Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 9, NUMBER 2 (2006) ISSN 1311-0454

ON TWO SAIGO'S FRACTIONAL INTEGRAL OPERATORS IN THE CLASS OF UNIVALENT FUNCTIONS

Virginia Kiryakova *

Dedicated to Professor Megumi Saigo,
on the occasion of his 70th anniversary

Abstract

Recently, many papers in the theory of univalent functions have been devoted to mapping and characterization properties of various linear integral or integro-differential operators in the class $\mathcal{S}$ (of normalized analytic and univalent functions in the open unit disk U), and in its subclasses (as the classes $\mathcal{S}^*$ of the starlike functions and $\mathcal{K}$ of the convex functions in U). Among these operators, two operators introduced by Saigo, one involving the Gauss hypergeometric function, and the other - the Appell (or Horn) F_3-function, are rather popular. Here we view on these Saigo's operators as cases of generalized fractional integration operators, and show that the techniques of the generalized fractional calculus and special functions are helpful to obtain explicit sufficient conditions that guarantee mappings as: $\mathcal{S} \mapsto \mathcal{S}$ and $\mathcal{K} \mapsto \mathcal{S}$, that is, preserving the univalency of functions.

2000 Mathematics Subject Classification: Primary 26A33, 30C45; Secondary 33A35

Key Words and Phrases: generalized fractional integrals; Saigo operators; classes of univalent, starlike and convex functions; Gauss and generalized hypergeometric functions

* Partially supported by National Science Fund (Bulg. Ministry of Educ. and Sci.) under Project MM 1305.

1. Definitions and introduction

Definition 1. For real numbers $\alpha > 0, \beta$ and η, the *Saigo hypergeometric fractional integral operator* $I_{0,z}^{\alpha,\beta,\eta} f(z)$ is defined by

$$I_{0,z}^{\alpha,\beta,\eta} f(z) = \frac{z^{-\alpha-\beta}}{\Gamma(\alpha)} \int\limits_0^z (z-\zeta)^{\alpha-1} {}_2F_1 \left(\begin{matrix} \alpha+\beta, -\eta; \\ \alpha; \end{matrix} \; 1 - \frac{\zeta}{z} \right) f(\zeta) d\zeta, \quad (1)$$

with the *Gauss hypergeomteric function* ${}_2F_1(a,b;c;z)$ in the kernel, as a special case of the *generalized hypergeometric function* (see for example, [7], Vol.1; [28]):

$${}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \cdot \frac{z^k}{k!}, \quad (a)_k := \Gamma(a+k)/\Gamma(a). \quad (2)$$

Here $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin (as such is the unit disk U), of the order

$$f(z) = O(|z|^{\varepsilon}) \qquad (z \to 0),$$

where

$$\varepsilon > \max\{0, \beta - \eta\} - 1,$$

and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z - \zeta > 0$.

This operator has been initially introduced by Saigo in a series of his papers for studying boundary value problems for partial differential equations, especially for the Euler-Darboux equation, see [30], [31], [32], [39], or equations of mixed type, as in [12], [34]. Later on, the Saigo hypergeometric operator and its modifications have been used in many papers by him and his collaborators, to study various problems of univalent functions theory, see for example [40], [24], [38], [6], [16], etc.

Operator (1) contains as special cases the *Riemann-Liouville (R-L)* and *Erdélyi-Kober (E-K) operators* of fractional integration of order $\alpha > 0$, in the *classical fractional calculus (FC)*, see [36], [14]:

$$R^{\alpha} f(z) = z^{\alpha} \int\limits_0^1 \frac{(1-\sigma)^{\alpha-1}}{\Gamma(\alpha)} f(z\sigma) d\sigma, \quad (3)$$

$$I_{\beta}^{\gamma,\alpha} f(z) = \int\limits_0^1 \frac{(1-\sigma)^{\alpha-1}}{\Gamma(\alpha)} \sigma^{\gamma} f(z\sigma^{1/\beta}) d\sigma \quad (\alpha > 0, \gamma \in \mathbb{R}, \beta > 0), \quad (4)$$

namely:

$$R^{\alpha} f(z) = I_{0,z}^{\alpha,-\alpha,-\alpha} f(z) \quad , \quad I_1^{\gamma,\alpha} f(z) = z^{-\alpha-\gamma}\, I_{0,z}^{\alpha,-\alpha-\gamma,-\alpha} f(z).$$

