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APPLICATIONS OF THE OWA-SRIVASTAVA OPERATOR TO THE CLASS OF k-UNIFORMLY CONVEX FUNCTIONS

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Abstract

By making use of the fractional differential operator Ω_z^λ $(0 \leq \lambda < 1)$ due to Owa and Srivastava, a new subclass of univalent functions denoted by $k-\mathcal{SP}_\lambda$ $(0 \leq k < \infty)$ is introduced. The class $k-\mathcal{SP}_\lambda$ unifies the concepts of k-uniformly convex functions and k-starlike functions. Certain basic properties of $k-\mathcal{SP}_\lambda$ such as inclusion theorem, subordination theorem, growth theorem and class preserving transforms are studied.

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1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions analytic in the *open* unit disc

$$\mathcal{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

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and let $\mathcal{A}_0$ be the class of functions f in $\mathcal{A}$ given by the *normalized* power series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathcal{U}). \tag{1.1}$$

The class $\mathcal{S}$ consists of *univalent* functions in $\mathcal{A}_0$. For fixed k $(0 \leq k < \infty)$, the function $f \in \mathcal{A}_0$ is said to be in $k - \mathcal{UCV}$; the class of *k-uniformly convex functions* in $\mathcal{U}$, if the image of every circular arc γ contained in $\mathcal{U}$, with center ξ, where $|\xi| \leq k$, is a convex arc (cf. [7]). The class $k - \mathcal{SP}$ is defined from $k - \mathcal{UCV}$ via the *Alexander's transform* (see [8]), i.e.

$$f \in k - \mathcal{UCV} \Longleftrightarrow g \in k - \mathcal{SP}, \text{ where } g(z) = zf'(z) \quad (z \in \mathcal{U}).$$

It is well known (cf. [7]) that $f \in k - \mathcal{UCV}$ (respectively $k - \mathcal{SP}$) if and only if the values of

$$p(z) = 1 + \frac{zf''(z)}{f'(z)} \left(\text{respectively } \frac{zf'(z)}{f(z)}\right) \quad (z \in \mathcal{U})$$

lie in the conic region Ω_k in the w-plane, where

$$\Omega_k := \{w = u + iv \in \mathbb{C} : u^2 > k^2(u-1)^2 + k^2v^2; 0 \leq k < \infty\}. \tag{1.2}$$

The purpose of the present note is to study some basic properties of the class $k - \mathcal{UCV}$ and $k - \mathcal{SP}$ in a more general setting of *fractional calculus*. We need to remind the following definitions.

If f and g are functions in $\mathcal{A}$ and given by the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ $(z \in \mathcal{U})$, then the *Hadamard product* (or *convolution*) of f and g denoted by $f * g$, is defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n = (g * f)(z) \quad (z \in \mathcal{U}).$$

Note that $f * g \in \mathcal{A}$. The Carlson-Shaffer [2] operator $\mathcal{L}(a, c)$ is defined in terms of the Hadamard product by

$$(\mathcal{L}(a,c)f)(z) := \Phi(a, c; z) * f(z) \quad (z \in \mathcal{U}, f \in \mathcal{A}), \tag{1.3}$$

where

$$\Phi(a, c; z) := \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1} \left(z \in \mathcal{U}, c \notin \mathbb{N}_0^- = \{0\} \cup \{-1, -2, -3, \dots\}\right) \tag{1.4}$$

and $(\lambda)_n$ is the Pochhammer symbol (or *shifted factorial*) defined in terms of the gamma function, by

$$(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0) \\ \lambda \ldots (\lambda+n-1) & (n \in \mathbb{N} := \{1,2,...\}). \end{cases}$$

