{n\to\infty}||\triangle^n_k f||^{\frac{1}{2n}}_{k,2} < \infty$.
ii) $PW^2_{k,R}(I\!R^d) := \{f \in PW^2_k(I\!R^d)/R^{\triangle_k}_f = R\}$.

The real L^2-Paley-Wiener theorem for the Dunkl transform can be formulated as follows:

THEOREM 3.4. *The Dunkl transform $\mathcal{F}_D$ is a bijection:*
i) From $PW^2_{k,R}(I\!R^d)$ onto $L^2_{k,R}(I\!R^d)$.
ii) From $PW^2_k(I\!R^d)$ onto $L^2_{k,c}(I\!R^d)$.

P r o o f. i) Let g be in $PW^2_{k,R}(I\!R^d)$. From the relation (20) the function $\mathcal{F}_D(\triangle^n_k g)(\xi) = (-1)^n||\xi||^{2n}\mathcal{F}_D(g)(\xi)$ belongs to $L^2_k(I\!R^d)$ for all $n \in I\!N$. Thus from Theorem 2.3 we deduce that

$$\lim_{n\to\infty}\left\{\int_{I\!R^d}||\xi||^{4n}|\mathcal{F}_D(g)(\xi)|^2\omega_k(\xi)d\xi\right\}^{\frac{1}{4n}} = \lim_{n\to\infty}\left\{\int_{I\!R^d}|\triangle_k g(x)|^2\omega_k(x)dx\right\}^{\frac{1}{4n}} = R.$$

Using Proposition 3.2, we conclude that $\mathcal{F}_D(g)$ has compact support with $R_{\mathcal{F}_D(g)} = R$.

Conversely, let f be in $L^2_{k,R}(I\!R^d)$. Then $||\xi||^n f(\xi) \in L^1_k(I\!R^d)$ for any $n \in I\!N$, and $\mathcal{F}^{-1}_D(f) \in \mathcal{E}(I\!R^d)$. On the other hand, from Theorem 2.3 we have

$$\lim_{n\to\infty}\left\{\int_{I\!R^d}|\triangle_k^n(\mathcal{F}_D^{-1}f)(x)|^2\omega_k(x)dx\right\}^{\frac{1}{4n}}$$
$$=\lim_{n\to\infty}\left\{\int_{I\!R^d}||\xi||^{4n}|f(\xi)|^2\omega_k(\xi)d\xi\right\}^{\frac{1}{4n}}=R.$$

Thus $\mathcal{F}_D^{-1}(f)\in PW^2_{k,R}(I\!R^d)$.

ii) We deduce ii) from i). ■

Corollary 3.5. *The Dunkl transform $\mathcal{F}_D$ is a bijection from $PW^2_k(I\!R^d)$ onto $\mathcal{H}_{L^2_k}(\mathbb{C}^d)$.*

P r o o f. We deduce the result from Theorem 3.4 ii) and Theorem 2.7. ■

Definition 3.6. i) The Paley-Wiener space $PW_k(I\!R^d)$ is the space of functions $f\in\mathcal{E}(I\!R^d)$ satisfying

a) $(1+||x||)^m\triangle_k^n f\in L^2_k(I\!R^d)$ for all $n,m\in I\!N$.

b) $R_f^{\triangle_k}:=\lim_{n\to\infty}||\triangle_k^n f||_{k,2}^{\frac{1}{2n}}<\infty$.

ii) We have $PW_{k,R}(I\!R^d):=\{f\in PW_k(I\!R^d)/R_f^{\triangle_k}=R\}$, for $R\geq 0$.

Remark. We notice that the only difference between $PW^2_k(I\!R^d)$ and $PW_k(I\!R^d)$ is the extra requirement of polynomial decay to help ensuring that $\mathcal{F}_D(f)$ belongs to $\mathcal{E}(I\!R^d)$.

The real Paley-Wiener theorem for the Dunkl transform of functions in the preceding spaces is the following:

Theorem 3.7. *The Dunkl transform $\mathcal{F}_D$ is a bijection:*

i) From $PW_{k,R}(I\!R^d)$ onto $D_R(I\!R^d)$.

ii) From $PW_k(I\!R^d)$ onto $D(I\!R^d)$.