Saigo's hypergeometric fractional integral, itself, can be represented as a composition of two E-K fractional integrals, for example:

$$I_{0,z}^{\alpha,\beta,\eta} f(z) = z^{-\beta} I_1^{\eta-\beta,-\eta} I_1^{0,\alpha+\eta} f(z). \tag{5}$$

For negative values of α, the operator (1) is extended as a fractional derivative operator similarly to the way of introducing of classical Riemann-Liouville and Erdélyi-Kober fractional derivatives, namely:

$$I_{0,z}^{\alpha,\beta,\eta} f(z) = \frac{d^n}{dz^n}\, I_{0,z}^{\alpha+n,\beta-n,\eta-n} f(z), \tag{6}$$

where $n = [-\Re(\alpha)] + 1$ with $[\alpha]$ denoting the integer part of α.

Further in this paper, for the sake of denotations' brevity, we shall omit the subindex $0, z$ and write the Saigo hypergeometric integral operator simply as

$$I^{\alpha,\beta,\eta} \quad \text{instead of} \quad I_{0,z}^{\alpha,\beta,\eta}.$$

Operator (1) is a typical representative of the so-called *hypergeometric fractional integral operators* of the general form

$$Hf(z) \tag{7}$$

$$= \int_0^1 \frac{\sigma^{\gamma_2}(1-\sigma)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)}\, {}_2F_1(\gamma_2+\alpha_2-\gamma_1, \alpha-1; \alpha_1+\alpha_2; 1-\sigma)\, f(z\sigma^{1/\beta}) d\sigma.$$

Such operators were studied also by Love [19], Kalla and Saxena [11], Srivastava and Buschman [37], McBride [23], Hohlov [8, 9], and other authors.

Definition 2. Let $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{C}$, $\Re(\gamma) > 0$, and F_3 denote the *Appell third function*, known also as *Horn's F_3-function*

$$F_3(\alpha, \alpha'; \beta, \beta'; \gamma; z, \xi) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n}} \frac{z^m \xi^n}{m!n!} \quad \text{for } |z| < 1, |\xi| < 1, \tag{8}$$

which reduces to the Gauss function, as: $F(\alpha, \beta; \gamma; z)$

$$= F_3(\alpha, \alpha'; \beta, \beta'; \gamma; z, 0) = F_3(\alpha, 0; \beta, \beta'; \gamma; z, \xi) = F_3(\alpha, \alpha'; \beta, 0; \gamma; z, \xi).$$

The F_3-*operator* is defined by

$$I(\alpha,\alpha',\beta,\beta';\gamma)f(z)$$
$$= z^{-\alpha}\int_0^z \frac{(z-\xi)^{\gamma-1}}{\Gamma(\gamma)}\xi^{-\alpha'}\ F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{\xi}{z},1-\frac{z}{\xi}\right)f(\xi)d\xi. \qquad (9)$$

Similar fractional integration operator involving the F_3-function was introduced first by Marichev [22] in a study on Volterra integral equations of convolutional type. Saigo [33] introduced operator (9) as a generalization of the hypergeometric integral operator (1), as well as a composition (in view of Mellin transform techniques) of two such operators, for example (if $\Re(\gamma) > \Re(\kappa) > 0$):

$$I(\alpha,\alpha',\beta,\beta';\gamma)f(z) = I^{\kappa,\alpha-\kappa,-\beta}\ I^{\gamma-\kappa,\alpha'-\gamma+\kappa,\beta'-\gamma+\kappa}f(z). \qquad (10)$$

Saigo and Maeda [35] studied some relations of operators (9) with the Mellin transforms, hypergeometric operators (1), their decompositions and properties in the McBride spaces ([23]) $F_{p,\mu}$.

Both Saigo's operators (1) and (9) can be considered as examples of the *operators for generalized fractional integration* (of Riemann-Liouville type), first introduced as such a notion by Kalla [10], in the form

$$Rf(z) = z^{-\gamma-1}\int_0^z \Phi(\frac{\zeta}{z})\zeta^{\gamma}\ f(\zeta)d\zeta = \int_0^1 \Phi(\sigma)\,\sigma^{\gamma}f(z\sigma)d\sigma, \qquad (11)$$

with a kernel-function $\Phi(z)$ an arbitrary continuous (resp. analytic) function so that the above integral makes sense. By special choices of $\Phi(z)$, as some elementary or special functions, the particular known operators of fractional calculus can be obtained. If one takes the kernel to be an arbitrary *Meijer's G-function* (details of definitions, conditions on the parameters and types of contour $\mathcal{L} \in \mathbb{C}$, see in [7], Vol.1; [28]; [14], App.):