Definition 1. (cf. [12],[13], see also [20],[21]) Let the function f be analytic in a simply connected region of the z-plane containing the origin. The *fractional derivative of f of order λ* is defined by

$$(D_z^\lambda f)(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\lambda} d\zeta \quad (0 \leq \lambda < 1),$$

where the multiplicity of $(z-\zeta)^\lambda$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Using Definition 1 and its known extensions involving fractional derivative and fractional integral, Owa and Srivastava [13] introduced the *fractional differintegral operator* $\Omega_z^\lambda : \mathcal{A}_0 \longrightarrow \mathcal{A}_0$ defined by

$$(\Omega_z^\lambda f)(z) = \Gamma(2-\lambda) z^\lambda (D_z^\lambda f)(z) \qquad (\lambda \neq 2,3,...; \;\; z \in \mathcal{U}).$$

Note that $\Omega_z^0 f(z) = f(z)$, $\Omega_z^1 f(z) = zf'(z)$ and

$$(\Omega_z^\lambda f)(z) = (\mathcal{L}(2, 2-\lambda)f)(z) \qquad (0 \leq \lambda < 1; \; z \in \mathcal{U}). \tag{1.5}$$

If f and g are functions in $\mathcal{A}$, we say that f is *subordinate* to g, written symbolically as $f \prec g$ in $\mathcal{U}$ or $f(z) \prec g(z)$ $(z \in \mathcal{U})$, if there exits a function $\omega \in \mathcal{A}$ satisfying the conditions of the Schwarz lemma such that $f(z) = g(\omega(z)), (z \in \mathcal{U})$. It is well known [4] that if g is univalent in $\mathcal{U}$, then $f \prec g$ in $\mathcal{U}$ is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$. We now introduce the following class of functions.

Definition 2. A function $f \in \mathcal{A}_0$ is said to be in the class $k-\mathcal{SP}_\lambda$ $(0 \leq \lambda < 1, \; 0 \leq k < \infty)$, if $\Omega_z^\lambda f \in k - \mathcal{SP}$. Or, equivalently:

$$\Re\left(\frac{z(\Omega_z^\lambda f)'(z)}{(\Omega_z^\lambda f)(z)} \right) > k \left| \frac{z(\Omega_z^\lambda f)'(z)}{(\Omega_z^\lambda f)(z)} - 1 \right| \qquad (z \in \mathcal{U}).$$

The class $k-\mathcal{SP}_\lambda$ unifies many classical and recently studied subclasses of $\mathcal{A}_0$ related to $\mathcal{S}$. Notably, for $k=0, \lambda=0 : 0-\mathcal{SP}_0 := \mathcal{S}^*$, the class of univalent starlike functions (see [4]); for $k=0, \lambda=1 : 0-\mathcal{SP}_1 := \mathcal{CV}$, the class of univalent convex functions (see [4]); for $k=0, \lambda \neq 0 : 0-\mathcal{SP}_\lambda := \mathcal{S}_\lambda$ (see [19]); for $k=1, \lambda \neq 0 : 1-\mathcal{SP}_\lambda := \mathcal{SP}_\lambda$ (see [18]); for $k=1, \lambda=0 : 1-\mathcal{SP}_0 := \mathcal{SP}$ (see [15]); for $k=1, \lambda=1 : 1-\mathcal{SP}_1 := \mathcal{UCV}$ (see [6]); for $k \neq 0, \lambda=0 : k-\mathcal{SP}_0 := k-\mathcal{SP}$ (see [8]) and for $k \neq 0, \lambda=1 : k-\mathcal{SP}_1 := k-\mathcal{UCV}$ (see [7]).

In the present article we investigate certain basic properties of the general class $k-\mathcal{SP}_\lambda$, such as inclusion theorem, subordination, growth theorem and class preserving transforms. Our results generalize and include some results found in [7], [8], [10] and [18].

2. Preliminary lemmas

We need the following results in our investigation:

Lemma 1. ([16]) *Let F and G be univalent convex functions in $\mathcal{U}$. Then their Hadamard product $F * G$ is also a univalent convex function in $\mathcal{U}$.*

Lemma 2. ([17]) *Let the functions F and G be univalent convex in $\mathcal{U}$. Also let $f \prec F$ and $g \prec G$ in $\mathcal{U}$. Then $f * g \prec F * G$ in $\mathcal{U}$.*

Lemma 3. ([16]) *Let each of the functions f and g be univalent starlike of order $1/2$. Then for every $F \in \mathcal{A}$*

$$\frac{f(z) * (g(z)F(z))}{f(z) * g(z)} \in \overline{CH}\{F(\mathcal{U})\}, \quad (z \in \mathcal{U}),$$

where $\overline{CH}$ denotes the closed convex hull.