P r o o f. i) Let g be in $PW_{k,R}(I\!R^d)\subset PW^2_{k,R}(I\!R^d)$. Then $\mathcal{F}_D(g)\in\mathcal{E}(I\!R^d)$ since g has polynomial decay, and by Theorem 3.4 the function $\mathcal{F}_D(g)$ has compact support with $R_{\mathcal{F}_D(g)}=R$.

Conversely, let $f\in D_R(I\!R^d)$, then $\mathcal{F}_D^{-1}(f)\in\mathcal{S}(I\!R^d)$ and $\mathcal{F}_D^{-1}(f)\in PW^2_{k,R}(I\!R^d)$ by Theorem 3.4. So it only remains to show that $\mathcal{F}_D^{-1}(f)$ satisfies the polynomial decay condition for any $f\in D_R(I\!R^d)$. For this, we use the following identity which is easy to prove:

For a suitable functions f and g, we deduce from relations (3),(4),(9),(13) that

$$(1+||x||^2)^n\mathcal{F}_D^{-1}(f)(x)=\int_{I\!R^d}(I-\triangle_k)^n f(\xi)K(ix,\xi)\omega_k(\xi)d\xi.$$

Thus we obtain the result.

ii) We deduce the result from the i). ■

4. Dunkl transform of functions with polynomial domain support

THEOREM 4.1. *Let $P(x)$ be a non-constant polynomial on $I\!R^d$. For any function $f \in \mathcal{S}(I\!R^d)$ the following relation holds:*

$$\lim_{n\to\infty} ||P^n(iT)f||_{k,p}^{\frac{1}{n}} = \sup_{y\in supp\mathcal{F}_D(f)} |P(y)|,\ 1 \le p \le \infty, \tag{29}$$

with $T = (T_1, ..., T_d)$.

P r o o f. We consider $f \neq 0$ in $\mathcal{S}(I\!R^d)$. Set $q = \frac{p}{p-1}$ if $1 < p < \infty$, and $q = 1$ or ∞ if $p = \infty$ or 1. The proof is divided in several steps.
In the following three steps we suppose that

$$0 < \sup_{y\in supp\,\mathcal{F}_D(f)} |P(y)| < \infty. \tag{30}$$

<u>*First step*</u>: In this step we shall prove that

$$\overline{\lim_{n\to\infty}} ||P^n(iT)f||_{k,p}^{\frac{1}{n}} \le \sup_{y\in supp\mathcal{F}_D(f)} |P(y)|,\ 1 \le p \le \infty.$$

• Let $2 \le p < \infty$. By applying Proposition 2.4, there exists a positive constant C such that

$$||P^n(iT)f||_{k,p} \le C||P^n(\xi)\mathcal{F}_D(f)||_{k,q}, \tag{31}$$

$$\le C(\sup_{y\in supp\mathcal{F}_D(f)} |P(y)|)^n ||\mathcal{F}_D(f)||_{k,q}. \tag{32}$$

Thus

$$\overline{\lim_{n\to\infty}} ||P^n(iT)f||_{k,p}^{\frac{1}{n}} \le \sup_{y\in supp\mathcal{F}_D(f)} |P(y)|. \tag{33}$$

• Suppose now that $1 \le p < 2$. For $r > 2\gamma + d$ Hölder's inequality gives

$$\begin{aligned} ||f||_{k,p}^p &= \int_{I\!R^d} (1 + ||x||^2)^{-rp}|(1 + ||x||^2)^r f(x)|^p \omega_k(x)dx \\ &\le ||(1 + ||x||^2)^r f||_{k,2}^p ||(1 + ||x||^2)^{-rp}||_{k,\frac{2}{2-p}}. \\ &\le C||(1 + ||x||^2)^r f||_{k,2}^p. \end{aligned} \tag{34}$$

Thus, from the relation (20) we obtain

$$||f||_{k,p}^p \le C||(I - \triangle_k)^r[\mathcal{F}_D(f)]||_{k,2}^p,$$

where C is a positive constant.

