Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 9, NUMBER 3 (2006) ISSN 1311-0454

# MEAN-PERIODIC FUNCTIONS ASSOCIATED WITH THE JACOBI-DUNKL OPERATOR ON $I\!R$

**N. Ben Salem [1], A. Ould Ahmed Salem [2], B. Selmi [3]**

*Dedicated to Professor Khalifa Trimèche,*
*on the occasion of his 60th anniversary*

**Abstract**

Using a convolution structure on the real line associated with the Jacobi-Dunkl differential-difference operator $\Lambda_{\alpha,\beta}$ given by:

$$\Lambda_{\alpha,\beta} f(x) = f^{'}(x) + ((2\alpha+1)\coth x + (2\beta+1)\tanh x)\left(\frac{f(x)-f(-x)}{2}\right),$$

$\alpha \geq \beta \geq -\frac{1}{2}$, we define mean-periodic functions associated with $\Lambda_{\alpha,\beta}$. We characterize these functions as an expansion series intervening appropriate elementary functions expressed in terms of the derivatives of the eigenfunction of $\Lambda_{\alpha,\beta}$. Next, we deal with the Pompeiu type problem and convolution equations for this operator.

*2000 Mathematics Subject Classification*: 34K99, 44A15, 44A35, 42A75, 42A63

*Key Words and Phrases*: Jacobi-Dunkl operator, mean periodic function, Jacobi-Dunkl expansion, Pompeiu problem

## 0. Introduction

For $\alpha \geq \beta \geq -\frac{1}{2}$, we consider the Jacobi-Dunkl operator $\Lambda_{\alpha,\beta}$ defined on $C^1(I\!R)$ by

$$\Lambda_{\alpha,\beta} f(x) = f^{'}(x) + ((2\alpha+1)\coth x + (2\beta+1)\tanh x)\left(\frac{f(x)-f(-x)}{2}\right).$$

We point out that the operator $\Lambda_{\alpha,\beta}$ coincides with the Heckman-Opdam operator, known also as the Dunkl-Heckman operator:

$$D_\xi = \partial_\xi + \frac{1}{2}\sum_{\alpha\in\mathcal{R}_+} k_a a(\xi)\frac{1+e^{-a}}{1-e^{-a}}(1-r_a),$$

on $I\!R$, with $\mathcal{R}_+ = \{2,4\}$ and suitable choice of $k_\alpha$'s (see [17] and the references cited therein).

The eigenfunction of this operator, satisfying $\Lambda_{\alpha,\beta}f(x) = i\lambda f(x),\ \lambda\in\mathbb{C}$ and $f(0)=1$, can be expressed in terms of the Jacobi functions $\varphi_\mu^{\alpha,\beta}$ and $\varphi_\mu^{\alpha+1,\beta+1}$, namely:

$$\Psi_\lambda^{\alpha,\beta}(x) = \varphi_\mu^{\alpha,\beta}(x) + \frac{i\lambda}{2(\alpha+1)}\sinh x\cosh x\ \varphi_\mu^{\alpha+1,\beta+1}(x),$$

where

$$\lambda^2 = \mu^2+\rho^2 \quad \text{with} \quad \rho = \alpha+\beta+1,$$

and

$$\varphi_\mu^{\alpha,\beta}(x) = {}_2F_1\Big(\frac{\rho+i\mu}{2},\frac{\rho-i\mu}{2};\alpha+1;-\sinh^2 x\Big).$$

We note that in [2], the authors have established a product formula for the eigenfunction $\Psi_\lambda^{\alpha,\beta},\ \ \lambda\in\mathbb{C}$ : For $x,y\in I\!R$

$$\Psi_\lambda^{\alpha,\beta}(x)\Psi_\lambda^{\alpha,\beta}(y) = \int_{I\!R}\Psi_\lambda^{\alpha,\beta}(z)d\mu_{x,y}^{\alpha,\beta}(z),$$

where $\mu_{x,y}^{\alpha,\beta}$ is a real uniformly bounded measure with compact support, which may not be positive. This leads in a natural way to define the translation operators, denoted $T_x^{\alpha,\beta}$, $x\in I\!R$, by

$$\forall y\in I\!R, T_x^{\alpha,\beta}f(y) = \int_{I\!R} f(z)d\mu_{x,y}^{\alpha,\beta}(z).$$

Here $f$ is a measurable function on $I\!R$.

We point out that in [10], the authors have shown that for all $\lambda\in\mathbb{C}$, the eigenfunction $\Psi_\lambda^{\alpha,\beta}$, admits an integral representation which permits to define an intertwining operator $V_{\alpha,\beta}$ on $\mathcal{E}(I\!R)$, the space of $C^\infty$-functions on $I\!R$, by

$$V_{\alpha,\beta}f(x) = \begin{cases} \displaystyle\int_{-|x|}^{|x|} K(x,y)f(y)dy, & \text{if} \quad x\in I\!R\setminus\{0\},\\ f(0), & \text{if} \quad x=0,\end{cases}$$

where $K(x,.)$ is a positive function on $\mathbb{R}$, continuous on $]-|x|,|x|[$ and supported in $[-|x|,|x|]$, $V_{\alpha,\beta}$ intertwines $\Lambda_{\alpha,\beta}$ and the usual derivative $D = \frac{d}{dx}$: $\Lambda_{\alpha,\beta}V_{\alpha,\beta} = V_{\alpha,\beta}D$.

This intertwining operator leads also to define $T_x^{\alpha,\beta}f$ as follows

$$T_x^{\alpha,\beta}f(y) = V_{\alpha,\beta,x}V_{\alpha,\beta,y}(V_{\alpha,\beta}^{-1}(f)(x+y)).$$

Of course, the two formulas defining $T_x^{\alpha,\beta}f$ coincide on $\mathcal{E}(\mathbb{R})$.

A function $f$ in $\mathcal{E}(\mathbb{R})$ is called mean-periodic associated with the operator $\Lambda_{\alpha,\beta}$, if there exists a non zero distribution $\mu \in \mathcal{E}'(\mathbb{R})$, such that

$$\mu *_{\alpha,\beta} f(x) = 0,$$

where, for all $x \in \mathbb{R}$,

$$\mu *_{\alpha,\beta} f(x) = \langle \mu_y \, , \, T_{-x}^{\alpha,\beta}\check{f}(y) \rangle,$$

here $\check{f}(u) = f(-u)$.

Using the operator $V_{\alpha,\beta}$ and the results of L. Schwartz in [19] for the classical case, we give a representation of a mean-periodic function $f$ in $\mathcal{E}(\mathbb{R})$, associated with $\Lambda_{\alpha,\beta}$ , in terms of a series intervening elementary functions $\Psi_{\lambda,l}^{\alpha,\beta}(x)$, defined by $\dfrac{d^l}{dt^l}(\Psi_{-it}^{\alpha,\beta}(x))_{t=i\lambda} = V_{\alpha,\beta}(x^l e^{i\lambda x})$, which we call exponential-monomials associated with $\Lambda_{\alpha,\beta}$. Namely, we have (formally)

$$f(x) = \sum_{(\lambda,l)} \sum_{0\le j\le l-1} c_{\lambda,j}\Psi_{\lambda,j}^{\alpha,\beta}(x), \quad c_{\lambda,j} \in \mathbb{C},$$

the summation is extended over the distinct roots of $\mathcal{F}_{\alpha,\beta}(\mu)$ counted with multiplicities $l$, where $\mathcal{F}_{\alpha,\beta}(\mu)$ is the Jacobi-Dunkl transform of $\mu$ defined by

$$\mathcal{F}_{\alpha,\beta}(\mu)(\lambda) = \langle \mu \, , \, \Psi_{-\lambda}^{\alpha,\beta} \rangle.$$

For general $\mu$ one can get convergence of the series expansion in the topology of $\mathcal{E}(\mathbb{R})$ only if one groups the terms and then uses the Abel summability. For a wide class of distributions $\mu$ the Abelian summation process is not necessary. More precisely, if one assumes that $\mu \in \mathcal{E}'(\mathbb{R})$ is slowly-decreasing in the following mean: there are $A, \varepsilon > 0$ such that for any $x \in \mathbb{R}$

$$\mathrm{Max}\{|\mathcal{F}_{\alpha,\beta}(\mu)(y)|, \ y \in \mathbb{R}, \ \ |x-y| \le A\log(1+|x|^2)\} \ge \varepsilon(1+|x|)^{-\frac{1}{\varepsilon}},$$

then the Abel summation procedure can be dispensed.