Lemma 4. ([9]) *Let the function $h(z) = 1+h_1z+h_2z^2+\dots$ be univalent convex in $\mathcal{U}$. For $0 \le \lambda < 1$, if $\frac{\Omega_z^\lambda f(z)}{z} \prec h(z)$, then*

$$\frac{f(z)}{z} \prec \frac{1}{z}\{\mathcal{L}(2-\lambda, 2)[zh(z)]\}.$$

The result is the best possible.

3. Main results

We have the following:

Theorem 1. (Inclusion Theorem) *If $0 \le \mu < \lambda < 1$ and $1 \le k < \infty$, then*

$$k-\mathcal{SP}_\lambda \subset k-\mathcal{SP}_\mu.$$

P r o o f. Let $f \in k-\mathcal{SP}_\lambda$ and $0 \le \mu < \lambda < 1$. Then, by (1.5) and (1.4)

$$(\Omega_z^\mu f)(z) = \Phi(2-\lambda, 2-\mu; z) * \Omega_z^\lambda f(z)$$

$$\text{and} \quad z(\Omega_z^\mu f)'(z) = \Phi(2-\lambda, 2-\mu; z) * z(\Omega_z^\lambda f)'(z) \quad (z \in \mathcal{U}),$$

where Φ is the function defined by (1.4).

It is well known (cf.[9]) that $\Phi(2-\lambda, 2-\mu; z) \in \mathcal{S}^*(1/2)$ and since $1 \leq k < \infty$, the function $\Omega_z^\lambda f \in \mathcal{S}^*(1/2)$. Moreover, Ω_k defined by (1.2) is a convex region. Hence by Lemma 3, we get

$$\frac{z(\Omega_z^\mu f)'(z)}{\Omega_z^\mu f(z)} = \frac{\Phi(2-\lambda, 2-\mu; z) * \left\{\frac{z(\Omega_z^\lambda f)'(z)}{\Omega_z^\lambda f(z)} \Omega_z^\lambda f(z)\right\}}{\Phi(2-\lambda, 2-\mu; z) * \Omega_z^\lambda f(z)} \in \Omega_k$$

for every $z \in \mathcal{U}$. Therefore $f \in k - \mathcal{SP}_\mu$. The proof of Theorem 1 is complete. ∎

By taking $\lambda \to 1$ and $\mu = 0$, we have the following:

Corollary 1. *If $1 \leq k < \infty$ and $0 \leq \upsilon < 1$, then*

$$k - \mathcal{UCV} \subset k - \mathcal{SP}_\upsilon \subseteq k - \mathcal{SP} \subseteq \mathcal{SP} \quad \text{and } k - \mathcal{UCV} \subset \mathcal{UCV} \subset \mathcal{SP}.$$

Corollary 2. *If $1 \leq l \leq k < \infty$ and $0 \leq \mu < \lambda < 1$, then $k - \mathcal{SP}_\lambda \subseteq l - \mathcal{SP}_\mu$.* In particular, $k - \mathcal{SP}_\lambda \subset k - \mathcal{SP}\mu$.

P r o o f. Since $\frac{k}{k+1}$ is an increasing function of k, the result follows from the geometry of the region Ω_k. ∎

Remark 1. Taking $k = 1$ in Theorem 1 we obtain an inclusion result due to [18].