$$G^{m,n}_{p,q}(z) = G^{m,n}_{p,q}\left[z\,\middle|\,\begin{matrix}a_1,\cdots,a_p\\ b_1,\cdots,b_q\end{matrix}\right] = G^{m,n}_{p,q}\left[z\,\middle|\,\begin{matrix}(a_j)_1^p\\ (b_j)_1^q\end{matrix}\right]$$
$$= \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=n+1}^{p}\Gamma(a_j-s)\prod_{j=m+1}^{q}\Gamma(1-b_j+s)}z^s\,ds \quad (z\neq 0), \qquad (12)$$

then a great extent of generality is achieved. But this has been also a trouble to develop an efficient and detailed theory of such operators of generalized fractional calculus that could be studied and used efficiently for applications.

However, using for $\Phi(\sigma)$ in (11) a suitable Meijer's G-function of a particular order $(m, 0, m, m)$, in Kiryakova [14], we have been able to develop a generalized fractional calculus that includes, as special cases, almost all the known operators of fractional integration and differentiation studied by many authors. This theory has already found various applications in solving problems in the theory of special functions, integral transforms and operational calculus, differential and integral equations, series expansions, etc.

Definition 3. Let $m > 1$ be an integer, $\beta > 0; \gamma_j$ $(j = 1, \dots, m)$ be real and $\delta_j \geqq 0$ $(j = 1, \dots, m)$. The set $\delta = (\delta_1, \dots, \delta_m)$ is considered as a fractional multiorder of integration. The following basic notion of a *generalized operator of fractional integration (generalized fractional integral operator)* is introduced:

$$I_{\beta,m}^{(\gamma_j),(\delta_j)} f(z)$$

$$= \begin{cases} \int\limits_0^1 G_{m,m}^{m,0}\left[\sigma \left| \begin{matrix} (\gamma_j+\delta_j)_1^m \\ (\gamma_j)_1^m \end{matrix} \right.\right] f\left(z\sigma^{1/\beta}\right) d\sigma, & \text{if } \sum\limits_{j=1}^m \delta_j > 0; \\ f(z), & \text{if } \sum\limits_{j=1}^m \delta_j = 0. \end{cases} \tag{13}$$

The corresponding *generalized fractional derivative* is denoted by $D_{\beta,m}^{(\gamma_j),(\delta_j)}$ and defined by means of a suitable explicit differintegral expression, see [14]. An important and useful characterization property of the operators of the *Generalized Fractional Calculus (GFC)* in [14] is their alternative representation as *products of commuting E-K fractional integrals (4)*, namely:

$$I_{\beta,m}^{(\gamma_j),(\delta_j)} f(z) = I_\beta^{\gamma_1,\delta_1} \ \dots \ I_\beta^{\gamma_m,\delta_m} f(z)$$

$$= \int\limits_0^1 \cdots \int\limits_0^1 \left[\prod_{j=1}^m \frac{(1-\sigma_j)^{\delta_j-1}\sigma_j^{\gamma_j}}{\Gamma(\delta_j)}\right] f\left[z(\sigma_1\dots\sigma_m)^{1/\beta}\right] d\sigma_1 \dots d\sigma_m. \tag{14}$$

Our operators $I_{\beta,m}^{(\gamma_j),(\delta_j)}$ and $D_{\beta,m}^{(\gamma_j),(\delta_j)}$ are shown to incorporate in the scheme of GFC all the other known operators of fractional integration and differentiation, studied by other authors. Especially, using the representation of the Gauss hypergeometric function ${}_2F_1$ as a Meijer $G^{2,0}_{2,2}$-function ([28], p. 720, §8.4.49, eq. (22)) as well as comparing the decompositions (5) and (14), one can easily see that *Saigo's hypergeometric integral operator (1) is a generalized fractional integral in the sense of (13)* with $m = 2$:

$$I^{\alpha,\beta,\eta}_{0,z} f(z) = z^{-\beta} I^{(\eta-\beta,0),(-\eta,\alpha+\eta)}_{1,2} f(z). \tag{15}$$

