Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 9, NUMBER 3 (2006) ISSN 1311-0454

COMMUTANTS OF THE DUNKL OPERATORS IN $C(\mathbb{R})$

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Dedicated to Professor Khalifa Trimèche,
on the occasion of his 60th anniversary

Abstract

The Dunkl operators $D_k f(x) = \dfrac{df(x)}{dx} + k\dfrac{f(x)-f(-x)}{x}$, $k \geq 0$, are considered in the space $C^1 = C^1(\mathbb{R})$ of the smooth functions on $\mathbb{R} = (-\infty, \infty)$, and the operators $M : C(\mathbb{R}) \to C(\mathbb{R})$ with $M : C^1(\mathbb{R}) \to C^1(\mathbb{R})$ such that $MD_k = D_k M$ in $C^1(\mathbb{R})$ are characterized ($C(\mathbb{R})$ being the space of continuous functions on $\mathbb{R}$). Next, for a non-zero linear functional $\Phi : C(\mathbb{R}) \to \mathbb{C}$ the continuous linear operators M with the invariant hyperplane $\Phi\{f\} = 0$ and commuting with D_k in it are also characterized. Further, mean-periodic functions for D_k with respect to the functional Φ are introduced and it is proved that they form an ideal in a corresponding convolutional algebra $(C(\mathbb{R}), *)$. As an application the mean-periodic solutions of differential-difference equations of the form $P(D_k)y = f$ with a polynomial P are considered.

2000 Mathematics Subject Classification: 44A35; 42A75; 47A16, 47L10, 47L80

Key Words and Phrases: Dunkl operator, commutant, invariant hyperplane, convolutional algebra, multiplier, cyclic element, mean-periodic function

[*] Supported by the Tunisian Research Foundation under 04/UR/15-02

1. Introduction

In the last two decades the differential-difference operators

$$D_k f(x) = \frac{df(x)}{dx} + k\frac{f(x) - f(-x)}{x}, \qquad k \geq 0 \tag{1}$$

introduced by C. F. Dunkl [12] had been the item of numerous studies. For a survey of most of them see e.g. M. Rösler [15].

Nevertheless, some basic problems, connected with the Dunkl operators (1), still remain out of the attention of the researchers. We will mention here for example the spectral theory of D_k. Compared with the differentiation operator, there is no systematic study of it.

Here we attempt to solve some problems connected with D_k which in a sense belongs to its spectral theory. For other operators their analogues are treated as application of their spectral theory. Here we prefer a direct approach, since till now no spectral theory of D_k is available.

In the Introduction we consider some auxiliary results for the Dunkl operators to be used later. Most of them are found by other authors, and maybe only the general Taylor expansion for the Dunkl operators and a convolution product $f * g$ in $C(\mathbb{R})$ are new.

1.1. A family of operators commuting with D_k: the translation operators T_k^y

The Dunkl translation (or shift) operators (see Trimeche [18] and M. Rösler [15]), are a class of operators $M : C(\mathbb{R}) \to C(\mathbb{R})$ commuting with D_k in $C^1(\mathbb{R})$, and in a sence, they are the simplest such operators. Let us remind their definition.

Definition 1. Let $f \in C(\mathbb{R})$ and $y \in \mathbb{R}$. Then $(T_k^y f)(x) = u(x, y)$ is the solution of the initial value problem

$$D_{k,x} u(x, y) = D_{k,y} u(x, y), \qquad u(x, 0) = f(x). \tag{2}$$

T_k^y is said to be a translation operator for the Dunkl operator D_k.

Such solution $u(x, y)$ exists for arbitrary $f \in C(\mathbb{R})$ and it has the following explicit form (see e.g. [1] and [15]):

$$T_k^y f(x) = \frac{\Gamma\left(k + \frac{1}{2}\right)}{\Gamma(k)\Gamma\left(\frac{1}{2}\right)} \left[\int_0^\pi f_e\left(\sqrt{x^2 + y^2 - 2|xy|\cos t}\right) h^e(x, y, t) \sin^{2k-1} t\, dt\right.$$

$$+\int_0^\pi f_o\left(\sqrt{x^2+y^2-2|xy|\cos t}\right) h^o(x,y,t)\sin^{2k-1} t\,dt\Bigg].$$

As usually, the subscripts "e" and "o" denote correspondingly the even and the odd part of a function g : $g_e(x) = \dfrac{g(x)+g(-x)}{2}$, $g_o(x) = \dfrac{g(x)-g(-x)}{2}$. If $h \in C(\mathbb{R})$, then

$$\begin{aligned} h^e(x,y,t) &= 1-\operatorname{sign}(xy)\cos t, \\ h^o(x,y,t) &= \begin{cases} \dfrac{(x+y)(1-\operatorname{sign}(xy)\cos t)}{\sqrt{x^2+y^2-2|xy|\cos t}} & \text{for } (x,y)\neq(0,0), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

LEMMA 1. *The translation operators satisfy:*

$$\text{(i) } T_k^y f(x) = T_k^x f(y) \qquad \textit{and} \qquad \text{(ii) } T_k^y T_k^z f(x) = T_k^z T_k^y f(x).$$

P r o o f. (i) It follows by interchanging x and y in the definition (2). Then the solution is the same, but it has to be denoted by $T_k^x f(y)$, while the initial notation is $T_k^y f(x)$.