Consequently for all $n \in I\!N$, we deduce that

$$||P^n(iT)f||_{k,p} \leq C^{\frac{1}{p}}||(I-\triangle_k)^r[P^n(\xi)\mathcal{F}_D(f)]||_{k,2}. \qquad (35)$$

To estimate the second member, we use the following relation given in Proposition 5.1 of [9]: For all $\mu \in I\!N^d\backslash\{0\}$, there exists a positive constant C satisfying: For all $x \in I\!R^d$ there exist $\xi_p(x,\alpha)$, $p = 1,...,|\mu|$, and $\alpha \in R_+$, such that

$$|T^\mu u(x)| \leq |D^\mu u(x)| + C\sum_{\alpha\in R^+}\sum_{|\beta|=|\mu|}\sum_{p=1}^{|\mu|}|D^\beta u(\xi_p(x,\alpha))|. \qquad (36)$$

From this relation one can deduce that

$$||P^n(iT)f||_{k,p} \leq C^{\frac{1}{p}}n^{2r}\sup_{y\in supp\mathcal{F}_D(f)}|P(y)|^{n-2r}||\varphi_n||_{k,2}$$

with $supp\,\varphi_n \subset supp\,\mathcal{F}_D(f)$ and $||\varphi_n||_{k,2} \leq C_1$, where C_1 is a constant independent of n. Hence,

$$||P^n(iT)f||_{k,p} \leq C^{\frac{1}{p}}C_1n^{2r}\sup_{y\in supp\mathcal{F}_D(f)}|P(y)|^{n-2r}. \qquad (37)$$

Thus

$$\overline{\lim_{n\to\infty}}||P^n(iT)f||_{k,p}^{\frac{1}{n}} \leq \sup_{y\in supp\mathcal{F}_D(f)}|P(y)|. \qquad (38)$$

- Let now $p=\infty$. From the relation (13) We have

$$||f||_{\infty,k} \leq \frac{c_k^2}{4^{\gamma+\frac{d}{2}}}||\mathcal{F}_D(f)||_{k,1}.$$

On the other hand, from Cauchy-Schawrz's inequality we obtain

$$||\mathcal{F}_D(f)||_{k,1} \leq C_0||(1+||\xi||^2)^{\frac{2\gamma+d}{2}}\mathcal{F}_D(f)(\xi)||_{k,2},$$

where C_0 is a positive constant.

Combining the previous inequalities and replacing f by $P^n(iT)f$, we deduce that there exists a positive constant C such that

$$||P^n(iT)f||_{k,\infty} \leq C||P^n(\xi)(1+||\xi||^2)^{\frac{2\gamma+d}{2}}\mathcal{F}_D(f)(\xi)||_{k,2}. \qquad (39)$$

Thus,

$$\overline{\lim_{n\to\infty}}||P^n(iT)f||_{k,\infty}^{\frac{1}{n}} \leq \sup_{y\in supp\,(1+||\xi||^2)^{\frac{2\gamma+d}{2}}\mathcal{F}_D(f)}|P(y)| = \sup_{y\in supp\,\mathcal{F}_D(f)}|P(y)|. \qquad (40)$$

Using (33), (38) and (40) we obtain

$$\overline{\lim_{n\to\infty}}||P^n(iT)f||_{k,p}^{\frac{1}{n}} \leq \sup_{y\in supp\,\mathcal{F}_D(f)} |P(y)|,\ 1\leq p\leq\infty. \tag{41}$$

Second step: In this step we want to prove that

$$\lim_{n\to\infty}||P^n(iT)f||_{k,2}^{\frac{1}{n}} = \sup_{y\in supp\,\mathcal{F}_D(f)} |P(y)|.$$

For any ε, $0<\varepsilon<\sup_{y\in supp\,\mathcal{F}_D(f)} |P(y)|$, there exists a point $x_0\in \operatorname{supp}\mathcal{F}_D(f)$ such that

$$|P(x_0)| > \sup_{y\in supp\,\mathcal{F}_D(f)} |P(y)| - \frac{\varepsilon}{2}$$

As P is a continuous function, there exists a neighborhood U_{x_0} such that

$$|P(x)| > \sup_{y\in supp\,\mathcal{F}_D(f)} |P(y)| - \varepsilon,\ x\in U_{x_0}$$

From Theorem 2.3 we deduce that

$$\begin{aligned}||P^n(iT)f||_{k,2} &= \frac{c_k^2}{4^{\gamma+\frac{d}{2}}}||P^n(\xi)\mathcal{F}_D(f)||_{k,2}\\ &\geq \frac{c_k^2}{4^{\gamma+\frac{d}{2}}}||P^n(\xi)\mathcal{F}_D(f)1_{U_{x_0}}||_{k,2},\end{aligned}$$

where $1_{U_{x_0}}$ is the characteristic function of U_{x_0}.