Similarly, the representation of the F_3-function (8) as a Meijer $G^{3,0}_{3,3}$-function (see [28], p. 727, §8.4.51, eq. (2)), shows *the Saigo F_3-operator (9) as a generalized fractional integral* with $m = 3$:

$$\begin{aligned} I(\alpha, \alpha', \beta, \beta'; \gamma) f(z) \\ = z^{-\alpha-\alpha'+\gamma} \int_0^1 G^{3,0}_{3,3} \left[\sigma \,\middle|\, \begin{matrix} \alpha - \alpha' + \beta, \gamma - 2\alpha', \gamma - \alpha' - \beta' \\ \alpha - \alpha', \beta - \alpha', \gamma - 2\alpha' - \beta' \end{matrix} \right] f(z\sigma) d\sigma \\ = z^{-\alpha-\alpha'+\gamma} I^{(\alpha-\alpha',\beta-\alpha',\gamma-2\alpha'-\beta'),(\beta,\gamma-\alpha'-\beta,\alpha')}_{1,3} f(z). \end{aligned} \tag{16}$$

Definition 4. By $\mathcal{A}$ we denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{17}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all functions which are also *univalent* in U. Further, a function $f(z)$ belonging to $\mathcal{S}$ is said to be *convex*, if it satisfies the inequality:

$$\operatorname{Re} \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} > 0 \qquad (z \in U) \tag{18}$$

and this subclass of $\mathcal{S}$ is denoted by $\mathcal{K}$.

Definition 5. The *Hadamard product (convolution)* of two analytic functions in U:

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \qquad \text{and} \qquad g(z) = \sum_{k=0}^{\infty} b_k z^k$$

is defined by

$$(f * g)(z) := \sum_{k=0}^{\infty} a_k b_k z^k. \tag{19}$$

Many papers studying the class of univalent functions and its subclasses, as those of the starlike and convex functions, make use of various linear integral or integro-differential operators. These include the familiar operators of Biernacki [3], Libera [20], Bernardi [2], Ruscheweyh [29], Carlson and Shaffer [5], Hohlov [8, 9], Srivastava and Owa [25, 26], and others. In [13, 14, 15, 16] we have shown that all such operators are special cases of the operators (13) of the generalized fractional calculus [14]. In a previous paper [16], joint with Saigo and Owa, we have found some distortion inequalities and other characterization theorems for the functions of the above-mentioned classes, thus showing that the classical techniques used in other papers (cf. [40, 24]) on univalent functions, work quite easily also for the class of our generalized fractional integrals and derivatives.

One of the important problems in the theory of univalent functions is the construction of linear operators preserving the class $\mathcal{S}$ and some of its subclasses. Biernacki [3] claimed that a certain integral operator maps $\mathcal{S}$ into itself, but later a counterexample by Krzyz and Lewandowski [18] showed that he was wrong. However, another linear integral operator, introduced by Libera [20], maps each of the subclasses of the convex, starlike and close-to-convex functions into itself. Bernardi [2] generalized Libera's operators, but also studied operators preserving only some subclasses of $\mathcal{S}$. Ruscheweyh [29] and Livingston [21] investigated differential operators, inverse to Biernacki's and Libera's ones, but could not find operators preserving the univalence of the whole class $\mathcal{S}$.

In this connection, the *works of Hohlov [8, 9] seem to be pioneering.* By means of a Hadamard convolution (19) with the Gauss hypergeometric function, he introduced a three-parameter family of operators $\mathbf{F}(a,b,c)$:

$$\mathbf{F}(a,b,c)f(z) = \Big(\{z\ {}_2F_1(a,b;c;z)\} * f\Big)(z). \tag{20}$$

These are hypergeometric operators of the form (7) that could be represented also as generalized fractional integrals (13) with $m=2$:

$$\begin{aligned}\mathbf{F}(a,b,c)f(z) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int\limits_0^1 \frac{(1-\sigma)^{c-a-b}}{\Gamma(c-a-b+1)}\sigma^{b-c} \qquad (21)\\ &\qquad \times\ {}_2F_1(c-a,1-a;c-a-b+1;1-\sigma)f(z\sigma)d\sigma\\ = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\ I_1^{a-2,1-a} I_1^{b-2,c-b} f(z) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\ I_{1,2}^{(a-2,b-2),(1-a,c-b)} f(z).\end{aligned}$$

Hohlov found sufficient conditions on the parameters a,b,c for the operators (19)-(20) to preserve the whole class $\mathcal{S}$ of univalent functions or to map its subclass $\mathcal{K}$ of convex functions into $\mathcal{S}$. Since these operators generalize the above-mentioned operators, he could easily explain the reasons for the failure of the previous authors.