To prove (ii), use (i) as follows

$$\begin{aligned} &(T_k^y T_k^z f)(x) = T_k^y(T_k^z f(x)) = T_k^y(T_k^x f(z)) = T_k^y(T_k^x f)(z) \\ &= T_k^z(T_k^x f)(y) = T_k^z(T_k^x f(y)) = T_k^z(T_k^y f(x)) = (T_k^z T_k^y f)(x), \end{aligned}$$

which gives the result. ■

LEMMA 2. *Each of the operators T_k^y commutes with D_k in $C^1(\mathbb{R})$, i.e.*

$$D_{k,x} T_k^y f(x) = T_k^y D_{k,x} f(x). \tag{3}$$

P r o o f. Applying D_k to the defining equations (2) of the translation operator, it follows that

$$\begin{aligned} &D_{k,x}(D_{k,x}u(x,y)) = D_{k,x}(D_{k,y}u(x,y)) = (D_{k,x}D_{k,y})u(x,y) \\ &\qquad\qquad = (D_{k,y}D_{k,x})u(x,y) = D_{k,y}(D_{k,x}u(x,y)), \\ &D_{k,x}u(x,0) = D_{k,x}f(x). \end{aligned}$$

Here the commutation $D_{k,y}D_{k,x}g(x,y) = D_{k,x}D_{k,y}g(x,y)$ for any function $g(x,y) \in C^1(\mathbb{R}\times\mathbb{R})$ was used. Its validity is a straightforward check. Denote $v(x,y) = D_{k,x}u(x,y) = D_{k,x}T_k^y f(x)$ and $w(x) = D_{k,x}f(x)$. Then the above equalities are in fact the definition of the translation $v(x,y) = T_k^y w(x)$. Hence, substituting v and w yields the desired equality (3). ■

1.2. The right inverse operators of D_k in $C(\mathbb{R})$ and their Taylor expansions

Let L_k denote an arbitrary right inverse operator of D_k in $C^1(\mathbb{R})$. Then, if $f \in C^1(\mathbb{R})$, $L_k f(x) = y(x)$ is the solution of the equation $D_k y(x) = f(x)$. Each such solution is determined up to an additive constant C.

First, let $y = L_k f$ be the solution of $D_k y(x) = f(x)$ with zero initial value condition, i.e. $y(0) = 0$. It is easy to find that

$$L_k f(x) = \int_0^x \left[f_o(t) + \left(\frac{t}{x}\right)^{2k} f_e(t) \right] dt, \tag{4}$$

where f_e and f_o are the even and the odd part of f, respectively.

Indeed, let $y(x) = L_k f(x)$ be represented as the sum of its even and odd parts, i.e. $y = y_e + y_o$. Then, in $D_k y(x) = f(x)$, which can be written as

$$y_e' + y_o' + \frac{2k}{x} y_o = f_e + f_o,$$

we separate the even and the odd part:

$$y_o' + \frac{2k}{x} y_o = f_e, \qquad y_e' = f_o.$$

Solving these two simultaneous equations and taking into account the initial condition $y(0) = 0$ yields

$$y_e(x) = \int_0^x f_o(t)dt \qquad \text{and} \qquad y_o(x) = \int_0^x \left(\frac{t}{x}\right)^{2k} f_e(t)dt,$$

which gives the desired representation.

In the general case the representation of an arbitrary right inverse operator L_k of D_k is

$$L_k f(x) = \int_0^x \left[f_e(t) + \left(\frac{t}{x}\right)^{2k} f_o(t) \right] dt + C.$$

In order L_k to be a linear operator, the additive constant C should depend on f, i.e. it has to be a linear functional $\Psi\{f\}$. Hence, an arbitrary linear right inverse operator of D_k in $C^1(\mathbb{R})$ has the form

$$L_k f(x) = \int_0^x \left[f_e(t) + \left(\frac{t}{x}\right)^{2k} f_o(t) \right] dt + \Psi\{f\},$$

with a linear functional Ψ.

According to the general theory of right invertible operators (Bittner [2], Przeworska-Rolewicz [14]) an important characteristic of L_k is its "initial projector"

$$Ff(x) = f(x) - L_k D_k f(x) = \Phi\{f\}. \tag{5}$$

It maps $C^1(\mathbb{R})$ onto $\ker D_k = \mathbb{R}$, i.e. it is a linear functional Φ on $C^1(\mathbb{R})$. Expressing Φ by Ψ, we obtain

$$\Phi\{f\} = f(0) - \Psi\{D_k f\}.$$

Let us note that $\Phi\{1\} = 1$ which expresses the projector property of F.

Considering the right inverse operator L_k of D_k, it is more convenient to obtain $L_k f$ as the solution of an elementary boundary value problem of the form

$$D_k y = f, \qquad \Phi\{y\} = 0, \tag{6}$$

where Φ is a given linear functional on $C^1(\mathbb{R})$ with $\Phi\{1\} = 1$.

The simplest case of such an operator is when Φ is the Dirac functional $\Phi\{f\} = f(0)$. Then L_k is the operator (4). The general solution of (6) is

$$L_k f(x) = \int_0^x \left[f_e(y) + \left(\frac{y}{x}\right)^{2k} f_o(y)\right] dy - \Phi_t \left\{ \int_0^t \left[f_e(y) + \left(\frac{y}{t}\right)^{2k} f_o(y)\right] dy \right\}. \tag{7}$$

Definition 2. The Dunkl-Appell polynomials $\{A_{k,n}(x)\}_{n=0}^{\infty}$ are introduced by the recurrences

$$A_{k,0}(x) \equiv 1, \quad \text{and} \quad D_k A_{k,n+1}(x) = A_{k,n}(x), \quad \Phi\{A_{k,n+1}\} = 0, \quad n \geq 0.$$

Lemma 3. *The Dunkl-Appell polynomials have the representation*

$$A_{k,n}(x) = L_k^n\{1\}(x),$$

where L_k is the right inverse (7) of the Dunkl operator D_k.

