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# ON MAXIMAL FUNCTION ON THE LAGUERRE HYPERGROUP

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*Dedicated to Professor Khalifa Trimèche, on the occasion of his 60th anniversary*

**Abstract**

Let $\mathbb{K} = [0, \infty) \times \mathbb{R}$ be the Laguerre hypergroup which is the fundamental manifold of the radial function space for the Heisenberg group. In this paper we consider the generalized shift operator, generated by Laguerre hypergroup, by means of which the maximal function is investigated. For $1 < p \leq \infty$ the $L_p(\mathbb{K})$-boundedness and weak $L_1(\mathbb{K})$-boundedness result for the maximal function is obtained.

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*Key Words and Phrases*: Laguerre hypergroup, generalized translation operator, Fourier-Laguerre transform, maximal function

## 1. Introduction

The Hardy–Littlewood maximal function is an important tool of harmonic analysis. It was first introduced by Hardy and Littlewood in 1930 (see [12]) for functions defined on the circle, and later it was extended to the Euclidean spaces, various Lie groups, symmetric spaces, and some weighted

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measure spaces (see [7], [8], [16], [19], [20]). In the setting of hypergroups, some versions of Hardy–Littlewood maximal functions were given in [5] for the Jacobi hypergroups of compact type, in [6] for the Jacobi-type hypergroups, and in [17] for the one-dimensional Bessel-Kingman hypergroups, in [2] for the one-dimensional Chebli-Trimeche hypergroups, and in [9] (see also [10]) for the $n$-dimensional Bessel-Kingman hypergroups ($n \geq 1$).

In the present work, we study the maximal function on the Laguerre hypergroup, so we fix $\alpha \geq 0$ and $\mathbb{K} = [0, \infty) \times \mathbb{R}$ and we define the maximal function using the harmonic analysis on the Laguerre hypergroup which can be seen as a deformation of the hypergroup of radial functions on the Heisenberg group (see, for example [1, 14, 15, 18]). For $1 < p \leq \infty$ the $L_p(\mathbb{K})$-boundedness and weak $L_1(\mathbb{K})$-boundedness result for the maximal function is obtained. The proof of this result extends ideas and techniques of [10].

## 2. Preliminaries

We consider the following partial differential operators:

$$\begin{cases} D_1 = \dfrac{\partial}{\partial t}, \\ D_2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{2\alpha+1}{x}\dfrac{\partial}{\partial x} + x^2\dfrac{\partial^2}{\partial t^2}, \\ (x,t) \in ]0,\infty[\times\mathbb{R} \quad \text{and} \quad \alpha \in [0,\infty[. \end{cases}$$

For $\alpha = n-1$, $n \in \mathbb{N} \setminus \{0\}$, the operator $D_2$ is the radial part of the sub-Laplacian on the Heisenberg group $\mathbb{H}_n$.

For $(\lambda, m) \in \mathbb{R} \times \mathbb{N}$, the initial value problem

$$\begin{cases} D_1 u = i\lambda u, \\ D_2 u = -4|\lambda|\left(m + \frac{\alpha+1}{2}\right) u; \\ u(0,0) = 1, \quad \dfrac{\partial u}{\partial x}(0,t) = 0 \quad \text{for all} \quad t \in \mathbb{R}, \end{cases}$$

has a unique solution $\varphi_{\lambda,m}$ given by

$$\varphi_{\lambda,m}(x,t) = e^{i\lambda t}\mathcal{L}_m^{(\alpha)}\left(|\lambda|x^2\right), \quad (x,t) \in \mathbb{K},$$

where $\mathcal{L}_m^{(\alpha)}$ is the Laguerre function defined on $\mathbb{R}_+$ by

$$\mathcal{L}_m^{(\alpha)}(x) = e^{-x/2}L_m^{(\alpha)}(x)/L_m^{(\alpha)}(0)$$

and $L_m^{(\alpha)}$ is the Laguerre polynomial of degree $m$ and order $\alpha$ (see [1]).