Thus,

$$||P^n(iT)f||_{k,2} \geq \frac{c_k^2}{4^{\gamma+\frac{d}{2}}}(\sup_{y\in supp\mathcal{F}_D(f)} |P(y)|-\varepsilon)^n||\mathcal{F}_D(f)1_{U_{x_0}}||_{k,2}\,.$$

This inequality implies

$$\begin{aligned}\underline{\lim}_{n\to\infty}||P^n(iT)f||_{k,2}^{\frac{1}{n}} &\geq (\sup_{y\in supp\mathcal{F}_D(f)} |P(y)|-\varepsilon)\lim_{n\to\infty}||\mathcal{F}_D(f)1_{U_{x_0}}||_{k,2}^{\frac{1}{n}}\\ &\geq \sup_{y\in supp\,\mathcal{F}_D(f)}(|P(y)|-\varepsilon).\end{aligned} \tag{42}$$

But ε can be chosen arbitrarily small, thus from (41) and (42) the relation (29) follows for $p=2$.

Third step: In this step we shall prove that

$$\underline{\lim}_{n\to\infty}||P^n(iT)f||_{k,p}^{\frac{1}{n}} \geq \sup_{y\in supp\,\mathcal{F}_D(f)} |P(y)|,\ 1\leq p\leq\infty.$$

Since $f \in \mathcal{S}(I\!R^d)$, the iteration of the relation (4) implies the relation

$$\int_{I\!R^d} \overline{P^n(iT)f(x)} P^n(iT)f(x)\omega_k(x)dx = \int_{I\!R^d} \overline{f(x)} P^{2n}(iT)f(x)\omega_k(x)dx. \tag{43}$$

Hence, by Hölder's inequality,

$$||P^n(iT)f||_{k,2}^2 \leq ||f||_{k,q}||P^{2n}(iT)f||_{k,p}. \tag{44}$$

where q is the conjugate exponent of p. Thus,

$$\lim_{n\to\infty} ||P^n(iT)f||_{k,2}^{\frac{1}{n}} \leq (\lim_{n\to\infty} ||f||_{k,q}^{\frac{1}{2n}})\underline{\lim}_{n\to\infty}||P^{2n}(iT)f||_{k,p}^{\frac{1}{2n}} \tag{45}$$

$$\leq \underline{\lim}_{n\to\infty}||P^{2n}(iT)f||_{k,p}^{\frac{1}{2n}}. \tag{46}$$

Applying now the relation (29) with $p = 2$, we conclude that

$$\sup_{y\in supp\,\mathcal{F}_D(f)} |P(y)| = \lim_{n\to\infty} ||P^n(iT)f||_{k,2}^{\frac{1}{n}} \leq \underline{\lim}_{n\to\infty}||P^{2n}(iT)f||_{k,p}^{\frac{1}{2n}}. \tag{47}$$

We replace in formula (44) the function f by $P(iT)f$ and we obtain

$$||P^{n+1}(iT)f||_{k,2}^2 \leq ||P(iT)f||_{k,q}||P^{2n+1}(iT)f||_{k,p}. \tag{48}$$

Thus,

$$\sup_{y\in supp\,\mathcal{F}_D(f)} |P(y)| = \lim_{n\to\infty} ||P^{n+1}(iT)f||_{k,2}^{\frac{1}{n+1}} \leq \underline{\lim}_{n\to\infty}||P^{2n+1}(iT)f||_{k,p}^{\frac{1}{2n+1}}. \tag{49}$$

Using (47) and (49) we deduce that

$$\sup_{y\in supp\,\mathcal{F}_D(f)} |P(y)| \leq \underline{\lim}_{n\to\infty}||P^n(iT)f||_{k,p}^{\frac{1}{n}}. \tag{50}$$

Then formulas (50) and (42) give (29). Thus we have proved the theorem under the condition (30).