P r o o f. By induction: If $n = 1$, then $D_k A_{k,1}(x) = A_{k,0}(x) \equiv 1 \equiv L_k^0\{1\}(x)$ and therefore $A_{k,1}(x) = L_k\{1\}(x)$. Now, suppose that the assertion is true for arbitrary $n \geq 0$. Then

$$D_k A_{k,n+1}(x) = A_{k,n}(x) = L_k^n\{1\}(x), \Phi\{A_{k,n+1}\} = 0,$$

hence $A_{k,n+1}(x) = L_k A_{k,n}(x) = L_k L_k^n\{1\}(x) = L_k^{n+1}\{1\}(x)$, which proves the lemma. ∎

Lemma 4. *If $f \in C^n(\mathbb{R})$, then*

$$f(x) = \sum_{j=0}^{n-1} \Phi\{D_k^j f\} A_{k,j}(x) + L_k^n (D_k^n f)(x) \qquad \text{and} \tag{8}$$

$$T_k^y f(x) = T_k^x f(y) = \sum_{j=0}^{n-1} \Phi\{T_k^x D_k^j f\} A_{k,j}(y) + L_k^n (T_k^x D_k^n f)(y), \tag{9}$$

where $A_{k,j}(y) = L_k^j\{1\}(y)$ are the Dunkl-Appell polynomials, related to the functional Φ.

P r o o f. Delsarte [5], Bittner [2], and Przeworska-Rolewicz [14] proposed variants of the Taylor formula for right invertible operators in linear spaces. In our case the general Taylor formula is the obvious operator identity

$$I = \sum_{j=0}^{n-1} L_k^j F D_k^j + L_k^n D_k^n,$$

where I is the identity operator and $F = I - L_k D_k$. In functional form the above identity takes the form

$$f(x) = \sum_{j=0}^{n-1} L_k^j F D_k^j f(x) + L_k^n D_k^n f(x),$$

where the initial projector F of L_k (5) is a linear functional Φ:

$$F f(x) = f(x) - L_k D_k f(x) = \Phi\{f\}.$$

F projects the space $C(\mathbb{R})$ into the space $\mathbb{R}$ of the constants. Hence the Taylor formula with remainder term for the Dunkl operator D_k is

$$f(x) = \sum_{j=0}^{n-1} \Phi\{D_k^j f\} L_k^j\{1\}(x) + L_k^n D_k^n f(x), \tag{10}$$

which gives the result. (9) follows from (8) if we substitute $f(x)$ by $T_k^y f(x)$. ∎

Corollary 1. *If f is a polynomial, then*

$$f(x) = \sum_{j=0}^{\infty} \Phi\{D_k^j f\} A_{k,j}(x) \qquad \text{and} \tag{11}$$

$$T_k^y f(x) = T_k^x f(y) = \sum_{j=0}^{\infty} \Phi\{T_k^x D_k^j f\} A_{k,j}(y), \tag{12}$$

where $A_{k,j} = L_k^j\{1\}$ *are the Dunkl-Appell polynomials.*

Further, we will use only the special case of the last formula, when $\Phi\{f\} = f(0)$. Then it takes the form

$$T_k^y f(x) = T_k^x f(y) = \sum_{j=0}^{\infty} D_k^j f(x) a_{k,j} y^j, \tag{13}$$

where $a_{k,j}$ are the constants

$$a_{k,j} = \begin{cases} \dfrac{1}{(2k+1)(2k+3)\ldots(2k+2m-1).2.4\ldots(2m-2)} & \text{if } j = 2m-1 \\ \dfrac{1}{(2k+1)(2k+3)\ldots(2k+2m-1).2.4\ldots 2m} & \text{if } j = 2m \end{cases}.$$

The values of the constants are not important for our purposes, but let us mention that they can be found using the representation (4), which implies for odd and even powers

$$L_k y^{2m-1} = \frac{y^{2m}}{2m} \qquad \text{and} \qquad L_k y^{2m} = \frac{y^{2m+1}}{2k+2m+1}.$$

Then, calculating consequently $L_k\{1\}, L_k^2\{1\}, \ldots, L_k^j\{1\}, \ldots,$ the formula for $a_{k,j}$ follows by induction.

1.3. The intertwining operator V_k for the Dunkl operator D_k

In Dunkl [12], Theorem 5.1, the similarity operator

$$V_k f(x) = b_k \int_{-1}^{1} f(xy)(1-y)^{k-1}(1+y)^k dy, \quad b_k = \frac{\Gamma(2k+1)}{2^{2k}\Gamma(k)\Gamma(k+1)} \tag{14}$$

is found, which transforms the differentiation operator $D = \dfrac{d}{dx}$ into D_k:

$$V_k D = D_k V_k.$$

Some properties of this operator and its adjoint operator are studied by K. Trimeche [17].