Let $\alpha \geq 0$ be a fixed number, $\mathbb{K} = [0, \infty) \times \mathbb{R}$ and $m_\alpha$ the weighted Lebesgue measure on $\mathbb{K}$, given by

$$dm_\alpha(x,t)=\frac{x^{2\alpha+1}dxdt}{\pi\Gamma(\alpha+1)},\quad \alpha\geq 0.$$

For every $1\leq p\leq\infty$, we denote by $L_p(\mathbb{K})=L_p(\mathbb{K};dm_\alpha)$ the spaces of complex-valued functions $f$, measurable on $\mathbb{K}$ such that

$$\|f\|_{L_p(\mathbb{K})}=\left(\int_{\mathbb{K}}|f(x,t)|^p\,dm_\alpha(x,t)\right)^{1/p}<\infty,\quad \text{if}\quad p\in[1,\infty),$$

and

$$\|f\|_{L_\infty(\mathbb{K})}=ess\sup_{(x,t)\in\mathbb{K}}|f(x,t)|.$$

Let $|(x,t)|_{\mathbb{K}}=(x^4+t^2)^{1/4}$ be the homogeneous norm of $(x,t)\in\mathbb{K}$ and $r>0$. We will denote by $\delta_r(x,t)=(rx,r^2t)$, the dilation of $(x,t)\in\mathbb{K}$ and by $B_r(x,t)$ the ball centered at $(x,t)$ of radius $r$, i.e., the set $B_r(x,t)=\{(y,s)\in\mathbb{K}:|(x-y,t-s)|_{\mathbb{K}}<r\}$. Let also $B_r=B_r(0,0)$.

We denote by

$$f_r(x,t)=r^{-(2\alpha+4)}f\big(\delta_{\frac{1}{r}}(x,t)\big)$$

the dilation of the function $f$ defined on $\mathbb{K}$ preserving the mean value of $f$ with respect to the measure $dm_\alpha$, in the sense that

$$\int_{\mathbb{K}}f_r(x,t)dm_\alpha(x,t)=\int_{\mathbb{K}}f(x,t)dm_\alpha(x,t),\quad \forall r>0\ \text{and}\ \ f\in L_1(\mathbb{K}).$$

The Fourier-Laguerre transform $\mathcal{F}$ is defined for $f\in L_1(\mathbb{K})$ by:

$$\mathcal{F}(f)(\lambda,m)=\int\limits_{\mathbb{K}}\varphi_{-\lambda,m}(x,t)f(x,t)dm_\alpha(x,t)$$

and we have (see [1, 15])

$$\|\mathcal{F}(f)\|_{L_\infty(\mathbb{K})}\leq\|f\|_{L_1(\mathbb{K})}.$$

For $(x,t),(y,s)\in\mathbb{K}$ and $\theta\in[0,2\pi)$, $r\in[0,1)$ let

$$((x,t),(y,s))_{\theta,r}=\left(\left(x^2+y^2+2xyr\cos\theta\right)^{1/2},t+s+xyr\sin\theta\right).$$

The generalized translation operators $T^{(\alpha)}_{(x,t)}$ on the Laguerre hypergroup are given for a suitable function $f$ by

$$T^{(\alpha)}_{(x,t)}f(y,s)=\begin{cases}\frac{1}{2\pi}\int_0^{2\pi}f\left(((x,t),(y,s))_{\theta,1}\right)d\theta, & \text{if } \alpha=0,\\ \frac{\alpha}{\pi}\int_0^1\left(\int_0^{2\pi}f\left(((x,t),(y,s))_{\theta,r}\right)d\theta\right)r(1-r^2)^{\alpha-1}dr, & \text{if } \alpha>0.\end{cases}$$