For $0 \leq k < \infty$, let q_k be the Riemann map of $\mathcal{U}$ onto the region Ω_k satisfying $q_k(0) = 1, q_k'(0) > 0$, where the region Ω_k is defined by (1.2). We define the function $\mathcal{G}$ on $\mathcal{U}$ by

$$\mathcal{G}(z) = \frac{1}{z}\left[\mathcal{L}(2-\lambda, 2)\left\{z \, \exp\left(\int_0^z \frac{q_k(t)-1}{t} dt\right)\right\}\right], \quad (z \in \mathcal{U}). \tag{3.1}$$

Theorem 2. *Let $0 \leq \lambda < 1, 1 \leq k < \infty$ and $\mathcal{G}$ be defined by (3.1). Then $\mathcal{G}$ is a univalent convex function. Furthermore, if $f \in k - \mathcal{SP}_\lambda$, then*

$$(i) \quad \frac{f(z)}{z} \prec \mathcal{G}(z) \quad (z \in \mathcal{U}), \tag{3.2}$$

$$(ii) \quad \mathcal{G}(-r) \leq \left|\frac{f(z)}{z}\right| \leq \mathcal{G}(r) \quad (|z| = r), \tag{3.3}$$

$$(iii) \quad \left|\arg\left(\frac{f(z)}{z}\right)\right| \leq \max_{\theta \in [0,2\pi]} \left\{\arg(\mathcal{G}(re^{i\theta}))\right\} \quad (|z| = r). \tag{3.4}$$

Equality holds in (3.3) and (3.4) for some $z \neq 0$ if and only if f is a rotation of $z\mathcal{G}$.

P r o o f. Write

$$g(z) = \exp\left(\int_0^z \frac{q_k(t)-1}{t}dt\right).$$

A calculation shows that for $z \in \mathcal{U}$

$$\Re\left\{1 + \frac{zg''(z)}{g'(z)}\right\} = \Re\left\{q_k(z) - 1 + \frac{zq_k'(z)}{q_k(z)-1}\right\} > \frac{k}{k+1} - 1 + \frac{1}{2} > 0.$$

Thus g is a univalent convex function. It is well known (cf. [3]) that $\frac{\Phi(2-\lambda,2;z)}{z}$ is a univalent convex function. Since $\mathcal{G}(z) = \frac{\Phi(2-\lambda,2;z)}{z} * g(z)$, $(z \in \mathcal{U})$, by an application of Lemma 1 we get that $\mathcal{G}$ is univalent convex.

Next, let $f \in k - \mathcal{SP}_\lambda, (0 \leq \lambda < 1, 1 \leq k < \infty)$, then by Definition 2,

$$\frac{z(\Omega_z^\lambda f)'(z)}{(\Omega_z^\lambda f)(z)} \prec q_k(z) \quad (z \in \mathcal{U}).$$

A result of Goluzin gives (cf. [5], also see [11, p.70], [14, p.51])

$$\frac{(\Omega_z^\lambda f)(z)}{z} \prec \exp\left(\int_0^z \frac{q_k(t)-1}{t}dt\right).$$

Now using Lemma 4, we get

$$\frac{f(z)}{z} \prec \mathcal{G}(z) \qquad (z \in \mathcal{U}).$$

This is precisely the assertion of (3.2). The estimates in (3.3) and (3.4) now follow as consequences of Lindelöf's principle of subordination. The proof of Theorem 2 is complete. ∎

Remark 2. Taking $k = 1$ and $k = 1, \lambda \to 1$ in Theorem 2, we get the subordination and growth theorems respectively in [18] and [10]. Theorem 2 also includes the subordination and growth theorems in [7] and [8], in particular case $\lambda \to 1$ and $\lambda = 0$ respectively.

Theorem 3. *If $f \in \mathcal{S}_\lambda(1/2)$ and $g \in k - \mathcal{SP}_\mu(\lambda \leq \mu, 1 \leq k < \infty)$, then $\Omega_z^\lambda f * \Omega_z^\mu g \in k - \mathcal{SP}$. In particular, if $f \in \mathcal{S}_\lambda(1/2)$ and $g \in k - \mathcal{SP}_\lambda$, then $\Omega_z^\lambda f * \Omega_z^\lambda g \in k - \mathcal{SP}$.*

P r o o f. By the definition of the class $\mathcal{S}_\lambda(1/2)$, the function $\Omega_z^\lambda f \in \mathcal{S}^*(1/2)$. Also, since $k \geq 1$, $\Omega_z^\mu g \in \mathcal{S}^*(1/2)$. Therefore by Lemma 3, $\Omega_z^\lambda f * \Omega_z^\mu g \in k - \mathcal{SP}$. The proof of Theorem 3 is complete. ∎