In Ben Salem and Kallel [1] the inverse operator V_k^{-1} of V_k is given. Using the denotation $Sf(x) = \dfrac{1}{2x}\dfrac{df(x)}{dx}$, it has the form:

(i) If $k = n + r$ with $n \in \mathbb{N}$ and $r \in (0,1)$, then for $x \neq 0$

$$V_k^{-1} f(x) = c_k \left[|x| S^{n+1} \int_0^{|x|} (x^2 - y^2)^{-r} f_e(y) y^{2k} dy \right. \\ \left. + \operatorname{sign} x . S^{n+1} \int_0^{|x|} (x^2 - y^2)^{-r} f_o(y) y^{2k+1} dy \right], \tag{15}$$

where

$$c_k = \frac{2\sqrt{\pi}}{\Gamma\left(n + r + \frac{1}{2}\right) \Gamma(1 - r)}.$$

(ii) If $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, then

$$V_k^{-1} f(x) = \frac{\sqrt{\pi}}{\Gamma\left(k + \frac{1}{2}\right)} [x S^k (x^{2k-1} f_e(x)) + S^k (x^{2k} f_o(x))], \quad x \neq 0. \tag{16}$$

V_k transforms $C(\mathbb{R})$ into a proper subspace $\widetilde{C_k}$ of it. We may expect that V_k is a similarity from a right inverse operator Λ of $D = \dfrac{d}{dx}$ to L_k. In order to specify the operator Λ let us define the linear functional

$$\widetilde{\Phi}\{f\} = (\Phi \circ V_k)\{f\} \tag{17}$$

in $\widetilde{C_k}$. Then define $\Lambda : \widetilde{C_k} \to \widetilde{C_k}$ to be the solution $y = \Lambda \widetilde{f}$ of the boundary value problem

$$Dy(x) = y'(x) = \widetilde{f}(x), \qquad \widetilde{\Phi}\{y\} = 0.$$

Thus we obtain

$$\Lambda \widetilde{f}(x) = \int_0^x \widetilde{f}(y) dy - \widetilde{\Phi_t} \left\{ \int_0^t \widetilde{f}(\tau) d\tau \right\}. \tag{18}$$

Lemma 5. *The following similarity relation holds:*

$$V_k \Lambda = L_k V_k. \tag{19}$$

P r o o f. Applying V_k to the defining equation $D(\Lambda \widetilde{f}) = \widetilde{f}$, one obtains

$$V_k D(\Lambda \widetilde{f}) = V_k \widetilde{f} = f$$

or

$$D_k (V_k \Lambda \widetilde{f}) = V_k \widetilde{f} = f.$$

In fact, the boundary value condition $\widetilde{\Phi}\{\Lambda\widetilde{f}\} = 0$ can be written as $\Phi\{V_k\Lambda\widetilde{f}\} = 0$. Hence $u = V_k\Lambda\widetilde{f}$ is the solution of the boundary value problem $D_k u = f, \Phi\{u\} = 0$, i.e. $u = L_k f$. Therefore

$$V_k\Lambda V_k^{-1} f = L_k f \qquad \text{or} \qquad V_k\Lambda = L_k V_k. \qquad \square$$

The similarity relation (19) allows to introduce a convolution structure $* : C(\mathbb{R}) \times C(\mathbb{R}) \to C(\mathbb{R})$, such that L_k to be the convolution operator $L_k = \{1\}*$ in $C(\mathbb{R})$.

The operator Λ is defined not only in $\widetilde{C}$, but in the whole space $C(\mathbb{R})$. This allows to introduce a convolution structure $\widetilde{*} : C(\mathbb{R}) \to C(\mathbb{R})$.

Lemma 6. (Dimovski [8], Theorem 2.1.1, p.52) *The operation*

$$(f\widetilde{*}g)(x) = \widetilde{\Phi}_t\left\{\int_t^x f(x+t-\tau)g(\tau)d\tau\right\} \tag{20}$$

is a bilinear, commutative and associative operation in $C(\mathbb{R})$, such that

$$\Lambda f = \{1\}\widetilde{*}f. \tag{21}$$

Lemma 7. *If $f, g \in C(\mathbb{R})$, then $f\widetilde{*}g \in C^n(\mathbb{R})$ where $n = k$ for integer k and $n = [k]+1$ for noninteger k.*

The proof follows from the results on the smoothness of convolution (20) in Bozhinov [3], Theorem 2.1.13, Corollary 4, since the linear functional $\widetilde{\Phi} = \Phi \circ V_k$ is a "smoothing" one.

Theorem 1. *The operation*

$$f * g = D_k^{2n} V_k[(V_k^{-1}L_k^n f)\widetilde{*}(V_k^{-1}L_k^n g)] \tag{22}$$

is a convolution of L_k in $C(\mathbb{R})$ such that

$$L_k f = \{1\} * f. \tag{23}$$

P r o o f. First of all, the operation $f*g$ is well defined in $C(\mathbb{R})$. Indeed, from the inversion formulae (15) and (16) for V_k it follows that $L_k^n f$ and $L_k^n g$ belong to the range of V_k, and hence $V_k^{-1}L_k^n f$ and $V_k^{-1}L_k^n g$ are functions from $C(\mathbb{R})$. From Lemma 7 it follows that $(V_k^{-1}L_k^n f)\widetilde{*}(V_k^{-1}L_k^n g) \in C^n(\mathbb{R})$.

The intertwining operator V_k increases the order of smoothness by n and hence (22) is well defined in $C(\mathbb{R})$.