Next, for $\mu, \nu \in \mathcal{E}'(I\!R)$ and $g, h \in \mathcal{E}(I\!R)$, we are interesting to establish the uniqueness and existence of solutions $f \in \mathcal{E}(I\!R)$ of the system

$$(S) \quad \begin{cases} \mu *_{\alpha,\beta} f = g, \\ \nu *_{\alpha,\beta} f = h, \end{cases}$$

where $*_{\alpha,\beta}$ is the convolution associated with $\Lambda_{\alpha,\beta}$.

The uniqueness turns out to prove that $f = 0$ is the unique solution in $\mathcal{E}(I\!R)$ of the system

$$(S_0) \quad \begin{cases} \mu *_{\alpha,\beta} f = 0, \\ \nu *_{\alpha,\beta} f = 0. \end{cases}$$

This leads naturally to study the Pompeiu problem in this context, which consists to characterize compactly supported distributions $\mu_1, \mu_2$ such that $f = 0$ is the unique smooth function satisfying

$$\mu_i *_{\alpha,\beta} f = 0, for \;\; i = 1, 2.$$

In other words, $f = 0$ is the unique smooth function which is mean periodic relatively to $\mu_1$ and $\mu_2$.

The paper is organized as follows. The first section is devoted to introduce some results about harmonic analysis associated with $\Lambda_{\alpha\beta}$ which will be used later. In Section 2, we introduce the notion of a mean periodic function associated to the operator $\Lambda_{\alpha,\beta}$ called $(\alpha, \beta)$-mean-periodic function. Next, we summarize the essential fact about these functions namely their series expansion in terms of $(\alpha, \beta)$-exponential monomial $\Psi_{\lambda,l}^{\alpha,\beta}$. Then we determine their coefficients, for that purpose we construct a biorthogonal system. In Sections 3 and 4, we introduce the Pompeiu problem related to $(\alpha, \beta)$-mean-periodic function and we give the resolution of a system of convolution equations associated with the Jacobi-Dunkl operator.

## 1. Preliminaries

In the following, we begin by introducing some useful spaces:

- $\mathcal{D}(R)$ is the spaces of $C^\infty$-functions on $I\!R$, with compact support, we have

$$\mathcal{D}(I\!R) = \bigcup_{a>0} \mathcal{D}_a(I\!R),$$

where $\mathcal{D}_a(\mathbb{R})$ is the space of $C^\infty$-functions on $\mathbb{R}$, with support in the closed interval $[-a, a]$. We provide $\mathcal{D}_a(\mathbb{R})$ with the topology of uniform convergence of functions and their derivatives. For this topology $\mathcal{D}_a(\mathbb{R})$ is a Fréchet space.
The space $\mathcal{D}(\mathbb{R})$ is provided with the inductive limit topology.
- $\mathcal{E}(\mathbb{R})$ the space of $C^\infty$-functions on $\mathbb{R}$, endowed with the usual topology of uniform convergence of the functions and their derivatives of all order on compact subsets of $\mathbb{R}$.
- $\mathcal{E}'(\mathbb{R})$ the space of distributions on $\mathbb{R}$ with compact support.
- $\mathbb{H}(\mathbb{C})$ the space of entire functions on $\mathbb{C}$, rapidly decreasing of exponential type. We have

$$\mathbb{H}(\mathbb{C}) = \bigcup_{a>0} \mathbb{H}_a(\mathbb{C}),$$

$$\mathbb{H}_a(\mathbb{C}) = \Big\{\psi \text{ entire}, \forall m \in \mathbb{N}, \quad \varrho_m(\psi) = \sup_{\lambda\in\mathbb{C}} |(1 + |\lambda|^2)^m \psi(\lambda) e^{-a|\mathcal{I}m\lambda|}| < +\infty\Big\}.$$

We provide $\mathbb{H}_a(\mathbb{C})$ with the topology defined by the seminorms $\varrho_m, m \in \mathbb{N}$.
The space $\mathbb{H}(\mathbb{C})$ is equipped with the inductive limit topology.
- $\mathcal{H}(\mathbb{C})$ the space of entire functions on $\mathbb{C}$, slowly increasing of exponential type, i.e. $\exists m \in \mathbb{N}, \exists R > 0$, such that

$$\sup_{\lambda\in\mathbb{C}} |(1 + |\lambda|^2)^{-m} \psi(\lambda) e^{-R|\mathcal{I}m\lambda|}| < +\infty.$$

### 1.1. The function $\Psi_\lambda^{\alpha,\beta}$

For $\alpha \geq \beta \geq -\frac{1}{2}$, we consider the Jacobi-Dunkl operator $\Lambda_{\alpha,\beta}$ given by

$$\Lambda_{\alpha,\beta} f(x) = f'(x) + ((2\alpha+1)\coth x + (2\beta+1)\tanh x)\left(\frac{f(x) - f(-x)}{2}\right), \; f \in C^1(\mathbb{R}).$$

The eigenfunction $\Psi_\lambda^{\alpha,\beta}$ of $\Lambda_{\alpha,\beta}$ satisfying

$$\begin{cases} \Lambda_{\alpha,\beta}\, u &= i\lambda\, u, \; \lambda \in \mathbb{C}, \\ u(0) &= 1, \end{cases}$$

is related to the Jacobi functions $\varphi_\mu^{\gamma,\delta}$ and it is given by

$$\Psi_\lambda^{\alpha,\beta}(x) = \varphi_\mu^{\alpha,\beta}(x) + i\frac{\lambda}{2(\alpha+1)} \sinh x \cosh x\, \varphi_\mu^{\alpha+1,\beta+1}(x), \quad x \in \mathbb{R},$$

where $\lambda^2 = \mu^2 + \rho^2$ and $\rho = \alpha + \beta + 1$. We recall that $\varphi_\mu^{\gamma,\delta}$ is defined in terms of the Gauss hypergeometric function ${}_2F_1$ by

$$\varphi_\mu^{\gamma,\delta}(x) = {}_2F_1(\frac{\gamma+\delta+1+i\mu}{2}, \frac{\gamma+\delta+1-i\mu}{2}; \gamma+1; -\sinh^2 x), \quad x \in I\!R.$$

For $x \in I\!R \setminus \{0\}$ and $\lambda \in \mathbb{C}$, the function $\Psi_\lambda^{\alpha,\beta}$ admits the following integral representation

$$\Psi_\lambda^{\alpha,\beta}(x) = \int_{-|x|}^{|x|} K(x,y) e^{i\lambda y} dy, \tag{1}$$

where $K(x,.)$ is a positive function on $I\!R$, continuous on $]-|x|,|x|[$, supported in $[-|x|,|x|]$. For the explicit form, one can see formula (3.4) in [10].

Also, the function $\Psi_\lambda^{\alpha,\beta}$ verifies the following properties (see, [3] and [10]):

i) For all $n \in I\!N \setminus \{0\}$, $x \in I\!R \setminus \{0\}$ and $\lambda \in \mathbb{C}$, then

$$\frac{d^n}{dx^n} \Psi_\lambda^{\alpha,\beta}(x) = P_x^n(\lambda)\, \Psi_\lambda^{\alpha,\beta}(x) + Q_x^{n-1}(\lambda)\, \Psi_\lambda^{\alpha,\beta}(-x),$$

where $P_x^n$ (resp. $Q_x^n$) is a polynomial in $\lambda$ of degree $n$ (resp. of degree $\leq n-1$). Its coefficients are bounded independently on $x$, $|x| \geq x_0$, where $x_0 > 0$.

ii) For all $n \in I\!N$, there exists a constant $c_n > 0$ such that for all $x \in I\!R$ and $\lambda \in I\!R \setminus \{0\}$,

$$|\frac{d^n}{dx^n} \Psi_\lambda^{\alpha,\beta}(x)| \leq c_n (1+|x|) \frac{(1+\rho+|\lambda|)^{n+1}}{|\lambda|}.$$

iii) For all $n \in I\!N$, $x \in I\!R$ and $\lambda \in \mathbb{C}$, we have

$$|\frac{d^n}{d\lambda^n} \Psi_\lambda^{\alpha,\beta}(x)| \leq |x|^n e^{|Im\,\lambda||x|}.$$

Formula (1) permits to define the Jacobi- Dunkl intertwining operator on $\mathcal{E}(I\!R)$ by

$$V_{\alpha,\beta} f(x) = \begin{cases} \displaystyle\int_{-|x|}^{|x|} K(x,y) f(y) dy, & \text{if} \quad x \in I\!R \setminus \{0\}, \\ f(0), & \text{if} \quad x = 0. \end{cases}$$

It is a topological automorphism of $\mathcal{E}(I\!R)$ verifying

$$V_{\alpha,\beta}(Df) = \Lambda_{\alpha,\beta}(V_{\alpha,\beta} f), \quad f \in \mathcal{E}(I\!R),$$

$D$ is the usual derivative operator.