They satisfy the following properties (see [1], [15]):

$$T^{(\alpha)}_{(x,t)}f(y,s)=T^{(\alpha)}_{(y,s)}f(x,t),\quad T^{(\alpha)}_{(0,0)}f(y,s)=f(y,s),$$

$$\|T^{(\alpha)}_{(x,t)}f\|_{L_p(\mathbb{K})}\leq\|f\|_{L_p(\mathbb{K})}\quad\text{for}\quad f\in L_p(\mathbb{K}),$$

$$\mathcal{F}\big(T^{(\alpha)}_{(x,t)}f\big)(\lambda,m)=\mathcal{F}(f)(\lambda,m)\ \varphi_{\lambda,m}(x,t).$$

In [14] the translation operator $T^{(\alpha)}_{(x,t)}$ is given by:

$$T^{(\alpha)}_{(x,t)}f(y,s)=\int_{\mathbb{K}}f(z,v)W_\alpha((x,t),(y,s),(z,v))z^{2\alpha+1}dzdv,$$

where $dzdv$ is the Lebesgue measure on $\mathbb{K}$, and $W_\alpha$ is an appropriate kernel satisfying

$$\int_{\mathbb{K}}W_\alpha((x,t),(y,s),(z,v))z^{2\alpha+1}dzdv=1.$$

For all $(\lambda,m)\in\mathbb{R}\times\mathbb{N}$, the function $\varphi_{\lambda,m}(x,t)$ satisfies the following product formula

$$\varphi_{\lambda,m}(x,t)\,\varphi_{\lambda,m}(y,s)=T^{(\alpha)}_{(x,t)}\varphi_{\lambda,m}(y,s).$$

Using the generalized translation operators $T^{(\alpha)}_{(x,t)}$, $(x,t)\in\mathbb{K}$, we define a generalized convolution product $*$ on $\mathbb{K}$ by

$$\left(\delta_{(x,t)}*\delta_{(y,s)}\right)(f)=T^{(\alpha)}_{(x,t)}f(y,s),$$

where $\delta_{(x,t)}$ is the Dirac measure at $(x,t)$.

The convolution product is defined on the space $M_b(\mathbb{K})$ of bounded Radon measures on $\mathbb{K}$ by

$$(\mu*\nu)(f)=\int_{\mathbb{K}\times\mathbb{K}}T^{(\alpha)}_{(x,t)}f(y,s)\,d\mu(x,t)\,d\nu(y,s).$$

When $\mu=h\cdot m_\alpha$ and $\nu=g\cdot m_\alpha$, with $h$ and $g$ in the space $L_1(\mathbb{K})$ of integrable functions on $\mathbb{K}$ with respect to the measure $dm_\alpha(x,t)$, we have

$$\mu*\nu=(f*\check{g})m_\alpha,\ \text{ with }\ \check{g}(y,s)=g(y,-s),$$

where $h*g$ is the convolution product of $h$ and $g$ defined by:

$$(h*g)(x,t)=\int_{\mathbb{K}}T^{(\alpha)}_{(x,t)}h(y,s)\,g(y,-s)\,dm_\alpha(y,s),\quad\text{for all}\quad(x,t)\in\mathbb{K}.$$

$(M_b(\mathbb{K}), *, i)$ is an involutive Banach algebra, where $i$ is the involution on $\mathbb{K}$ given by $i(x,t) = (x,-t)$ and the convolution product $*$ satisfies all the conditions of Jewett (see [13]), hence $(\mathbb{K}, *, i)$ is a hypergroup in the sense of Jewett (see [3], [13]) and the functions $\varphi_{\lambda,m}$ are characters of $\mathbb{K}$. If $\alpha = n-1$ is a nonnegative integer, then the Laguerre hypergroup $\mathbb{K}$ can be identified with the hypergroup of radial functions on the Heisenberg group $\mathcal{H}_n$.

## 3. Polar coordinates in Laguerre hypergroup $\mathbb{K}$ and some lemmas

Let $\Sigma = \Sigma_2$ denote the unit sphere in $\mathbb{K}$, $\omega_2$ its surface area and $\Omega_2$ its volume (see [11]). For $\xi = (x,t) \in \mathbb{K}$, we consider the transformation given by

$$x = r(\cos\varphi)^{1/2},\ t = r^2 \sin\varphi,$$

where $-\pi/2 \le \varphi \le \pi/2$, $r = |\xi|_{\mathbb{K}}$ and $\xi' = ((\cos\varphi)^{1/2}, \sin\varphi) \in \Sigma$.