The operator $T = V_k^{-1} L_k^n$ is a transmutation operator of L_k into Λ, i.e.

$$TL_k = \Lambda T. \tag{24}$$

Indeed, (24) is equivalent to (19). According to Dimovski [8], Theorem 1.3.6, p.26, the operation

$$f \widehat{*} g = T^{-1}[Tf \widetilde{*} Tg]$$

is a convolution of the operator $L_k = T^{-1}\Lambda T$ in $C(\mathbb{R})$, such that $f \widehat{*} g$ is in $C^n(\mathbb{R})$. Then $f * g = D_k^n (f \widehat{*} g)$ is well defined in $C(\mathbb{R})$ and (23) holds. ■

2. General commutant

The main result in this section is the following theorem.

THEOREM 2. *Let $M : C(\mathbb{R}) \to C(\mathbb{R})$ be a continuous linear operator with $M : C^1(\mathbb{R}) \to C^1(\mathbb{R})$. Then the following assertions are equivalent:*
(i) *M commutes with the Dunkl operator*

$$D_k f(x) = \frac{df(x)}{dx} + k \frac{f(x) - f(-x)}{x} \quad in \quad C^1(\mathbb{R});$$

(ii) *$MT_k^y = T_k^y M$ for every $y \in \mathbb{R}$;*
(iii) *M admits a representation of the form*

$$(Mf)(x) = \Phi_t\{T_k^t f(x)\} \tag{25}$$

with a continuous linear functional $\Phi : C(\mathbb{R}) \to \mathbb{C}$.

P r o o f. (i)⇒(ii)
Suppose that M commutes with the Dunkl operator D_k in $C^1(\mathbb{R})$, i.e. $MD_k f = D_k M f$ for $f \in C^1(\mathbb{R})$. Then, for arbitrary $y \in \mathbb{R}$ and any polynomial $f(x)$, Taylor formula (13) implies

$$\begin{aligned}
(MT_k^y f)(x) = (MT_k^x f)(y) = M_x \sum_{n=0}^{\infty} (D_k^n f)(x) a_{k,n} y^n \\
= \sum_{n=0}^{\infty} (MD_k^n f)(x) a_{k,n} y^n = \sum_{n=0}^{\infty} (D_k^n M f)(x) a_{k,n} y^n \\
= \sum_{n=0}^{\infty} (D_k^n (Mf))(x) a_{k,n} y^n = (T_k^x M f)(y) = (T_k^y M f)(x).
\end{aligned}$$

Since $MT_k^y = T_k^y M$ is true for polynomials, then this is true for arbitrary $f \in C^1(\mathbb{R})$. One should simply use approximation by polynomials.

(ii)⇒(i)

Suppose $MT_k^t = T_k^t M$ for every $t \in \mathbb{R}$. For arbitrary polynomial $f(x)$ reverse the order in the above chain of equalities as follows:

$$\sum_{n=0}^{\infty}(MD_k^n f)(x)a_{k,n}y^n = (M(T_k^t f))(x)$$

$$= (T_k^t(Mf))(x) = \sum_{n=0}^{\infty}(D_k^n Mf)(x)a_{k,n}y^n$$

The sums have to coincide for every x and hence the coefficients of y^n are equal for arbitrary n. For $n = 1$ it follows that

$$(M(D_k f))(x) = (D_k(Mf))(x). \tag{26}$$

Assuming that (26) is true for polynomials, it follows that it is true for arbitrary $f \in C^1(\mathbb{R})$ since f could be approximated by polynomials.

(ii)⇒(iii)

Let

$$MT_k^y f(x) = T_k^y Mf(x), \qquad \forall y \in \mathbb{R}. \tag{27}$$

The property $T_k^y f(x) = T_k^x f(y)$ applied to the right hand side of (27) gives

$$(M(T_k^y f))(x) = (T_k^x(Mf))(y). \tag{28}$$

Define the linear functional Φ as

$$\Phi\{f\} := (Mf)(0).$$

Then, substituting $y = 0$ in (28) and taking into account that T_k^0 is the identity operator, one has

$$(M(T_k^y f))(0) = (T_k^0(Mf))(y) = (Mf)(y). \tag{29}$$

The left hand side is the value of the functional Φ for the function $g(x) = (T_k^y f)(x)$, and hence

$$(Mf)(y) = \Phi_t\{(T_k^y f)(t)\} = \Phi_t\{(T_k^t f)(y)\}$$

using (28) and property (i) from Lemma 1. This in fact is the desired representation (25) of the commutant of D_k with y for x, and with the dumb variable t instead of y.

(iii)⇒(ii)

It is a matter of a direct check to show that the operators of the form (25) commute with T_k^y for every $y \in \mathbb{R}$:

$$\begin{aligned} MT_k^y f(x) &= \Phi_t\{(T_k^t T_k^y f)(x)\} = \Phi_t\{(T_k^y T_k^t f)(x)\} \\ &= T_k^y \Phi_t\{T_k^t f)(x)\} = T_k^y M f(x). \end{aligned}$$

This completes the proof. ■

Remark 1. From Kahane [13] it follows that the commutant of the differentiation operator D_0 in $C^1([-a,a])$ consists only of the trivial operators $Mf(x) = cf(x)$, where $c = const$. The same is true for D_k, $k > 0$, too. For the proof, the intertwining operator V_k has to be used.