In our paper [17], joint with Saigo and Srivastava, we generalized the approach of Hohlov to the operators (13)-(14) of the generalized fractional calculus, finding there explicit sufficient conditions (inequalities that should be satisfied by the parameters γ_k, δ_k, $k = 1, \dots, m$) for preserving the class $\mathcal{S}$ or mapping class $\mathcal{K}$ into class $\mathcal{S}$, see Theorems 3 and 4 therein.

Here, we show the corollaries of these rather general results, for the specific cases of the Saigo operators (1) and (9). In order to keep in the frames of a survey paper, we shall omit the proofs. They can be done easily as consequences of the general scheme in [17] (for the operators of GFC) or following a pattern similar to this in Hohlov [8, 9], but by using another case of Gauss hypergeometric function.

2. Saigo's hypergeometric fractional integration operator in the classes $\mathcal{A}$, $\mathcal{S}$, $\mathcal{K}$

One can easily obtain the following lemma, due to Srivastava, Saigo and Owa [40], that is a simple corollary also from our general Lemma 1 in [17]:

Lemma 0. *Let $\alpha > 0, \beta$ and η be real, and let $\kappa > \beta - \eta - 1$. Then*

$$I_{0,z}^{\alpha,\beta,\eta} z^{\kappa} = c_k \; z^{\kappa-\beta} \quad \textit{with} \quad c_k = \frac{\Gamma(\kappa+1)\Gamma(\kappa-\beta+\eta+1)}{\Gamma(\kappa-\beta)\Gamma(\kappa+\alpha+\eta+1)} > 0. \tag{22}$$

Since we consider preserving the functions in the class $\mathcal{A}$, it is suitable to normalize the operator (1), according to (22), by means of multiplication by $[c_1]^{-1} z^{\beta}$. Thus, further we consider the *normalized Saigo's fractional integrals* (using the same name for the normalized version, but stressing this fact *by a tilde* in its notation: $\widetilde{I}_{0,z}^{\alpha,\beta,\eta} := [c_1]^{-1} z^{\beta} I_{0,z}^{\alpha,\beta,\eta}$),

$$\widetilde{I}_{0,z}^{\alpha,\beta,\eta} f(z) := \frac{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)}{\Gamma(2-\beta+\eta)} \; z^{\beta} \, I_{0,z}^{\alpha,\beta,\eta} f(z). \tag{23}$$

Then, from Lemma 0 and the more general results in [17], we easily obtain

Theorem 1. *Under the parametric constraints*

$$\alpha > -\eta > 0, \;\; \beta - \eta < 2, \tag{24}$$

Saigo's (normalized) hypergeometric fractional integral $\widetilde{I}_{0,z}^{\alpha,\beta,\eta}$ *maps the class* $\mathcal{A}$ *into itself, and the image of a power series (17) has the form*:

$$\widetilde{I}f(z)=\widetilde{I}_{0,z}^{\alpha,\beta,\eta}\left\{z+\sum_{k=2}^{\infty}a_k z^k\right\}=z+\sum_{k=2}^{\infty}\Psi(k)a_k z^k\in\mathcal{A}, \tag{25}$$

where the multiplier sequence is given by

$$\Psi(k)=\frac{(2-\beta+\eta)_{k-1}(1)_k}{(2-\beta)_{k-1}(2+\alpha+\eta)_{k-1}}>0\quad(k=2,3,4,\dots) \tag{26}$$

with $(a)_k=\Gamma(a+k)/\Gamma(a)$ *denoting the Pochhammer symbol.*

THEOREM 2. *In the class* $\mathcal{A}$*, Saigo's (normalized) hypergeometric fractional integral operator (23) can be represented by the Hadamard product*

$$\widetilde{I}_{0,z}^{\alpha,\beta,\eta}f(z)=(h*f)(z), \tag{27}$$

where the function $h(z)\in\mathcal{A}$ *is the following* ${}_3F_2$*- generalized hypergeometric function (2)*:

$$h(z)=z+\sum_{k=2}^{\infty}\Psi(k)z^k=z\ {}_3F_2\left(\begin{matrix}1,-\beta+\eta+2,2;\\ -\beta+2,\alpha+\eta+2;\end{matrix}\ z\right). \tag{28}$$

Using the above representation, and the more general results in [17], or in the particular case of hypergeometric fractional integral operators, the lines of proof analogous to these done by Hohlov [8], we can state the following