The Jacobian associated with the above transformation is equal to $r^{2\alpha+3}(\cos\varphi)^\alpha$, thus if $f$ is integrable in $\mathbb{K}$,

$$\int_{\mathbb{K}} f(x,t)\, dm_\alpha(x,t) = \frac{1}{2\pi\Gamma(\alpha+1)} \int_{-\pi/2}^{\pi/2} \int_0^\infty f\big(r(\cos\varphi)^{1/2}, r^2\sin\varphi\big)\, r^{2\alpha+3}(\cos\varphi)^\alpha dr d\varphi.$$

We write

$$\frac{1}{2\pi\Gamma(\alpha+1)} \int_{-\pi/2}^{\pi/2} (\cos\varphi)^\alpha d\varphi = \int_\Sigma d\xi'$$

and thus

$$\int_{\mathbb{K}} f(x,t)\, dm_\alpha(x,t) = \int_\Sigma \int_0^\infty r^{2\alpha+3} f(\delta_r \xi')\, dr d\xi'. \tag{1}$$

Here $d\xi'$ is called the surface area element on $\Sigma$.

LEMMA 1. *The following formulas*

$$\omega_2 = \frac{\Gamma(\frac{\alpha+1}{2})}{2\sqrt{\pi}\Gamma(\alpha+1)\Gamma(\frac{\alpha}{2}+1)}$$

*and*

$$\Omega_2 = \frac{\Gamma(\frac{\alpha+1}{2})}{4\sqrt{\pi}(\alpha+2)\Gamma(\alpha+1)\Gamma(\frac{\alpha}{2}+1)}$$

*are valid.*

P r o o f. We have

$$\omega_2 = \int_\Sigma d\xi' = \frac{1}{2\pi\Gamma(\alpha+1)} \int_{-\pi/2}^{\pi/2} (\cos\varphi)^\alpha d\varphi = \frac{\Gamma(\frac{\alpha+1}{2})}{2\sqrt{\pi}\Gamma(\alpha+1)\Gamma(\frac{\alpha}{2}+1)}.$$

By (1),

$$\Omega_2 = \int_{B_1} dm_\alpha(x,t) = \int_\Sigma \int_0^1 r^{2\alpha+3} dr d\xi' = \frac{\omega_2}{2\alpha+4}$$

$$= \frac{\Gamma(\frac{\alpha+1}{2})}{4\sqrt{\pi}(\alpha+2)\Gamma(\alpha+1)\Gamma(\frac{\alpha}{2}+1)}.$$

■

Observe that for any $(x,t) \in \mathbb{K}$ and $r > 0$ the area of the sphere $S_r(x,t)$ of center $(x,t)$ and radius $r$ is equal to $r^{2\alpha+3}\omega_2$ and its volume is equal to $r^{2\alpha+4}\Omega_2 = r^{2\alpha+4}\frac{\omega_2}{2\alpha+4}$.

LEMMA 2. *The function $f(x,t) = |(x,t)|_{\mathbb{K}}^\lambda$ is integrable in any neighborhood of the origin if and only if $\lambda > -2\alpha - 4$, and it is integrable in the complement of any neighborhood of the origin if and only if $\lambda < -2\alpha - 4$.*

P r o o f. For a fixed $\lambda \neq -2\alpha - 4$ and $0 < a < b < \infty$, we have

$$\int_{a<|(x,t)|_{\mathbb{K}}<b} |(x,t)|_{\mathbb{K}}^\lambda \, dm_\alpha(x,t)$$

$$= \int_\Sigma \int_a^b r^{2\alpha+3+\lambda} dr d\xi' = \frac{\omega_2}{2\alpha+4}(b^{2\alpha+4+\lambda} - a^{2\alpha+4+\lambda}),$$

and thus the thesis follows. ■

LEMMA 3. *For every $(x,t), (y,s) \in \mathbb{K}$ and $r > 0$ the function $T^{(\alpha)}_{(x,t)}\chi_{B_r}(y,s)$ is supported in $\widetilde{B}_r(x,t)$ and the following inequalities are valid:*

$$m_\alpha B_r(x,t) \leq C r^{2\alpha+4} \max\{1, (x/r)^{2\alpha+1}\},$$

$$m_\alpha \widetilde{B}_r(x,t) \leq C r^{2\alpha+4} \max\{1, (x/r)^{2\alpha+3}\},$$