3. The commutant of D_k in an invariant hyperplane

A hyperplane in $C(\mathbb{R})$ can be defined by an arbitrary nonzero linear functional Φ in $C(\mathbb{R})$. Such a functional has an explicit Riesz-Markov representation

$$\Phi\{f\} = \int_\alpha^\beta f(t)d\mu(t)$$

with $-\infty < \alpha \leq \beta < +\infty$ and with a Radon measure μ on $[\alpha,\beta]$. This measure μ is generated by a complex valued function of bounded variation and we will denote it also by μ. It is convenient to assume that $\mu(t)$ is normalized in such a way that the representation to be unique. To this end we may assume that $\mu(t)$ is right-continuous and $\mu(\alpha) = 0$. The case $\alpha = \beta$ is a special case of the Dirac functional $\Phi\{f\} = Af(\alpha)$. Here we do not consider this case separately (which it deserves on its own right) since it is embraced in the general case with the only assumption that $\Phi\{1\} = \mu([\alpha,\beta]) \neq 0$.

Definition 3. A hyperplane in $C(\mathbb{R})$ defined by a nonzero linear functional Φ is said to be the subset of $C(\mathbb{R})$

$$C_\Phi = \{f : f \in C(\mathbb{R}), \Phi\{f\} = 0\}.$$

Our main task in this section is to characterize the linear operators $M : C(\mathbb{R}) \to C(\mathbb{R})$ for which C_Φ is an invariant hyperplane, i.e. with $M(C_\Phi) \subset C_\Phi$, and which commute with a Dunkl operator D_k in C_Φ^1. As usual, by C^1_Φ we denote the subspace of C_Φ, consisting of smooth functions. The set of all such operators will be denoted by $\mathrm{Comm}_\Phi\{D_k\}$.

Since the commutation relation $MD_k = D_kM$ should be satisfied in C^1_Φ, we will consider only operators $M : C(\mathbb{R}) \to C(\mathbb{R})$, for which $M : C^1(\mathbb{R}) \to C^1(\mathbb{R})$. This restriction allows to give a complete constructive characterization of the commutant $\mathrm{Comm}_\Phi\{D_k\}$. Such restriction is not assumed in the next theorem.

THEOREM 3. *A linear operator* $M : C(\mathbb{R}) \to C(\mathbb{R})$ *belongs to* Comm_Φ $\{D_k\}$ *iff it commutes with a right inverse operator* R_{λ_0} *of* $D_k^{(\lambda_0)} = D_k - \lambda_0$, *defined by the boundary value condition* $\Phi\{R_{\lambda_0}f\} = 0$ *provided* $\lambda_0 \in \mathbb{C}$ *is such that* R_{λ_0} *exists.*

P r o o f. a) Let $\lambda_0 \in \mathbb{C}$ be such that R_{λ_0} exists. This means that the boundary value problem

$$(D_k - \lambda_0)y = f, \qquad \Phi\{y\} = 0$$

has a unique solution $y = R_{\lambda_0}f$ for arbitrary $f \in C(\mathbb{R})$. The operator R_{λ_0} is such that $R_{\lambda_0} : C(\mathbb{R}) \to C^1_\Phi$. We are to prove that if for an operator $M : C(\mathbb{R}) \to C(\mathbb{R})$ one has $MR_{\lambda_0} = R_{\lambda_0}M$ and $M(C_\Phi) \subset C_\Phi$, then $MD_k = D_kM$ in C^1_Φ.

Consider the function $h = (MD_k^{(\lambda_0)} - D_k^{(\lambda_0)}M)f$ for $f \in C^1_\Phi$. We will show that $R_{\lambda_0}h = 0$. Indeed,

$$\begin{aligned} R_{\lambda_0}h &= R_{\lambda_0}MD_k^{(\lambda_0)}f - R_{\lambda_0}D_k^{(\lambda_0)}Mf \\ &= M(R_{\lambda_0}D_k^{(\lambda_0)})f - R_{\lambda_0}D_k^{(\lambda_0)}Mf. \end{aligned}$$

According to Przeworska-Rolewicz [14] we have $R_{\lambda_0}D_k^{(\lambda_0)}f = f$ on C^1_Φ. Then by the assumption $f \in C^1_\Phi \Rightarrow Mf \in C^1_\Phi$ and hence $R_{\lambda_0}D_k^{(\lambda_0)}Mf = Mf$. Thus we obtain $R_{\lambda_0}h = Mf - Mf = 0$, which implies $h = 0$.

b) Conversely, let $M : C_\Phi(\mathbb{R}) \to C_\Phi(\mathbb{R})$ and $MD_k^{(\lambda_0)} = D_k^{(\lambda_0)}M$ in C^1_Φ. If $f \in C(\mathbb{R})$, consider the function $g = (MR_{\lambda_0} - R_{\lambda_0}M)f$. It is easy to verify that $D_k^{(\lambda_0)}g = 0$ and $\Phi\{g\} = 0$. Indeed,

$$\begin{aligned} D_k^{(\lambda_0)}g &= D_k^{(\lambda_0)}MR_{\lambda_0}f - (D_k^{(\lambda_0)}R_{\lambda_0})Mf \\ &= M(D_k^{(\lambda_0)}R_{\lambda_0})f - Mf = Mf - Mf = 0, \end{aligned}$$

and

$$\Phi\{g\} = \Phi\{MR_{\lambda_0}f\} - \Phi\{R_{\lambda_0}Mf\} = 0$$

since $R_{\lambda_0} : C(\mathbb{R}) \to C^1_\Phi(\mathbb{R})$, i.e. $\Phi\{R_{\lambda_0} f\} = 0$. Since $M : C_\Phi \to C_\Phi$, then $\Phi\{MR_{\lambda_0} f\} = 0$. We proved that g is the solution of the boundary value problem

$$D_{\lambda_0} g = 0, \qquad \Phi\{g\} = 0,$$

and hence $g = 0$.