THEOREM 4. *Let* $f \in k-\mathcal{SP}$ *and* $g \in k-\mathcal{SP}_\lambda (0 \leq \lambda < 1, 1 \leq k < \infty)$. *Then:*

(a) $f * g \in k-\mathcal{SP}_\lambda$,

(b) *the function* $\mathcal{I}(g)$ *defined by the integral transform*

$$\mathcal{I}(g)(z) := \frac{\gamma+1}{z^\gamma}\int_0^z t^{\gamma-1} g(t)dt \qquad (z \in \mathcal{U},\ \gamma > -1) \tag{3.7}$$

is also in the class $k-\mathcal{SP}_\lambda$.

P r o o f. (a) Since $k \geq 1, f \in \mathcal{S}^*(1/2)$ and $\Omega_z^\lambda g \in \mathcal{S}^*(1/2)$. Therefore, by Lemma 3, $f * g \in k-\mathcal{SP}_\lambda$.

(b) The integral transform defined by (3.7) can be written in terms of the Carlson-Shaffer operator (cf. [2]) as $\mathcal{I}(g)(z) = \mathcal{L}(\gamma+1,\gamma+2)g(z)$. Therefore,

$$z(\Omega_z^\lambda \mathcal{I}(g))'(z) = \Phi(\gamma+1,\gamma+2;z) * z(\Omega_z^\lambda g)'(z).$$

Using a result of Bernardi [1], it can be verified that $\Phi(\gamma+1,\gamma+2;z) \in \mathcal{S}^*(1/2)$ and by hypothesis, $\Omega_z^\lambda g \in \mathcal{SP} \subset \mathcal{S}^*(1/2)$ $(1 \leq k < \infty)$. Therefore by Lemma 3, $\mathcal{I}(g) \in k-\mathcal{SP}_\lambda$. The proof of Theorem 4 is complete. ■

Taking $\lambda \to 1$ in Theorem 4(a), we get the following:

COROLLARY 3. *Let the functions* f *and* g *be* k*-uniformly convex in* $\mathcal{U}$ $(k \geq 1)$. *Then their Hadamard product* $f * g$ *is also* k*-uniformly convex in* $\mathcal{U}$.

REMARK 3. Taking $k = 1$ in Theorem 4, we readily get results found in [18].

THEOREM 5. *Let* $f_j \in k-\mathcal{SP}_\lambda (j = 1,\ldots,n;\ 0 \leq \lambda < 1;\ 0 \leq k < \infty)$ *and let* g *be defined by*

$$\Omega_z^\lambda g = \prod_{j=1}^n (\Omega_z^\lambda f_j)^{\alpha_j}, \tag{3.8}$$

with $\alpha_j > 0$ *and* $\sum_{j=1}^n \alpha_j = 1$. *Then* $g \in k-\mathcal{SP}_\lambda$.

P r o o f. An application of the triangle inequality gives

$$k\left|\frac{z(\Omega_z^\lambda g)'(z)}{(\Omega_z^\lambda g)(z)} - 1\right| \leq \alpha_1 k\left|\frac{z(\Omega_z^\lambda f_1)'(z)}{(\Omega_z^\lambda f_1)(z)} - 1\right| + \cdots + \alpha_n k\left|\frac{z(\Omega_z^\lambda f_n)'(z)}{(\Omega_z^\lambda f_n)(z)} - 1\right|$$
$$< \Re\Big(\sum_{j=1}^n \alpha_j \frac{z(\Omega_z^\lambda f_j)'(z)}{(\Omega_z^\lambda f_j)(z)}\Big) = \Re\Big(\frac{z(\Omega_z^\lambda g)'(z)}{(\Omega_z^\lambda g)(z)}\Big) \quad (z \in \mathcal{U}).$$

Thus by definition, $g \in k-\mathcal{SP}_\lambda$. The proof of Theorem 5 is complete. ■

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