*where $\widetilde{B}_r(x,t) = \{(y,s) \in \mathbb{K} \, : \, |x-y| < r, \, |t-s| < x(x+r)\}$.*

P r o o f. Note that $T^{(\alpha)}_{(x,t)}\chi_{B_r}(y,s) = 0$ for any $(y,s) \in \mathbb{K} \setminus \widetilde{B}_r(x,t)$, and this means that the support of function $T^{(\alpha)}_{(x,t)}\chi_{B_r}(y,s)$ belongs to $\widetilde{B}_r(x,t)$.

Further,

$$m_\alpha B_r(x,t) = m_\alpha B_r(x,0)$$

$$= \int\limits_{\{(y,s)\in\mathbb{K}:\,|(x-y,\,s)|_{\mathbb{K}}<r\}} dm_\alpha(y,s) \le \int\limits_{(x-r)_+}^{x+r} y^{2\alpha+1}dy \int\limits_{-r^2}^{r^2} ds$$

$$\le C \begin{cases} r^{2\alpha+4}, & x \le r \\ r^3 x^{2\alpha+1}, & x > r \end{cases} = Cr^{2\alpha+4}\max\{1, (x/r)^{2\alpha+1}\},$$

and

$$m_\alpha \widetilde{B}_r(x,t) = \int\limits_{\{(y,s)\in\mathbb{K}:\,|x-y|<r,\,|t-s|<x(x+r)\}} dm_\alpha(y,s)$$

$$\le \int\limits_{(x-r)_+}^{x+r} y^{2\alpha+1}dy \int\limits_{-x(x+r)}^{x(x+r)} ds$$

$$\le C \begin{cases} r^{2\alpha+4}, & x \le r \\ rx^{2\alpha+3}, & x > r \end{cases} = Cr^{2\alpha+4}\max\{1, (x/r)^{2\alpha+3}\},$$

where $(x-r)_+ = \max\{x-r, 0\}$.

Thus the proof of Lemma 3 is completed. ∎

## 4. $L_p$-boundedness of maximal function on the Laguerre hypergroup

We now consider the maximal operator

$$Mf(x,t) = \sup_{r>0} \frac{1}{m_\alpha B_r} \int_{B_r} T^{(\alpha)}_{(x,t)}|f(y,s)|\, dm_\alpha(y,s).$$

THEOREM 1. *1) If $f \in L_1(\mathbb{K})$, then for every $\beta > 0$*

$$m_\alpha\{(x,t) \in \mathbb{K} : Mf(x,t) > \beta\} \le \frac{C}{\beta}\int_{\mathbb{K}} |f(x,t)|\, dm_\alpha(x,t),$$

*where $C > 0$ is independent of $f$.*

*2) If* $f \in L_p(\mathbb{K}), \quad 1 < p \le \infty$, *then* $Mf \in L_p(\mathbb{K})$ *and*

$$\|Mf\|_{L_p(\mathbb{K})} \le C_p \|f\|_{L_p(\mathbb{K})},$$

*where* $C_p > 0$ *is independent of* $f$.