Further, we may choose a $\lambda_0 \in \mathbb{C}$ for which there exists R_{λ_0} and look for the commutant of R_{λ_0} in $C(\mathbb{R})$. For the sake of simplicity we assume that $\lambda_0 = 0$.

Let us find the operator $R_0 = L_k$. It is the right inverse of D_k defined by the solution of the boundary value problem

$$D_k y = f, \qquad \Phi\{y\} = 0. \tag{30}$$

The solution was given in (7) in the second part of the introduction and it is

$$L_k f(x) = \int_0^x \left[f_o(y) + \left(\frac{y}{x}\right)^{2k} f_e(y) \right] dy - \Phi_t \left\{ \int_0^t \left[f_o(y) + \left(\frac{y}{t}\right)^{2k} f_e(y) \right] dy \right\},$$

where it is assumed that $\Phi\{1\} = 1$ and f_e and f_o are the even and the odd part of f correspondingly. ■

According to Theorem 3 the problem of characterization of the commutant of D_k in the invariant hyperplane C_Φ reduces to the problem of characterization of the the commutant of L_k in $C(\mathbb{R})$. This is a more or less standard problem in the frames of the convolutional calculus (see Dimovski [8] and Bozhinov [3]). The general scheme is the following:

1. Find a separately continuous convolution of L_k in $C(\mathbb{R})$, i.e. a bilinear, commutative and associative operation $f * g$, such that L_k to be a convolutional operator $L_k = \{\varphi\}*$ for some $\varphi \in C(\mathbb{R})$.

2. Show that the commutant of L_k in $C(\mathbb{R})$ coincides with the ring of the multipliers of the convolutional algebra $(C(\mathbb{R}), *)$. A sufficient condition is L_k to have a cyclic element in $C(\mathbb{R})$ (see Dimovski [8], p.32, Theorem 1.3.10).

3. Then an operator $M : C(\mathbb{R}) \to C(\mathbb{R})$ commutes with L_k in $C(\mathbb{R})$ iff it has the form

$$Mf = D_k(\varphi * f).$$

In our case under the additional restriction $M : C^1(\mathbb{R}) \to C^1(\mathbb{R})$ we obtain the following complete characterization of the commutant:

$$M \in \operatorname{Comm}_{C(\mathbb{R})}\{L_k\} \Longleftrightarrow Mf = m * f + \mu f$$

with $m \in C(\mathbb{R})$ and $\mu \in \mathbb{C}$.

Now we are ready to characterize the operators $M : C(\mathbb{R}) \to C(\mathbb{R})$ with $M : C^1(\mathbb{R}) \to C^1(\mathbb{R})$ commuting with L_k in $C^1(\mathbb{R})$.

THEOREM 4. *A continuous linear operator $M : C(\mathbb{R}) \to C(\mathbb{R})$ with $M : C^1(\mathbb{R}) \to C^1(\mathbb{R})$ commutes with L_k iff it admits a representation of the form*

$$Mf = m * f + \mu f \tag{31}$$

with $m \in C(\mathbb{R})$ and $\mu \in \mathbb{C}$. This representation is uniquely determined.

P r o o f. The operator L_k has the constant function $e(x) \equiv 1$ as a cyclic element in $C(\mathbb{R})$. Indeed, from the representation (23) it is seen that $L_k^n\{1\}$ is a polynomial of degree exactly n. Due to Weierstrass' approximation theorem, each $f \in C(\mathbb{R})$ can be approximated almost uniformly by linear combinations of $\{L_k^n\{1\}\}_{n=0}^\infty$, i.e. $\overline{\text{span}}\{L_k^n\{1\}\}_{n=0}^\infty = C(\mathbb{R})$.

Hence the commutant of L_k in $C(\mathbb{R})$ coincides with the ring of multipliers of the convolutional algebra $(C(\mathbb{R}), *)$ (see Dimovski [8], Theorem 1.3.11, p. 33).

It remains to characterize the multipliers $M : C(\mathbb{R}) \to C(\mathbb{R})$ of the convolutional algebra $(C(\mathbb{R}), *)$. Apply M to $L_k f = \{1\} * f$ to obtain

$$ML_k f = (M\{1\}) * f.$$

Since $ML_k = L_k M$, then

$$L_k(Mf) = (M\{1\}) * f.$$

Hence, applying D_k

$$Mf = D_k(M\{1\} * f).$$

Continue by using the formula

$$D_k(M\{1\} * f) = (D_k M\{1\}) * f + \Phi\{M\{1\}\} f$$

(see Dimovski [8], Theorem 1.3.8, p. 31). Thus the desired representation (31) is established with $m = D_k M\{1\}$ and $\mu = \Phi\{M\{1\}\}$.

In order to prove the uniqueness, assume that

$$M = m_1 * f + \mu_1 f = m_2 * f + \mu_2 f \qquad \text{or} \qquad (m_1 - m_2) * f = (\mu_2 - \mu_1) f.$$

Take $f(x) \equiv 1$:

$$L_K(m_1 - m_2) = \mu_2 - \mu_1.$$

Apply Φ:

$$0 = \mu_2 - \mu_1,$$

hence $\mu_1 = \mu_2$.