Fourth step: Suppose now $\sup_{y\in supp\mathcal{F}_D(f)} |P(y)| = +\infty$. Then for any $N > 0$ there exists a point $x_0 \in supp\mathcal{F}_D(f)$ such that $|P(x_0)| \geq 2N$. Since P is a continuous function, there exists a neighborhood U_{x_0} of x_0 on which $|P(x)| > N$. Similarly to the previous calculation in the second step, we obtain

$$\underline{\lim}_{n\to\infty}||P^n(iT)f||_{k,2}^{\frac{1}{n}} \geq \frac{c_k^2}{4^{\gamma+\frac{d}{2}}}\underline{\lim}_{n\to\infty}||P^n(\xi)\mathcal{F}_D(f)1_{U_{x_0}}||_{k,2}^{\frac{1}{n}},$$

$$\geq N\underline{\lim}_{n\to\infty}||f1_{U_{x_0}}||_{k,2}^{\frac{1}{n}} = N.$$

We choose N large, and we obtain

$$\lim_{n\to\infty} ||P^n(iT)f||_{k,2}^{\frac{1}{n}} = \infty.$$

Finally, if $\sup_{y\in supp\mathcal{F}_D(f)} |P(y)| = 0$, the identity (29) is clear.

Hence the proof of the theorem is finished. ■

DEFINITION 4.2. Let P be a non-constant polynomial and $U_p = \{x \in I\!R^d,\ |P(x)| \le 1\}$. The set U_P is called a polynomial domain in $I\!R^d$.

REMARK. A disk is a polynomial domain. A polynomial domain may be unbounded and nonconvex, for example the domain U_P with $P(\xi) = \xi_1^2 - \xi_2^2 - ... - \xi_d^2$ is neither bounded nor convex.

We have the following result.

COROLLARY 4.3. *Let f be in $\mathcal{S}(I\!R^d)$. The Dunkl transform $\mathcal{F}_D(f)$ vanishes outside a polynomial domain U_P, if and only if,*

$$\overline{\lim_{n\to\infty}} ||P^n(iT)f||_{k,p}^{\frac{1}{n}} \le 1,\ 1 \le p \le \infty. \tag{51}$$

REMARKS. i) If we take $P(y) = -||y||^2$, then $P(iT) = \triangle_k$, and Theorem 4.1 and Corollary 4.3 characterize functions such that the support of their Dunkl transform is a ball.

ii) Corollary 4.3 has been obtained for $p = 2$ by Vu Kim Tuan in [14].

iii) Theorem 4.1 and Corollary 4.3 generalize also the result obtained in [4].

COROLLARY 4.4. *Let g be in $\mathcal{S}'(I\!R^d)$. We assume that g and its Dunkl transform $\mathcal{F}_D(g)$ both have compact support. Then $g = 0$.*

P r o o f. Assume first that $g \ne 0$ in $D(I\!R^d)$, with supp$g \in B(0,R)$ for some $0 < R < \infty$, and $R_{\mathcal{F}_D(g)} < \infty$. Choose ξ_0 such that $||\xi_0|| > R_{\mathcal{F}_D(g)}$. Then

$$\overline{\lim_{n\to\infty}} |\int_{I\!R^d} \triangle_k^n g(x) K(i\xi_0, x)\omega_k(x)dx|^{\frac{1}{n}} \le \overline{\lim_{n\to+\infty}} \|\triangle_k^n g\|_{k,1}^{\frac{1}{n}} = R^2_{\mathcal{F}_D(g)}.$$

On the other hand, we have

$$\overline{\lim_{n\to\infty}} |\int_{I\!R^d} \triangle_k^n g(x) K(i\xi_0, x)\omega_k(x)dx|^{\frac{1}{n}} = \overline{\lim_{n\to\infty}} |\int_{I\!R^d} g(x) \triangle_k^n K(i\xi_0, x)\omega_k(x)dx|^{\frac{1}{n}}$$

$$= ||\xi_0||^2 \overline{\lim_{n\to\infty}} |\int_{I\!R^d} g(x) K(i\xi_0, x)\omega_k(x)dx|^{\frac{1}{n}} = ||\xi_0||^2 > R^2_{\mathcal{F}_D(g)}.$$