The dual operator ${}^tV_{\alpha,\beta}$ of the operator $V_{\alpha,\beta}$ is defined on $\mathcal{D}(I\!R)$ by

$$ {}^tV_{\alpha,\beta}(g)(y) = \int_{|x|\geq|y|} K(x,y)g(x)A_{\alpha,\beta}(x)dx. $$

It is a topological isomorphism of $\mathcal{D}(I\!R)$ and satisfies the transmutation relation

$$ D({}^tV_{\alpha,\beta}g) = {}^tV_{\alpha,\beta}(\Lambda_{\alpha,\beta}g),\ g\in\mathcal{D}(I\!R). $$

To complete, we recall that the dual operator ${}^tV_{\alpha,\beta}$ of the operator $V_{\alpha,\beta}$ is defined on $\mathcal{E}'(I\!R)$ by

$$ \langle\, {}^tV_{\alpha,\beta}(\mu),\ f\,\rangle = \langle\, \mu,\ V_{\alpha,\beta}(f)\,\rangle, $$

and it is an isomorphism of $\mathcal{E}'(I\!R)$.

We point out that from the properties of the intertwining operator $V_{\alpha,\beta}$ and its inverse, we have

$$ \forall\mu\in\mathcal{E}', \mathrm{supp}\mu\subset[-a,a] \Longleftrightarrow \mathrm{supp}\,{}^tV_{\alpha,\beta}(\mu)\subset[-a,a]. \tag{2} $$

### 1.2. The Jacobi-Dunkl transform and the convolution product

We recall some notions related to the Jacobi-Dunkl transform, which will be used later (see [2], [3] and [10]).

The Jacobi-Dunkl transform is defined on $\mathcal{D}(I\!R)$, (resp. $\mathcal{E}'(I\!R)$) by

$$ \mathcal{F}_{\alpha,\beta}(f)(\lambda) = \int_{I\!R} f(x)\Psi^{\alpha,\beta}_{-\lambda}(x)A_{\alpha,\beta}(x)dx,\ \lambda\in\mathbb{C}, $$

$$ (\text{resp.}\quad \mathcal{F}_{\alpha,\beta}(\mu)(\lambda) = \langle\mu\,,\ \Psi^{\alpha,\beta}_{-\lambda}\rangle\ ). $$

Here, $A_{\alpha,\beta}(x) = 2^{2\rho}(\sinh|x|)^{2\alpha+1}(\cosh x)^{2\beta+1}$.

It is connected to the usual Fourier transform $\mathcal{F}$ by the relations

$$ \begin{array}{lcrcl} \forall f\in\mathcal{D}(I\!R) & , & \mathcal{F}_{\alpha,\beta}(f) & = & \mathcal{F}\circ{}^tV_{\alpha,\beta}(f), \\ \forall\mu\in\mathcal{E}'(I\!R) & , & \mathcal{F}_{\alpha,\beta}(\mu) & = & \mathcal{F}\circ{}^tV_{\alpha,\beta}(\mu), \end{array} \tag{3} $$

where

$$ \mathcal{F}(f)(\lambda) = \int_{I\!R} f(x)\,e^{-i\lambda x}dx,\ \lambda\in\mathbb{C}, $$

$$ (\text{resp.}\quad \mathcal{F}(\mu)(\lambda) = \langle\mu,\ e^{-i\lambda\cdot}\rangle,\ \lambda\in\mathbb{C}). $$

From the relations (3) and the classical Paley-Wiener type theorems associated with the transformation $\mathcal{F}$, we deduce the following Paley-Wiener type theorems associated with the operator $\Lambda_{\alpha,\beta}$ (see [10]):

*The Jacobi-Dunkl transform $\mathcal{F}_{\alpha,\beta}$ is a topological isomorphism from $\mathcal{D}(\mathbb{R})$, (resp. $\mathcal{E}'(\mathbb{R})$) onto $\mathbb{H}(\mathbb{C})$, (resp. $\mathcal{H}(\mathbb{C})$).*

In [2] the authors have established a product formula for the function $\Psi_\lambda^{\alpha,\beta}$, $\lambda \in \mathbb{C}$:

$$\Psi_\lambda^{\alpha,\beta}(x)\Psi_\lambda^{\alpha,\beta}(y) = \int_{\mathbb{R}} \Psi_\lambda^{\alpha,\beta}(z)d\mu_{x,y}^{\alpha,\beta}(z),\ x, y \in \mathbb{R},$$

where $\mu_{x,y}^{\alpha,\beta}$ is a real uniformly bounded measure with compact support, which may not be positive.

The translation operators $T_x^{\alpha,\beta}$, $x \in \mathbb{R}$, associated with the Jacobi-Dunkl operator is defined by

$$T_x^{\alpha,\beta}f(y) = \int_{\mathbb{R}} f(z)d\mu_{x,y}^{\alpha,\beta}(z),\ x, y \in \mathbb{R},$$

here $f$ is a measurable function.

This formula coincides on $\mathcal{E}(\mathbb{R})$ with

$$\forall y \in \mathbb{R}\,,\ T_x^{\alpha,\beta}f(y) = V_{\alpha,\beta,x}V_{\alpha,\beta,y}(V_{\alpha,\beta}^{-1}(f)(x+y)). \tag{4}$$

The last formula was given as a definition of $T_x^{\alpha,\beta}$ on $\mathcal{E}(\mathbb{R})$, see [10].

Also, the operator $T_x^{\alpha,\beta}$, satisfies:

(i) For all $x \in \mathbb{R}$, $T_x^{\alpha,\beta}$ is linear and continuous from $\mathcal{E}(\mathbb{R})$ into itself.

(ii) For all $f \in \mathcal{E}(\mathbb{R})$, we have

$$\begin{array}{rclcl} T_x^{\alpha,\beta}f(y) & = & T_y^{\alpha,\beta}f(x) & , & T_0^{\alpha,\beta}f(y) = f(y) \\ T_x^{\alpha,\beta}T_y^{\alpha,\beta} & = & T_y^{\alpha,\beta}T_x^{\alpha,\beta} & , & T_x^{\alpha,\beta}\Lambda_{\alpha,\beta} = \Lambda_{\alpha,\beta}T_x^{\alpha,\beta}. \end{array}$$

(iii) For all $x, y \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, we have the following product formula:

$$T_x^{\alpha,\beta}(\Psi_\lambda^{\alpha,\beta})(y) = \Psi_\lambda^{\alpha,\beta}(x)\Psi_\lambda^{\alpha,\beta}(y).$$

(iv) For all $f \in \mathcal{D}_a(\mathbb{R})$ , $a > 0$, we have

$$\forall x \in \mathbb{R}\,,\ T_x^{\alpha,\beta}f \in \mathcal{D}_{a+|x|}(\mathbb{R}),$$

$$\forall \lambda \in \mathbb{C}\,,\ \mathcal{F}_{\alpha,\beta}(T_x^{\alpha,\beta}f)(\lambda) = \Psi_\lambda^{\alpha,\beta}(x)\mathcal{F}_{\alpha,\beta}(f)(\lambda).$$

To complete this section, we give the following definitions:

(i) We define the convolution of two distributions $\mu, \nu \in \mathcal{E}'(I\!R)$ by

$$\langle \mu *_{\alpha,\beta} \nu \, , \, f\rangle = \langle \mu_x \, , \, \langle \nu_y \, , \, T_x^{\alpha,\beta} f(y)\rangle\rangle \, , \; f \in \mathcal{E}(I\!R).$$

(ii) The convolution of $\mu \in \mathcal{E}'(I\!R)$ and $f \in \mathcal{E}(I\!R)$ is the function $\mu *_{\alpha,\beta} f \in \mathcal{E}(I\!R)$ given by

$$\mu *_{\alpha,\beta} f(x) = \langle \mu_y \, , \, T_{-x}^{\alpha,\beta} \check{f}(y)\rangle,$$

here $\check{f}(u) = f(-u)$.