P r o o f. The maximal function $Mf(x,t)$ may be interpreted as a maximal function defined on a space of homogeneous type. By this we mean a topological space $X$ equipped with a continuous pseudometric $\rho$ and a positive measure $\nu$ satisfying

$$\nu(E(\xi, 2r)) \le C_1 \nu(E(\xi, r)) \tag{2}$$

with a constant $C_1$ independent of $\xi$ and $r > 0$. Here $E(\xi, r) = \{\eta \in X : \rho(\xi, \eta) < r\}$, $\rho(\xi, \eta) = |\xi - \eta|$. Let $(X, \rho, \nu)$ be a space of homogeneous type. Define

$$M_\nu f(x) = \sup_{r>0} \nu(E(\xi, r))^{-1} \int_{E(\xi,r)} |f(\eta)| d\nu(\eta).$$

It is well known that the maximal operator $M_\nu$ is of weak type (1, 1) and is bounded on $L_p(X, d\nu)$ for $1 < p < \infty$ (see [4]). We shall use this result in the case in which $X = \mathbb{K}$, $\xi = (x,t)$, $\eta = (y,s) \in \mathbb{K}$, $\rho(\xi, \eta) = \max\{|x - y|, (t - s)^2\}$, $d\nu(\xi) = dm_\alpha(x,t)$. It is clear that this measure satisfies the doubling condition (2).

We will shall show that

$$Mf(x,t) \le C M_\nu f(\xi). \tag{3}$$

From the definition of the generalized shift operator it follows that $T^{(\alpha)}_{(x,t)} \chi_{B_r}(y,s)$ is supported in $\widetilde{B}_r(x,t)$.

Moreover, there exists a constant $C_2$ such that

$$0 \le T^{(\alpha)}_{(x,t)} \chi_{B_r}(y,s) \le \min\{1, C_2 (r/x)^{2\alpha+3}\}, \qquad \forall (y,s) \in \widetilde{B}_r(x,t). \tag{4}$$

In the case $x \le r$ this follows from the simple inequality $0 \le T^{(\alpha)}_{(x,t)} \chi_{B_r}(y,s) \le 1$.

Thus

$$Mf(x,t) \le M_1 f(x,t) + M_2 f(x,t),$$

where

$$M_1 f(x,t) = \sup_{r \ge x} \frac{1}{m_\alpha B_r} \int_{B_r} T^{(\alpha)}_{(x,t)} |f(y,s)| \, dm_\alpha(y,s),$$

$$M_2 f(x,t) = \sup_{r < x} \frac{1}{m_\alpha B_r} \int_{B_r} T^{(\alpha)}_{(x,t)} |f(y,s)| \, dm_\alpha(y,s).$$

If $r \geq x$, then from Lemma 3, $m_\alpha \widetilde{B}_r(x,t) \leq Cr^{2\alpha+4}$, also $m_\alpha B_r = \Omega_2 r^{2\alpha+4}$ and $T^{(\alpha)}_{(x,t)}\chi_{B_r}(y,s) \leq 1$ for all $(y,s) \in \widetilde{B}_r(x,t)$. Thus we have

$$M_1 f(x,t) \leq \sup_{r\geq x} \frac{1}{m_\alpha B_r} \int_{\widetilde{B}_r(x,t)} |f(y,s)|\, T^{(\alpha)}_{(x,t)}\chi_{B_r}(y,s)\, dm_\alpha(y,s)$$
$$\leq C \sup_{r\geq x} \frac{1}{m_\alpha \widetilde{B}_r(x,t)} \int_{\widetilde{B}_r(x,t)} |f(y,s)|\, dm_\alpha(y,s) \leq CM_\nu f(\xi).$$

If $r < x$, then by Lemma 3 and (4), $m_\alpha \widetilde{B}_r(x,t) \leq Crx^{2\alpha+3}$ and $T^{(\alpha)}_{(x,t)}\chi_{B_r}(y,s) \leq (r/x)^{2\alpha+3}$ for all $(y,s) \in \widetilde{B}_r(x,t)$. This yields

$$M_2 f(x,t) \leq \sup_{r\geq x} \frac{1}{m_\alpha B_r} \int_{\widetilde{B}_r(x,t)} |f(y,s)|\, T^{(\alpha)}_{(x,t)}\chi_{B_r}(y,s)\, dm_\alpha(y,s)$$
$$\leq C \sup_{r<x} \frac{1}{m_\alpha \widetilde{B}_r(x,t)} \int_{\widetilde{B}_r(x,t)} |f(y,s)|\, dm_\alpha(y,s) \leq CM_\nu f(\xi).$$