This gives a contradiction. ∎

Now let g be in $\mathcal{S}'(I\!R^d)$ with compact support. Let ψ be in $D(I\!R^d)$. Then from Theorem 5.4 of [17] $g *_D \psi$ belongs to $D(I\!R^d)$ and $\mathcal{F}_D(g *_D \psi) = \mathcal{F}_D(\psi)\mathcal{F}_D(g)$ has compact support. So $g *_D \psi = 0$ for all ψ in $D(I\!R^d)$, which implies that $g = 0$ (see [12],[16],[17] for the properties of the Dunkl convolution).

5. Dunkl transform of functions vanishing on a ball

The following theorem gives the radius of the large disk on which the Dunkl transform of functions in $L^2_k(I\!R^d)$ vanishes every where.

Let E_n, $n \in I\!N\backslash\{0\}$ be the Gauss kernel associated with the Dunkl operators defined by

$$\forall\, y \in I\!R^d,\ E_n(y) = \frac{c_k}{(4n)^{\gamma+\frac{d}{2}}} e^{-\frac{||y||^2}{4n}},$$

and $E_n *_D f$ the Dunkl convolution of E_n and f in $L^2_k(I\!R^d)$ given by

$$\forall\, x \in I\!R^d,\ E_n *_D f(x) = \int_{I\!R^d} \tau_x(E_n)(y) f(y)\omega_k(y)dy, \tag{52}$$

with

$$\tau_x(E_n)(y) = \frac{c_k}{(4n)^{\gamma+\frac{d}{2}}} e^{-\frac{(||x||^2+||y||^2)}{4n}} K(\frac{x}{\sqrt{2}n}, \frac{y}{\sqrt{2}n}). \tag{53}$$

THEOREM 5.1. *Let f be in $L^2_k(I\!R^d)$. We consider the sequence*

$$f_n(x) = E_n *_D f(x),\ x \in I\!R^d,\ n \in I\!N\backslash\{0\}. \tag{54}$$

Then,

$$\lim_{n\to\infty} \sqrt{-\frac{1}{n}\ln ||f_n||_{k,2}} = \lambda_{\mathcal{F}_D(f)}, \tag{55}$$

where

$$\lambda_{\mathcal{F}_D(f)} = \inf\left\{||\xi||,\ \xi \in supp\mathcal{F}_D(f)\right\}. \tag{56}$$

P r o o f. First we remark that from (52),(53) the function f_n is well defined. We assume that $||f||_{k,2} > 0$, otherwise the result is trivial. To prove (55) it is sufficient to verify the equivalent identity

$$\lim_{n\to\infty} ||f_n||_{k,2}^{\frac{1}{n}} = \exp(-\lambda^2_{\mathcal{F}_D(f)}). \tag{57}$$

By a simple calculation we see that the Dunkl transform of $f_n(x)$ is $\exp(-n||\xi||^2)\mathcal{F}_D(f)(\xi)$ and then by applying Theorem 2.3 we obtain

$$\begin{aligned} ||f_n||_{k,2} &= \frac{c_k}{2^{\gamma+\frac{d}{2}}}||\exp(-n||\xi||^2)\mathcal{F}_D(f)(\xi)||_{k,2} \\ &= \frac{c_k}{2^{\gamma+\frac{d}{2}}}||f||_{k,2}\{\int_{supp\mathcal{F}_D(f)} \exp(-2n||\xi||^2)\frac{|\mathcal{F}_D(f)(\xi)|^2}{||f||_{k,2}^2}\omega_k(\xi)d\xi\}^{\frac{1}{2}}. \end{aligned} \tag{58}$$

On the other hand it is known that if m is the Lebesque measure on $I\!R^d$ and U a subset of $I\!R^d$ such that $m(U) = 1$, then for all ϕ in the Lebesgue space $L^p(U, dm)$, $1 \leq p \leq +\infty$, we have

$$\lim_{p\to\infty} ||\phi||_{L^p(U;dm)} = ||\phi||_{L^\infty(U;dm)}. \tag{59}$$

By applying formula (59) with

$$U = supp\mathcal{F}_D(f),\ \phi = \exp(-||\xi||^2),\ p = 2n,\ \text{and } dm(\xi) = \frac{|\mathcal{F}_D(f)(\xi)|^2}{||f||_{k,2}^2}\omega_k(\xi)d\xi,$$

and use the fact that $\lim_{n\to+\infty}(\frac{c_k||f||_{k,2}}{2^{\gamma+\frac{d}{2}}})^{\frac{1}{n}} = 1$.