(iii) The convolution of two functions $f$ and $g$ in $\mathcal{D}(I\!R)$ is defined by the relation

$$f *_{\alpha,\beta} g(x) = \int_{I\!R} T_{-x}^{\alpha,\beta} \check{f}(y) g(y) A_{\alpha,\beta}(y) dy.$$

Obviously, we have the following properties:

**I**) Let $\mu, \nu$ be two distributions in $\mathcal{E}'(I\!R)$ and let $f, g$ be two functions in $\mathcal{D}(I\!R)$, then we have

$$\begin{array}{rcl} \mathcal{F}_{\alpha,\beta}(\mu *_{\alpha,\beta} \nu) & = & \mathcal{F}_{\alpha,\beta}(\mu)\mathcal{F}_{\alpha,\beta}(\nu), \\ \mathcal{F}_{\alpha,\beta}(\mu *_{\alpha,\beta} f) & = & \mathcal{F}_{\alpha,\beta}(\mu)\mathcal{F}_{\alpha,\beta}(f), \\ \mathcal{F}_{\alpha,\beta}(f *_{\alpha,\beta} g) & = & \mathcal{F}_{\alpha,\beta}(f)\mathcal{F}_{\alpha,\beta}(g). \end{array} \tag{5}$$

**II**) Let $\mu$ , $\nu$ be two distributions in $\mathcal{E}'(I\!R)$ and $f$ be in $\mathcal{E}(I\!R)$ we have

$$\mu *_{\alpha,\beta} (\nu *_{\alpha,\beta} f) = (\mu *_{\alpha,\beta} \nu) *_{\alpha,\beta} f. \tag{6}$$

Also, this convolution and the ordinary convolution $*$ are related by the following:

*Let $\mu$ , $\nu$ be two distributions in $\mathcal{E}'(I\!R)$ and $f$ be a function in $\mathcal{E}(I\!R)$ we have*

$$\begin{array}{rcl} ({}^tV_{\alpha,\beta})^{-1}(\mu) *_{\alpha,\beta} V_{\alpha,\beta}(f) & = & V_{\alpha,\beta}(\mu * f), \\ {}^tV_{\alpha,\beta}(\mu) * V_{\alpha,\beta}^{-1}(f) & = & V_{\alpha,\beta}^{-1}(\mu *_{\alpha,\beta} f), \\ {}^tV_{\alpha,\beta}(\mu *_{\alpha,\beta} \nu) & = & {}^tV_{\alpha,\beta}(\mu) * {}^tV_{\alpha,\beta}(\nu). \end{array} \tag{7}$$

## 2. Mean-periodic functions associated with the Jacobi-Dunkl operators

### 2.1. Mean-periodic functions

DEFINITION 2.1. A function $f$ in $\mathcal{E}(I\!R)$ is called mean-periodic associated with the operator $\Lambda_{\alpha,\beta}$, if there exists a non zero distribution $\mu \in \mathcal{E}'(I\!R)$, such that for all $x \in I\!R$

$$\mu *_{\alpha,\beta} f(x) = 0. \tag{8}$$

Henceforth we shall denote $(\alpha,\beta)$-mean-periodic function for the mean periodic function associated with $\Lambda_{\alpha,\beta}$. If we want to emphasize the equation satisfied by $f$ we will say that $f$ is $(\alpha,\beta)-$mean-periodic with respect to $\mu$ or $\mu$-$(\alpha,\beta)-$mean-periodic function.

If $\alpha = \beta = -1/2$, we recover the definition of the classical mean-periodic function, (see [19]).

As in [5], Proposition 6.1.2, we can prove the following proposition.

PROPOSITION 2.2. *The set* $\mathcal{M}_{\alpha,\beta} = \{f \in \mathcal{E}(I\!R),\ \mu *_{\alpha,\beta} f = 0\}$ *is a closed subspace of* $\mathcal{E}(I\!R)$ *which is invariant under translations* $T_x^{\alpha,\beta}$, $x \in I\!R$.

According to the Hahn-Banach theorem, we have the following proposition.

PROPOSITION 2.3. *A function* $f \in \mathcal{E}(I\!R)$ *is* $(\alpha,\beta)-$*mean periodic for at least one* $\mu \neq 0$, $\mu \in \mathcal{E}'(I\!R)$, *if and only if* $\mathcal{Z}^{\alpha,\beta}(f) \neq \mathcal{E}(I\!R)$.

$\mathcal{Z}^{\alpha,\beta}(f)$ *is the closure of the subspace of* $\mathcal{E}(I\!R)$ *spanned by* $T_{-x}^{\alpha,\beta}\check{f}$, $x \in I\!R$.

**Examples:**

(i) Given $a \in I\!R, a \neq 0$, every function $f$ in $\mathcal{E}(I\!R)$ such that

$$T_{-x}^{\alpha,\beta}\check{f}(a) = f(x)\ ,\ \text{for all } x \in I\!R$$

is $(\alpha,\beta)-$mean-periodic with respect to $\mu = \delta_a - \delta_0$, where $\delta_x$ denotes the Dirac point measure at $x$.

(ii) If $f \in \mathcal{D}(I\!R)$, $f \neq 0$, then $f$ is not mean periodic.

**Notations:** For $\lambda \in \mathbb{C}$, $x \in I\!R$ and $l \in I\!N$, we put

- $\Psi_\lambda(x) := \Psi_\lambda^{-\frac{1}{2},-\frac{1}{2}}(x) = e^{i\lambda x}$.
- $\Psi_{\lambda,l}(x) := \dfrac{d^l}{dt^l}(\Psi_{-it}(x))_{t=i\lambda} = x^l e^{i\lambda x}$.
- $\Psi_{\lambda,l}^{\alpha,\beta}(x) := V_{\alpha,\beta}(\Psi_{\lambda,l})(x) = \dfrac{d^l}{dt^l}(\Psi_{-it}^{\alpha,\beta}(x))_{t=i\lambda}$.

For $\mu \in \mathcal{E}'(\mathbb{R})$, as in the classical case, (see [5]), one can shows that

$$\mu *_{\alpha,\beta} \Psi_{\lambda,j}^{\alpha,\beta}(x)=\sum_{s=0}^{j}\binom{j}{s}\Psi_{\lambda,j-s}^{\alpha,\beta}(x)(-i)^s\mathcal{F}_{\alpha,\beta}^{(s)}(\mu)(\lambda)\ ,\ \mu \in \mathcal{E}'(\mathbb{R}),\ 0 \le j \le l-1. \tag{9}$$

Thus, if we choose $\mu \in \mathcal{E}'(\mathbb{R})$ such that $\lambda$ is a root of order at least $l$ of $\mathcal{F}_{\alpha,\beta}(\mu)$, we conclude that $x \longrightarrow \Psi_{\lambda,l}^{\alpha,\beta}(x)$ is $\mu-(\alpha,\beta)-$mean-periodic.

Definition 2.4. $\Psi_{\lambda,l}^{\alpha,\beta}$ are called $(\alpha,\beta)-$exponential-monomials.

By using relation (9), one can see the following proposition.

Proposition 2.5. *The functions $\Psi_{\lambda,j}^{\alpha,\beta}$, $0 \le j \le l-1$, belong to $\mathcal{Z}^{\alpha,\beta}(\check{f})$ if and only if, for each distribution $\mu \in \mathcal{E}'(\mathbb{R})$ verifying*

$$\mu *_{\alpha,\beta} f = 0,\ \textit{for all } x \in \mathbb{R},$$

*we have*

$$\mathcal{F}_{\alpha,\beta}^{(j)}(\mu)(\lambda) = 0,\quad \textit{for } j = 0,\cdots,l-1.$$

Definition 2.6. We call spectrum of a $(\alpha,\beta)-$mean-periodic function $f$ in $\mathcal{E}(\mathbb{R})$, denoted by sp$(f)$, the set of pairs $(\lambda,l)\,,\lambda \in \mathbb{C}\,,l \in \mathbb{N}$, such that the functions $\Psi_{\lambda,j}^{\alpha,\beta}$ belong to $\mathcal{Z}^{\alpha,\beta}(\check{f})$ for $0 \le j \le l-1$ and not for $j = l$ .