Therefore we get (3), which completes the proof. ∎

COROLLARY 1. *If $f \in L_{loc}(\mathbb{K})$, then*

$$\lim_{r\to 0} \frac{1}{m_\alpha B_r} \int_{B_r} \left|T^{(\alpha)}_{(x,t)} f(y,s) - f(x,t)\right| dm_\alpha(y,s) = 0$$

*for a. e. $(x,t) \in \mathbb{K}$.*

As an application, we give a result about approximations of the identity. The maximal function can be used to study almost everywhere convergence of $f * \varphi_\varepsilon$ as they can be controlled by the Hardy-Littlewood maximal function $Mf$ under some conditions on $\varphi$.

THEOREM 2. *Let $\psi$ a nonnegative and decreasing function on $[0,\infty)$, $|\varphi(x,t)| \leq \psi(|(x,t)|)$ and $\psi(|(x,t)|) \in L_1(\mathbb{K})$. Then there exists a constant $C > 0$ such that*

$$M_\varphi f(x,t) = \sup_{r>0} |(f * \varphi_r)(x,t)| \leq CMf(x,t).$$

P r o o f. We have

$$|(f*\varphi_r)(x,t)| = \left|\int_{\mathbb{K}} T^{(\alpha)}_{(x,t)} f(y,s)\varphi_r(y,-s)\,dm_\alpha(y,s)\right|$$
$$\le \sum_{k=-\infty}^{\infty} r^{-2\alpha-4}\int_{B_{2^{k+1}r}\setminus B_{2^k r}} \psi\big(r^{-1}|(y,s)|_{\mathbb{K}}\big)T^{(\alpha)}_{(x,t)}|f(y,s)|\,dm_\alpha(y,s)$$
$$\le \sum_{k=-\infty}^{\infty} r^{-2\alpha-4}\psi(2^k)\int_{B_{2^{k+1}r}\setminus B_{2^k r}} T^{(\alpha)}_{(x,t)}|f(y,s)|\,dm_\alpha(y,s)$$
$$\le \left(\Omega_2\sum_{k=-\infty}^{\infty} 2^{(k+1)(2\alpha+4)}\psi(2^k)\right)Mf(x,t)$$
$$= 2^{2\alpha+4}\left(\Omega_2\sum_{k=-\infty}^{\infty} 2^{k(2\alpha+4)}\psi(2^k)\right)Mf(x,t),$$

where we have used the fact that $m_\alpha B_r = \Omega_2 r^{2\alpha+4}$.

On the other hand, we have

$$\int_{\mathbb{K}}\psi(|(x,t)|_{\mathbb{K}})dm_\alpha(x,t) = \sum_{k=-\infty}^{\infty}\int_{B_{2^{k+1}r}\setminus B_{2^k r}}\psi(|(x,t)|_{\mathbb{K}})\,dm_\alpha(x,t)$$
$$\ge \Omega_2\sum_{k=-\infty}^{\infty}\left(2^{k(2\alpha+4)} - 2^{(k-1)(2\alpha+4)}\right)\psi(2^k)$$
$$= \left(1-2^{-2\alpha-4}\right)\left(\Omega_2\sum_{k=-\infty}^{\infty}2^{k(2\alpha+4)}\psi(2^k)\right).$$

The proposition is proved, as $\psi(|(x,t)|)\in L_1(\mathbb{K})$. ∎

Corollary 2. *Let $\varphi\in L_1(\mathbb{K})$ and assume $\int_{\mathbb{K}}\varphi(x,t)\,dm_\alpha(x,t)=1$. Then for $f\in L_p(\mathbb{K})$, $1\le p<\infty$*

$$\lim_{r\to 0}\|f*\varphi_r - f\|_{L_p(\mathbb{K})} = 0.$$

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