Thus we obtain

$$\lim_{n\to\infty} ||f_n||_{k,2}^{\frac{1}{n}} = \sup_{\xi\in supp\mathcal{F}_D(f)} \exp(-||\xi||^2) = \exp(-\lambda^2_{\mathcal{F}_D(f)}), \tag{60}$$

which is the relation (57). ■

A function $f \in L^2_k(I\!R^d)$ is the Dunkl transform of a function vanishing in a neighborhood of the origin, if and only if, $\lambda_{\mathcal{F}_D(f)} > 0$, or equivalently, if and only if the limit (57) is less than 1. Thus we have proved the following result.

Corollary 5.2. *The condition*

$$\lim_{n\to\infty} ||f_n||_{k,2}^{\frac{1}{n}} < 1, \tag{61}$$

is necessary and sufficient for a function $f \in L^2_k(I\!R^d)$ to have its Dunkl transform vanishing in a neighborhood of the origin.

Remark. From Theorem 3.3 and Corollary 5.2, it follows that the support of the Dunkl transform of a function in $L^2_k(I\!R^d)$ is in the tore $\lambda_{\mathcal{F}_D(f)} \leq ||\xi|| \leq R_{\mathcal{F}_D(f)}$, if and only if,

$$\lambda_{\mathcal{F}_D(f)} \leq \lim_{n\to\infty} \sqrt{-\frac{1}{n}\ln||f_n||_{k,2}} \leq \lim_{n\to\infty} ||\triangle_k^n f||_{k,2}^{\frac{1}{2n}} \leq R_{\mathcal{F}_D(f)}. \tag{62}$$

Theorem 5.3. *For any function $f \in \mathcal{S}(\mathbb{R}^d)$ the following relation holds*

$$\lim_{n\to\infty} \| \sum_{m=0}^{\infty} \frac{(n\triangle_k)^m f}{m!} \|_{k,p}^{\frac{1}{n}} = \exp(-\lambda^2_{\mathcal{F}_D(f)}),\ 1 \le p \le \infty. \tag{63}$$

In particular, a function $f \in \mathcal{S}(\mathbb{R}^d)$ is the Dunkl transform of a function in $\mathcal{S}(\mathbb{R}^d)$ vanishing in the ball $B(o,r)$ of center o and radius r, if and only if we have

$$\lim_{n\to\infty} \| \sum_{m=0}^{\infty} \frac{(n\triangle_k)^m f}{m!} \|_{k,p}^{\frac{1}{n}} \le \exp(-r^2),\ 1 \le p \le \infty. \tag{64}$$

P r o o f. It is similar to the proof of Theorem 4.1. ■

6. Dunkl transform of functions, vanishing outside a symmetric body

A subset K of $\mathbb{R}^d$ is called a symmetric body, if $-x \in K$ for all $x \in K$. The set $K^* := \{y \in \mathbb{R}^d,\ \langle x, y\rangle \le 1 \text{ for all } x \in K\}$ is called the polar set of K. We state now another real Paley-Wiener theorem given in [14]:

Theorem 6.1. *A function $f \in \mathcal{E}(\mathbb{R}^d)$ is the Dunkl transform of a function in $L^2_k(\mathbb{R}^d)$ vanishing outside a symmetric body K, if and only if, $T^\mu f$ belongs to $L^2_k(\mathbb{R}^d)$ for all $\mu = (\mu_1, ..., \mu_d) \in \mathbb{N}^d$, and for all $n \in \mathbb{N}$ we have*

$$\sup_{a \in K^*} \|(\langle a, T\rangle)^n f\|_{k,2} \le \|f\|_{k,2}, \tag{65}$$

where $T = (T_1, ..., T_d)$.

P r o o f. See [14], p.365-366. ■

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