From the last proposition, we can conclude that the spectrum is composed by the common zeros of the Jacobi-Dunkl transform of elements in $\mathcal{E}'(\mathbb{R})$, which are orthogonal to $\mathcal{Z}^{\alpha,\beta}(f)$, each zero being counted with its order of multiplicity.

There is a relationship between $(\alpha,\beta)-$mean-periodic functions and classical mean-periodic as shown in the following result which is deduced from the relation

$${}^tV_{\alpha,\beta}(\mu) * V_{\alpha,\beta}^{-1}(f) = V_{\alpha,\beta}^{-1}(\mu *_{\alpha,\beta} f).$$

Proposition 2.7. *A function $f$ in $\mathcal{E}(\mathbb{R})$ is $(\alpha,\beta)-$mean-periodic with respect to $\mu$ if and only if the function $V_{\alpha,\beta}^{-1}(f)$ is classical mean periodic with respect to ${}^tV_{\alpha,\beta}(\mu)$.*

From the work of L. Schwartz ([19]) about classical mean-periodic functions on $\mathbb{R}$ and Proposition 2.7, we deduce the following characterization of the $(\alpha,\beta)-$mean-periodic functions.

Theorem 2.8. *Every $(\alpha,\beta)-$mean-periodic function $f$ in $\mathcal{E}(\mathbb{R})$, can be approximated in the topology of $\mathcal{E}(\mathbb{R})$ by finite linear combinations of* *functions of the type* $\Psi^{\alpha,\beta}_{\lambda,l}$ , $(\lambda, l) \in sp(f)$. *More precisely, we can find finite sets* $\Lambda_n \subseteq sp(f)$ *and* $c_{\lambda,j} \in \mathbb{C}$*, such that*

$$f(x) = \lim_{n\to+\infty} \sum_{(\lambda,l)\in\Lambda_n} \sum_{0\le j\le l-1} c_{\lambda,j} \Psi^{\alpha,\beta}_{\lambda,j}(x), \tag{10}$$

*the coefficients* $c_{\lambda,j}$ *are uniquely determined.*

Remark. One can see that the space $\mathcal{M}_{\alpha,\beta}$, defined in Proposition 2.2, is generated by the $(\alpha, \beta)$-exponential monomials $\Psi^{\alpha,\beta}_{\lambda,j}$, for $j \in \{0, 1, 2...l-1\}$ and $(\lambda, l) \in \mathrm{sp}(f)$.

Example. Let $\Lambda^n_{\alpha,\beta}\delta_0$, $n \in \mathbb{N}$, the element of $\mathcal{E}'(\mathbb{R})$ defined by

$$\langle \Lambda^n_{\alpha,\beta}\delta_0 \, , \, f\rangle = (-1)^n(\Lambda^n_{\alpha,\beta}f)(0) \quad \text{and} \quad \mu = \sum_{n=0}^{N} c_n\Lambda^n_{\alpha,\beta}\delta_0, \; c_n \in \mathbb{C}, \; N \in \mathbb{N},$$

then $\mu *_{\alpha,\beta} f = 0$ means that $f$ is a solution of the homogeneous differential-difference equation with constant coefficients, namely the equation

$$c_N\Lambda^N_{\alpha,\beta}f(x) + \cdots c_0f(x) = 0.$$

Then the solution of this equation is given by finite sums of the form

$$f(x) = \sum_{(\lambda,l)\in\mathrm{sp}(f)} \sum_{0\le j\le l-1} a_{\lambda,j}\Psi^{\alpha,\beta}_{\lambda,j}(x) \quad , \quad a_{\lambda,j} \in \mathbb{C},$$

where $\mathrm{sp}(f)$ is the set of the roots $\lambda$ of the algebraic equation

$$\mathcal{F}_{\alpha,\beta}(\mu)(\lambda) = \sum_{n=0}^{N} c_n(i\lambda)^n = 0.$$

If $\alpha = \beta = -1/2$, this corresponds to the classical result of Euler.

## 2.2. Biorthogonal system associated with $(\alpha, \beta)-$exponential monomials

**Notation.** Let $\mu \in \mathcal{E}'(\mathbb{R})$, $\mu \neq 0$, we put
- $Z(\mathcal{F}_{\alpha,\beta}(\mu)) = \{(\lambda_n, l_n) \, , \, n \in \mathbb{N} \, , \, l_n \in \mathbb{N}\}$ ,
where $\lambda_n$ is a zero of order $l_n$ of the entire function $\mathcal{F}_{\alpha,\beta}(\mu)$.

As in the classical case [12], [19] and [5] (see also [20] and [4]), we construct a family of distributions $\mu_{n,m} \in \mathcal{E}'(\mathbb{R})$ verifying

$$\langle \mu_{n,m} \, , \, \Psi^{\alpha,\beta}_{-\lambda_s,j}\rangle = (-1)^j\delta_{n,s}\delta_{m,j}, \tag{11}$$

where $(\lambda_n, l_n) \in Z(\mathcal{F}_{\alpha,\beta}(\mu))$, $0 \leq m \leq l_n - 1$ and $0 \leq j \leq l_s - 1$, here $\delta_{r,s}$ denotes the Kronecker symbol. This formula permits to compute the coefficients $c_{\lambda,j}$ of the development of a $\mu$-$(\alpha, \beta)-$mean-periodic function $f \in \mathcal{E}(I\!R)$, respect to the $(\alpha, \beta)-$exponential monomials defined in (10).

Let $f$ be a function in $\mathcal{E}(I\!R)$, for all $n \in N$, we put

$$I_n(f)(x) := \int_0^x f(t)e^{-i\lambda_n(x-t)}dt.$$

It is known that the general solution in $\mathcal{E}(I\!R)$ of the equation

$$(D + i\lambda_n)^{l_n} g = f,$$

is given by

$$g(x) = \sum_{j=0}^{l_n-1} \beta_j\, x^j\, e^{-i\lambda_n x} + \overbrace{I_n \circ \cdots \circ I_n}^{l_n \,\text{times}}(f)(x) \quad , \quad \beta_j \in \mathbb{C}.$$

It follows that the general solution in $\mathcal{E}(I\!R)$ of the equation

$$(\Lambda_{\alpha,\beta} + i\lambda_n)^{l_n} g = f,$$

is given by

$$g(x) = \sum_{j=0}^{l_n-1} \beta_j \Psi^{\alpha,\beta}_{-\lambda_n,j}(x) + V_{\alpha,\beta} \overbrace{I_n \circ \cdots \circ I_n}^{l_n \,\text{times}}(V^{-1}_{\alpha,\beta}(f))(x),\ \beta_j \in \mathbb{C},$$

**Notation.**
- If $G$ is a meromorphic function, having $\gamma$ as a pole, we denote by $[G(\lambda)]_\gamma$ the singular part of $G(\lambda)$ in a neighborhood of $\gamma$, hence $G(\lambda) - [G(\lambda)]_\gamma$ is holomorphic in a neighborhood of $\gamma$.

LEMMA 2.9.
*(i) The distribution $q_n$ in $\mathcal{E}'(I\!R)$ whose the Jacobi-Dunkl transform is*

$$\mathcal{F}_{\alpha,\beta}(q_n)(\lambda) = (\lambda - \lambda_n)^{l_n} \left[\frac{1}{\mathcal{F}_{\alpha,\beta}(\mu)(\lambda)}\right]_{\lambda_n}$$

*has a support concentrated at the origin.*

*(ii) The distribution $\mu_{n,0} \in \mathcal{E}'(I\!R)$, $n \in I\!N$, whose the Jacobi-Dunkl transform is given by*

$$\mathcal{F}_{\alpha,\beta}(\mu_{n,0})(\lambda) = \begin{cases} \mathcal{F}_{\alpha,\beta}(\mu)(\lambda) \left[\dfrac{1}{\mathcal{F}_{\alpha,\beta}(\mu)(\lambda)}\right]_{\lambda_n} & , \text{ if } \lambda \neq \lambda_n, \\ 1 & , \text{ if } \lambda = \lambda_n, \end{cases}$$

*satisfies*

$$\langle \mu_{n,0}\, ,\, f\rangle = (-i)^{l_n} \langle q_n *_{\alpha,\beta} \mu\, ,\, V_{\alpha,\beta} \overbrace{I_n \circ \cdots \circ I_n}^{l_n \text{ times}}(V_{\alpha,\beta}^{-1}(f))\rangle, \text{ for all } f \in \mathcal{E}(I\!R).$$

P r o o f.