From $(m_1 - m_2) * f = 0$ we obtain $m_1 = m_2$, since the convolutional algebra $(C(\mathbb{R}), *)$ is annihilators-free. The proof is completed. ■

Combining Theorems 3 and 4, we can state the following characterization result:

THEOREM 5. *Let $M : C(\mathbb{R}) \to C(\mathbb{R})$ be a linear operator with an invariant hyperplane $C_\Phi = \{f : f \in C(\mathbb{R}), \Phi\{f\} = 0\}$ where Φ is a linear functional in $C(\mathbb{R})$ with $\Phi\{1\} = 1$. If $M : C^1(\mathbb{R}) \to C^1(\mathbb{R})$, then the following assertions are equivalent:*

(i) *M commutes with D_k in C_Φ;*

(ii) *M admits a representation of the form*

$$Mf = m * f + \mu f, \tag{32}$$

with $m \in C(\mathbb{R})$ and $\mu \in \mathbb{C}$.

4. Mean-periodic functions associated with the Dunkl operator

In Ben Salem and Kallel [1] the theory of the mean-periodic functions in $C(\mathbb{R})$, associated with the Dunkl operator D_k, is developed. This theory becomes more transparent if we use the explicit representations (25) and (32) of the commutants of D_k and its right inverses.

DEFINITION 4. A function $f \in C(\mathbb{R})$ is said to be mean-periodic for D_k if it belongs to the kernel space (null-space) of a linear operator $M : C(\mathbb{R}) \to C(\mathbb{R})$ with $M : C^1(\mathbb{R}) \to C^1(\mathbb{R})$ commuting with D_k.

As we have seen in Section 2, Theorem 2, each operator $M : C(\mathbb{R}) \to C(\mathbb{R})$ with $MD_k = D_kM$ in $C^1(\mathbb{R})$ has the form

$$(Mf)(x) = \Phi_t\{T_k^t f(x)\}$$

with a linear functional Φ in $C(\mathbb{R})$, where T_k^t is the generalized translation operator (2).

It is most natural to consider a class of mean-periodic functions depending on a fixed linear functional Φ. Further we denote such a class by MP_Φ.

To each class MP_Φ there corresponds the convolutional algebra $(C(\mathbb{R}), *)$, where $*$ is the operation (22).

LEMMA 8. *If $f \in MP_\Phi$, then $L_k f \in MP_\Phi$.*

P r o o f. Denote

$$\varphi(x) = \Phi_t\{T_k^t L_k f(x)\}$$

We use the commutation relation $D_k T_k^t = T_k^t D_k$ (Lemma 2) to obtain

$$D_k\varphi(x) = \Phi_t\{D_k T_k^t L_k f(x)\} = \Phi_t\{T_k^t D_k L_k f(x)\} = \Phi_t\{T_k^t f(x)\} = 0.$$

Hence $\varphi(x) = C = const.$ But $\Phi\{\varphi\} = \Phi_t\{T_k^t \Phi_x\{L_k f(x)\}\} = 0$ since $\Phi_x\{(L_k f(x)\}\} = 0$. We proved that $C = 0$. ■

THEOREM 6. *The class of mean-periodic functions MP_Φ is an ideal in the convolutional algebra $(C(\mathbb{R}), *)$.*

P r o o f. Assume that $f \in MP_\Phi$, i.e.

$$\Phi_t\{T_k^t f(x)\} = 0.$$

From Lemma 7, it follows that $L_k^{n+1} f \in MP_\Phi$ for $n = 0, 1, 2, \ldots$, i.e.

$$\Phi_t\{T_k^t L_k^{n+1} f(x)\} = 0.$$

Since $L_k f = \{1\} * f$, then $L_k^{n+1} f = A_{k,n} * f$, where the Dunkl-Appell polynomial $A_{k,n}$ is of degree exactly n. We have

$$\Phi_t\{T_k^t (A_{k,n} * f)(x)\} = 0$$

and then we can assert that

$$\Phi_t\{T_k^t (P * f)(x)\} = 0$$

for any polynomial P. By an approximation argument it follows that

$$\Phi_t\{T_k^t (g * f)(x)\} = 0$$

for an arbitrary function $g \in C(\mathbb{R})$, i.e. that $g * f \in MP_\Phi$. This completes the proof. ■

This theorem could be used to study the problem for solving differential-difference equations of the form

$$P(D_k)y = f, \tag{33}$$

with a polynomial P in a space MP_Φ of mean-periodic functions.

Here we will state only a typical result, leaving the complete study for a next publication.

Theorem 7. *In order a linear differential-difference equation of the form* (33) *to have a unique solution in* MP_Φ*, it is necessary and sufficient no one of the eigenvalues of the problem* $D_k u - \lambda u = 0$*,* $\Phi(u) = 0$*, to be a root of the polynomial* P*.*

Remark 2. Throughout of this paper (with the exception of Theorem 3) we assume that $\lambda = 0$ is not an eigenvalue of the problem $D_k u - \lambda u = 0$, $\Phi(u) = 0$, and this allows some technical simplifications. Nevertheless, the main results remain valid without such assumption. How should be settled the singular case, when $\lambda = 0$ is an eigenvalue, is shown in Dimovski and Hristov [9]. Representations (25) and (32) and the convolution (22) remain without any changes but one should use Theorem 3.

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Received: August 1, 2006

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