(i) Since the function $(\lambda - \lambda_n)^{l_n} \left[\dfrac{1}{\mathcal{F}_{\alpha,\beta}(\mu)(\lambda)}\right]_{\lambda_n}$ is a polynomial and using the relation (2), we can conclude that the distribution $q_n$ has a support concentrated at the origin.

(ii) By the Jacobi-Dunkl transform, it is clear that

$$(-i)^{l_n} (\Lambda_{\alpha,\beta} - i\lambda_n)^{l_n} \mu_{n,0} = q_n *_{\alpha,\beta} \mu.$$

For all $g$ in $\mathcal{E}(I\!R)$, we have

$$\langle q_n *_{\alpha,\beta} \mu\, ,\, g\rangle = (i)^{l_n} \langle \mu_{n,0}\, ,\, (\Lambda_{\alpha,\beta} + i\lambda_n)^{l_n} g\rangle.$$

Now the general solution of the equation

$$(\Lambda_{\alpha,\beta} + i\lambda_n)^{l_n} g = f \quad , \quad f \in \mathcal{E}(I\!R),$$

is given by

$$g(x) = \sum_{j=0}^{l_n - 1} \beta_j \Psi_{-\lambda_n, j}^{\alpha,\beta}(x) \quad + V_{\alpha,\beta} \overbrace{I_n \circ \cdots \circ I_n}^{l_n \text{ times}}(V_{\alpha,\beta}^{-1}(f))(x)\, ,\, \beta_j \in \mathbb{C}.$$

Hence,

$$\langle \mu_{n,0}\, ,\, f\rangle = (-i)^{l_n} \langle q_n *_{\alpha,\beta} \mu\, ,\, V_{\alpha,\beta} I_n \circ \cdots \circ I_n(V_{\alpha,\beta}^{-1}(f))\rangle.$$

REMARK. If the zeros $\lambda_n$ of $\mathcal{F}_{\alpha,\beta}(\mu)$ are simple, then the distribution $q_n$ is given by

$$q_n = \frac{1}{(\mathcal{F}_{\alpha,\beta}(\mu))'(\lambda_n)} \delta_0$$

and

$$\mathcal{F}_{\alpha,\beta}(\mu_{n,0})(\lambda) = \begin{cases} \dfrac{\mathcal{F}_{\alpha,\beta}(\mu)(\lambda)}{(\mathcal{F}_{\alpha,\beta}(\mu))'(\lambda_n)(\lambda - \lambda_n)} & , \text{if } \lambda \neq \lambda_n, \\ 1 & , \text{if } \lambda = \lambda_n. \end{cases}$$

Then, for all $f \in \mathcal{E}(I\!R)$,

$$\langle \mu_{n,0}\,,\, f\rangle = \frac{(-i)}{(\mathcal{F}_{\alpha,\beta}(\mu))'(\lambda_n)}\langle \mu\,,\, V_{\alpha,\beta} I_n(V_{\alpha,\beta}^{-1}(f))\rangle.$$

In the same way as in [19] (see also [5] and [4]), we can prove the following proposition.

Proposition 2.10. *For $\mu \in \mathcal{E}'(I\!R), \mu \neq 0$, there exists a distribution $\mu_{n,m}, 0 \leq m \leq l_n - 1$, in $\mathcal{E}'(I\!R)$ satisfying the relation (11). It is given by*

$$\mu_{n,m} = \frac{1}{m!}(\Lambda_{\alpha,\beta} - i\lambda_n)^m \mu_{n,0} + \tau_{n,m} *_{\alpha,\beta} \mu,$$

*where $\tau_{n,m}$ is the distribution in $\mathcal{E}'(I\!R)$, with support concentrated at the origin, for $m \neq 0$ its Jacobi-Dunkl transform is given by*

$$\mathcal{F}_{\alpha,\beta}(\tau_{n,m})(\lambda) = \frac{(i)^m}{m!}\left\{\left[\frac{(\lambda-\lambda_n)^m}{\mathcal{F}_{\alpha,\beta}(\mu)(\lambda)}\right]_{\lambda_n} - (\lambda-\lambda_n)^m\left[\frac{1}{\mathcal{F}_{\alpha,\beta}(\mu)(\lambda)}\right]_{\lambda_n}\right\}.$$

*Moreover, if $[a,b]$ is the smallest closed interval containing the support of $\mu$, then supp$(\mu_{n,m}) \subset [a,b]$.*

Corollary 2.11. *Let $f \in \mathcal{E}(I\!R)$ and $\mu \in \mathcal{E}'(I\!R)$, assume that*

$$f(x) = \sum_{n\geq 0}\sum_{0\leq l\leq l_n-1} c_{n,l}\Psi^{\alpha,\beta}_{\lambda_n,l}(x),$$

*with $(\lambda_n, l_n) \in Z(\mathcal{F}_{\alpha,\beta}(\mu))$ and the series converges in the topology of $\mathcal{E}(I\!R)$. Then $f$ is $\mu$-$(\alpha,\beta)-$mean-periodic and the coefficients $c_{n,l}$ can be computed by the formula*

$$c_{n,l} = \langle \mu_{n,l}\,,\, \check{f}\rangle = \frac{1}{l!}\langle(\Lambda_{\alpha,\beta} - i\lambda_n)^l \mu_{n,0}\,,\, \check{f}\rangle. \tag{12}$$

### 2.3. Jacobi-Dunkl expansion of $(\alpha,\beta)-$mean-periodic functions

Let $f$ be a $(\alpha,\beta)-$mean-periodic function in $\mathcal{E}(I\!R)$ with respect to $\mu \in \mathcal{E}'(I\!R)$. We will be interested in the convergence, which will be defined, of the series expansion

$$\sum_{(\lambda_n,l_n)\in Z(\mathcal{F}_{\alpha,\beta}(\mu))} \sum_{0\le l\le l_n-1} c_{n,l}\Psi^{\alpha,\beta}_{\lambda_n,l}, \tag{13}$$

to $f$, where the coefficients $c_{n,l}, 0 \le l \le l_n - 1, n \in I\!N$, are given by the relation (12).

Definition 2.12. The series (13) converges to $f$ in $\mathcal{E}(I\!R)$ by means of grouping of terms and Abel convergence factors, if there are disjoint finite subsets $Z_j$ ( groupings ) such that $Z(\mathcal{F}_{\alpha,\beta}(\mu)) = \bigcup_1^{+\infty} Z_j$ and for every $\varepsilon > 0$ the series expansion

$$\sum_{j=1}^{+\infty}\left[\sum_{(\lambda_n,l_n)\in Z_j}\left(\sum_{0\le l\le l_n-1} c_{n,l}\Psi^{\alpha,\beta}_{\lambda_n,l}(x+i\sigma\varepsilon)\right)\right],$$

converges to a function $f_\varepsilon$, satisfying $\lim_{\varepsilon\longrightarrow 0} f_\varepsilon = f$, where both the series and the limit are in the topology of $\mathcal{E}(I\!R)$.

Here, $\sigma = -1$ for $\mathcal{R}e\lambda_n > 0$, $\sigma = 1$ for $\mathcal{R}e\lambda_n < 0$ and $\sigma = 0$ for $\mathcal{R}e\lambda_n = 0$.

From the results of L. Schwartz ([19]) about the Fourier-expansion of classical mean-periodic functions on $I\!R$, we deduce, similarly as in [4] the following result for the Jacobi-Dunkl expansion of $(\alpha,\beta)-$mean-periodic functions relatively to a distribution in $\mathcal{E}'(I\!R)$.

Theorem 2.13. *Let $f$ be a $(\alpha,\beta)-$mean-periodic function with respect to $\mu \in \mathcal{E}'(I\!R)$, then the series expansion defined in (13) whose coefficients are given by the relation (12), converges to $f$ in $\mathcal{E}(I\!R)$ by means of grouping of terms and Abel convergence factors.*

L. Ehrenpreis [13]) (see also [7]), showed in the classical case that for a wide class of distributions $\mu$ the abelian summation process is not necessary. Naturally, we can extend it for $(\alpha,\beta)-$ mean-periodic functions.

Definition 2.14. A distribution $\mu \in \mathcal{E}'(I\!R)$ is called $(\alpha,\beta)$-slowly-decreasing, if there are positive constants $A, \varepsilon$, such that for any $x \in I\!R$

$$\mathrm{Max}\,\{|\mathcal{F}_{\alpha,\beta}(\mu)(y)|\,,\, y\in I\!R\,,\, |x-y|\le A\log(1+|x|)\} \ge \varepsilon(1+|x|)^{-1/\varepsilon}.$$

It turns out that:

i) if $\alpha = \beta = -1/2$, we have the definition of slowly decreasing distribution (see [7] p.123).

ii) $\mu$ is $(\alpha, \beta)$-slowly-decreasing if and only if ${}^tV_{\alpha,\beta}(\mu)$ is a slowly decreasing.

By using the result of L. Ehrenpreis for the Fourier expansion of classical mean-periodic functions on $\mathbb{R}$ (see [13]) and Proposition 2.7, we deduce the following theorem.

THEOREM 2.15. *If $\mu$ is $(\alpha, \beta)-$slowly-decreasing, there exists a finite grouping $Z_j$ of $Z(\mathcal{F}_{\alpha,\beta}(\mu))$ such that for any $f \in \mathcal{E}(\mathbb{R})$ satisfying $\mu *_{\alpha,\beta} f = 0$, the series*

$$\sum_{j=1}^{+\infty} \left[ \sum_{(\lambda_n, l_n) \in Z_j} \left( \sum_{0 \le l \le l_n - 1} c_{n,l} \Psi^{\alpha,\beta}_{\lambda_n,l}(x) \right) \right], \tag{14}$$

*converges to $f$ in $\mathcal{E}(\mathbb{R})$, where the coefficients $c_{n,l}$, $0 \le l \le l_n - 1$, $n \in \mathbb{N}$ are given by the relation (12).*

By imposing other conditions on the $(\alpha, \beta)-$slowly-decreasing distribution $\mu$, we can show as follows that the series expansion defined in the relation (13) converges in $\mathcal{E}(\mathbb{R})$ without the grouping of terms (i.e., for which we have $\text{card}(Z_j) = 1$ in (14) for all $j$).

It follows from the results of C. A. Berenstein and B. A. Taylor [7] (see also [6], p.214), in the classical case and Proposition 2.7, the following result.

THEOREM 2.16. *Given a $(\alpha, \beta)-$mean-periodic function $f$ in $\mathcal{E}(\mathbb{R})$ relatively to a distribution $\mu$ in $\mathcal{E}'(\mathbb{R})$ which is $(\alpha, \beta)-$slowly-decreasing, a necessary and sufficient condition for which the Jacobi-Dunkl series representation defined in the relation (13) converges to $f$ in $\mathcal{E}(\mathbb{R})$ without groupings is that for some $\varepsilon, c > 0$, we have*

$$|\mathcal{F}^{(l)}_{\alpha,\beta}(\mu)(\lambda)| \ge \varepsilon \frac{\exp(-c|\mathcal{I}m\lambda|)}{(1 + |\lambda|)^c}, \tag{15}$$

*where $(\lambda, l) \in Z(\mathcal{F}_{\alpha,\beta}(\mu))$.*

## 3. Pompeiu problem associated with the Jacobi-Dunkl operators

The Pompeiu problem extensively studied by several authors (see, [9] and [1]), is very closely related to the theory of mean periodic functions, see [6].

Let us first recall that a family $\mathcal{R}$ of compactly supported Radon measures is said to have the Pompeiu property associated with the Jacobi-Dunkl operator, (see [18]), if there is no non trivial function $f \in \mathcal{C}(\mathbb{R})$ or $\mathcal{E}(\mathbb{R})$ satisfying

$$f *_{\alpha,\beta} \mu = 0, \quad \text{for all } \mu \in \mathcal{R}. \tag{16}$$

Similarly, a collection $\mathcal{K}$ of bounded measurable subsets of $\mathbb{R}$ is said to have the Pompeiu property, if there is no non trivial function $f \in C(\mathbb{R})$ such that:

$$f *_{\alpha,\beta} 1_D(x) = \int_D T_{-x}^{\alpha,\beta} \check{f}(y) A_{\alpha,\beta}(y) dy \equiv 0, \quad \text{for all } D \in \mathcal{K}.$$

Analogously, we say that a family $\mathcal{R}$ of compactly supported distributions has the Pompeiu property, if there is no non trivial smooth function satisfying: $\mu *_{\alpha,\beta} f = 0$, for all $\mu \in \mathcal{R}$.

THEOREM 3.1. *Two distributions $\mu, \nu$ of $\mathcal{E}'(\mathbb{R})$ have the Pompeiu property if and only if $\mathcal{F}(\mu)$ and $\mathcal{F}(\nu)$ have no common zero.*

P r o o f. The system

$$\begin{cases} \mu *_{\alpha,\beta} f = 0, \\ \nu *_{\alpha,\beta} f = 0, \end{cases}$$

is clearly equivalent to the following

$$\begin{cases} {}^t V_{\alpha,\beta}(\mu) * V_{\alpha,\beta}^{-1}(f) = 0, \\ {}^t V_{\alpha,\beta}(\nu) * V_{\alpha,\beta}^{-1}(f) = 0, \end{cases}$$

which leads to $V_{\alpha,\beta}^{-1}(f) = 0$, (see [6], p. 206), and to $f = 0$.

Conversely, let $\lambda \in \mathbb{C}$ such that $\mathcal{F}_{\alpha,\beta}(\mu)(\lambda) = \mathcal{F}_{\alpha,\beta}(\nu)(\lambda) = 0$, then

$$\mu *_{\alpha,\beta} \Psi_\lambda^{\alpha,\beta}(x) = \nu *_{\alpha,\beta} \Psi_\lambda^{\alpha,\beta}(x) = 0, \quad \text{for all } x \in \mathbb{R}.$$

On the other hand, $\Psi_\lambda^{\alpha,\beta}(0) = 1$, hence $\mu$ and $\nu$ do not satisfy the Pompeiu problem, this finished the proof. ∎

**Applications**

PROPOSITION 3.2. *For $r_1 > r_2 > 0$ a necessary and sufficient condition such that there is no non trivial $f \in C(\mathbb{R})$ satisfying*

$$\forall x \in \mathbb{R}, \int_{-r_i}^{r_i} T_{-x}^{\alpha,\beta} \check{f}(y) A_{\alpha,\beta}(y) dy = 0, \quad (i = 1, 2), \tag{17}$$

*is that the functions $\mu \to \varphi_\mu^{\alpha+1,\beta+1}(r_1)$ and $\mu \to \varphi_\mu^{\alpha+1,\beta+1}(r_2)$, have no common zeros.*

P r o o f. If we denote by $\sigma_r = 1_{]-r,r[}(x) A_{\alpha,\beta}(x) dx$, the relation (17) is equivalent to the following

$$\forall x \in \mathbb{R}, \sigma_{r_i} *_{\alpha,\beta} f(x) = 0 \quad (i = 1, 2). \tag{18}$$

So by Theorem 3.1 there is no non trivial function satisfying (17) if and only if the functions $\lambda \to \mathcal{F}_{\alpha,\beta}(\sigma_{r_1})(\lambda)$ and $\lambda \to \mathcal{F}_{\alpha,\beta}(\sigma_{r_2})(\lambda)$ have no common zero. But we have

$$\frac{d}{dx}\left[(\sinh 2x)^{-1} A_{\alpha+1,\beta+1}(x) \varphi_\mu^{\alpha+1,\beta+1}(x)\right] = 16(\alpha+1) A_{\alpha,\beta}(x) \varphi_\mu^{\alpha,\beta}(x),$$

([15], p.148). It follows that ($\lambda^2 = \mu^2 + \rho^2$),

$$\mathcal{F}_{\alpha,\beta}(\sigma_{r_i})(\lambda) = \frac{2^{2\rho}}{\alpha+1} (\sinh r_i)^{2(\alpha+1)} (\cosh r_i)^{2(\beta+1)} \varphi_\mu^{\alpha+1,\beta+1}(r_i),$$

and this gives the result. ∎

By applying Theorem 3.1, for the measures $\mu_i = \frac{1}{2}(\delta_{r_i} + \delta_{-r_i}),\ \ i = 1, 2,$ we obtain the following proposition.

PROPOSITION 3.3. *For $r_1, r_2 > 0$, a necessary and sufficient condition such that there is no non trivial function $f \in C(\mathbb{R})$ satisfying*

$$\forall x \in \mathbb{R}, T_{r_i} f(x) + T_{-r_i} f(x) = 0, \quad (i = 1, 2)$$

*is that the functions $\mu \to \varphi_\mu^{\alpha+1,\beta+1}(r_1)$ and $\mu \to \varphi_\mu^{\alpha+1,\beta+1}(r_2)$, have no common zeros.*

## 4. Convolution equations

For $\mu, \nu \in \mathcal{E}'(I\!R)$, $\mu$ being $(\alpha,\beta)-$slowly-decreasing, we consider the system of $(\alpha,\beta)$-convolution equation

$$(S) \quad \begin{cases} \mu *_{\alpha,\beta} f = g, \\ \nu *_{\alpha,\beta} f = h, \end{cases}$$

where $g, h \in \mathcal{E}(I\!R)$ and the unknown function $f$ is also sought in $\mathcal{E}(I\!R)$. For the classical case one can see [7].

There is clearly compatibility condition, namely:

$$\mu *_{\alpha,\beta} h = \nu *_{\alpha,\beta} g.$$

PROPOSITION 4.1. *Assume that $\mathcal{F}_{\alpha,\beta}(\mu)$ and $\mathcal{F}_{\alpha,\beta}(\nu)$ have no common zeros. The necessary and sufficient condition for the previous system (S) to have a solution for every pair $(g,h)$ satisfying the compatibility condition, is the existence of $\mu_1, \nu_1 \in \mathcal{E}'(I\!R)$ such that*

$$\mu_1 *_{\alpha,\beta} \mu + \nu_1 *_{\alpha,\beta} \nu = \delta_0. \tag{19}$$

*In this case the solution is unique.*

P r o o f. From relation (7), the system (S) is equivalent to

$$\begin{cases} {}^tV_{\alpha,\beta}(\mu) * V_{\alpha,\beta}^{-1}(f) = V_{\alpha,\beta}^{-1}(g), \\ {}^tV_{\alpha,\beta}(\nu) * V_{\alpha,\beta}^{-1}(f) = V_{\alpha,\beta}^{-1}(h). \end{cases}$$

Using the classical result ([6], p. 219) and the fact that ${}^tV_{\alpha,\beta}^{-1}(\delta_0) = \delta_0$, we deduce the result. ■

By using Theorem 3.1, one can remark that if $f$ is a solution of the system (S), then $f$ is unique.

COROLLARY 4.2. *If the hypotheses of Proposition 4.1 hold, then the necessary and sufficient condition for the existence of a solution $f$ of the previous system (S) for every pair satisfying the compatibility condition is the existence of constants $\varepsilon > 0$ and $C > 0$, such that*

$$|\mathcal{F}_{\alpha,\beta}(\mu)(\xi)| + |\mathcal{F}_{\alpha,\beta}(\nu)(\xi)| \geq \varepsilon \frac{\exp(-C|\mathcal{I}m\xi|)}{(1+|\xi|)^C}, \text{ for all } \quad \xi \in \mathbb{C}. \tag{20}$$

P r o o f. The proof of the corollary is based on Proposition 4.1, the relation (7) and the result in the classical case (see [6] p. 220). ■

Remarks.

i) If we denote by

$$P_{\mu,\nu} : \mathcal{E} \to \mathcal{E}(\mathbb{R}) \times \mathcal{E}(\mathbb{R}), \quad f \to (\mu *_{\alpha,\beta} f, \nu *_{\alpha,\beta} f),$$

called Pompeiu transform under the assumption that $\mathcal{F}_{\alpha,\beta}(\mu), \mathcal{F}_{\alpha,\beta}(\nu)$ have no common zero, then $P_{\mu,\nu}$ is injective.

ii) Let $\mu, \nu \in \mathcal{E}'(\mathbb{R})$ satisfying (19), the solution $f$ of the system (S) is given by

$$f = \mu_1 *_{\alpha,\beta} g + \nu_1 *_{\alpha,\beta} h.$$

## References


[1] S.C. Bagchi and A. Sitaram, The Pompeiu problem revisited. *L'enseignement mathématique*, **36** (1990), 67-91.

[2] N. Ben Salem and A. Ould Ahmed Salem, Convolution structure associated with the Jacobi-Dunkl operator on $\mathbb{R}$. *Ramanujan J.* **12**, No 3 (2006), 357-376.

[3] N. Ben Salem and A. Ould Ahmed Salem, Sobolev type spaces associated with the Jacobi-Dunkl operator. *Fract. Calc. Appl. Anal.* **7**, No 1 (2004), 37-60.

[4] N. Ben Salem and S. Kallel, Mean-periodic functions associated with the Dunkl operators. *Integral Transforms Spec. Funct.* **15**, No 2 (2004), 155-179.

[5] C.A. Berenstein and Roger Gay, *Complex Analysis and Special Topics in Harmonic Analysis.* Springer-Verlag, New York-Berlin-Heidelberg (1995).

[6] C.A. Berenstein and B.A. Taylor, Mean-periodic functions. *Internat. J. Math. Math. Sci.* **3**, No 2 (1980), 199-235.

[7] C. A. Berenstein and B. A. Taylor, A new look at interpolation theory for entire functions of one variable. *Adv. Math.* **33** (1979), 109-143.

[8] C.A. Berenstein and L. Zalcman, Pompeiu's problem on symmetric spaces. *Comment. Math. Helvetici* **55** (1980), 593-621.

[9] L. Brown, M. Schreiber and B.A. Taylor, Spectral synthesis and the Pompeiu problem. *Ann. Inst. Fourier, Grenoble* **23**, No 3 (1973), 125-154.

[10] F. Chouchane, M. Mili and K. Trimèche, Positivity of the intertwining operator and harmonic analysis associated with the Jacobi-Dunkl operator on $\mathbb{R}$. *Analysis and Applications* **1**, No 4 (2003), 387-412.

[11] J. Delsarte, Les fonctions moyenne-périodiques. *J. Math. Pures et Appl.* **14** (1935), 403-453.

[12] J. Delsarte, *Lectures on Topics in Mean-periodic Functions and the Two-Radius Theorem.* Tata Institute of Fundamental Research, Bombay (1961).

[13] L. Ehrenpreis, Solutions of some problems of division, IV. *Amer. J. of Math.* **82** (1960), 522-588.

[14] J.P. Kahane, *Lectures on Mean Periodic Functions.* Tata Institute of Fundamental Research, Bombay (1957).

[15] T.H. Koornwinder, A new proof of a Paley-Wiener type theorem for the Jacobi transform. *Ark. Math.* **13** (1975), 145-159.

[16] T.H. Koornwinder, Jacobi functions and analysis on noncompact semi simple Lie groups. In: *Special Functions: Group Theoretical Aspects and Applications* (Eds. R.A. Askey, T. H. Koornwinder and W. Schempp) Dordrecht (1984).

[17] E.M. Opdam, *Lectures on Dunkl operators.* arXiv:math.RT:9812007 v1.

[18] A. Ould Ahmed Salem and B. Selmi, Pompeiu-type theorem associoated with the Jacobi-Dunkl operator on $I\!R$. Submitted paper.

[19] L. Schwartz, Théorie générale des fonctions moyenne-périodiques. *Ann. of Math.* **48** (1947), 857-929.

[20] K. Trimèche, *Transmutation Operators and Mean-Periodic Functions Associated with Differential Operators.* Mathematical Reports **4**, Part 1 (1988), Harwood Academic Publishers.

[1] *Faculté des Sciences de Tunis*
*Département de Mathématiques*
*Tunis, TUNISIA*

*e-mail: Nejib.BenSalem@fst.rnu.tn*

*Received: November 8, 2006*

[2,3] *Faculté des Sciences de Bizerte*
*Département de Mathématiques*
*Bizerte, TUNISIA*

*e-mails:* [2] *Ahmedou.OuldAhmedSalem@fst.rnu.tn*
[3] *Belgacem.Selmi@fsb.rnu.tn*