THEOREM 3. *Criteria for univalence of Saigo's hypergeometric fractional integral operators* $\widetilde{I}_{0,z}^{\alpha,\beta,\eta}$: *The conditions*

$$\alpha\geqq-\eta\geqq0\ ,\ \beta-\eta<2\ ,\ \alpha>3$$

and

$$\begin{aligned}&\frac{12(\eta-\beta+2)(\eta-\beta+3)}{(-\beta+2)(-\beta+3)(\alpha+\eta+2)(\alpha+\eta+3)}\ {}_3F_2\left(\begin{matrix}3,\eta-\beta+4,4;\\ -\beta+4,\alpha+\eta+4;\end{matrix}\ 1\right)\\ &\quad+\frac{6(\eta-\beta+2)}{(-\beta+2)(\alpha+\eta+2)}\ {}_3F_2\left(\begin{matrix}2,\eta-\beta+3,3;\\ -\beta+3,\alpha+\eta+3;\end{matrix}\ 1\right)\\ &\quad+\ {}_3F_2\left(\begin{matrix}1,\eta-\beta+2,2;\\ -\beta+2,\alpha+\eta+2;\end{matrix}\ 1\right)<2,\end{aligned} \tag{29}$$

imply that $\widetilde{I}=\widetilde{I}_{0,z}^{\alpha,\beta,\eta}:\mathcal{S}\mapsto\mathcal{S}$.

THEOREM 4. *The parameters' conditions*

$$\alpha \geqq -\eta \geqq 0\ ,\ \beta - \eta < 2\ ,\ \alpha > 2$$

and

$$\frac{2(\eta-\beta+2)}{(-\beta+2)(\alpha+\eta+2)}\ {}_3F_2\left(\begin{matrix} 2, \eta-\beta+3, 3; \\ -\beta+3, \alpha+\eta+3; \end{matrix}\ 1\right) + {}_3F_2\left(\begin{matrix} 1, \eta-\beta+2, 2; \\ -\beta+2, \alpha+\eta+2; \end{matrix}\ 1\right) < 2 \qquad (30)$$

imply that $\widetilde{I} = \widetilde{I}^{\alpha,\beta,\eta} : \mathcal{K} \mapsto \mathcal{S}$.

P r o o f. The proofs of Theorems 3 and 4 are based on the following ideas: For the operator $\widetilde{I}$ to preserve the class $\mathcal{S}$ of univalent functions, we require for the image-function

$$\widetilde{I}f(z) = z + \sum_{k=2}^{\infty} b_k z^k,$$

where $b_k = \Psi(k)a_k$ and $\Psi(k)$ as in (26), that the following sufficient condition (see [1])

$$\sigma_1 = \sum_{k=2}^{\infty} k|b_k| = \sum_{k=2}^{\infty} k\Psi(k)|a_k| < 1 \qquad (31)$$

is satisfied. In the case of Theorem 3, we use the known estimate given by de Branges' theorem [4], formerly known as a Bieberbach's conjecture: $|a_k| \leqq k$. Thus we can estimate the sum σ_1 as

$$\sigma_1 = \sum_{k=2}^{\infty} k\Psi(k)|a_k| \leqq \sum_{k=2}^{\infty} k^2\Psi(k) = \sum_{k=2}^{\infty} \frac{k^2}{(1)_{k-1}}[\Psi(k)(1)_{k-1}] < 1.$$

Then, following the lines of Hohlov's proof [8], we transform the above series into a sum of three terms including values at the point $z = 1$ of ${}_3F_2$-hypegeometric functions, all of which are representable by convergent series, due to the imposed parameters' conditions.

The proof of Theorem 4 is much akin to above, again requiring (31), but in this case, instead of the estimate $|a_k| \leqq k$, we use the estimate $|a_k| \leqq 1$ (see [27]) for the coefficients of convex functions $f(z)$ defined by (1). ■

Here, unlike the case of Hohlov operators and the criteria given in [8, 9], the functions ${}_3F_2(1)$ in Theorems 3 and 4 cannot be reduced to ${}_2F_1(1)$, and thus evaluated explicitly. However, for some special choices of the parameters α, β, η, it is possible to simplify the conditions in Theorems 3 and 4 to explicit inequalities for the parameters α, β, η.

COROLLARY 5. *For $\beta = 1$, the normalized Saigo operator (23) turns into a special case of the Hohlov operators (200-(21) with $a = \eta + 1, b = 2, c = \alpha + \eta + 2$ in the class $\mathcal{A}$:*

$$\widetilde{I}^{\alpha,1,\eta} f(z) = \mathbf{F}(\eta + 1, 2; \alpha + \eta + 2) f(z). \tag{32}$$

Then, if $-1 < \eta \leqq 0,\ \alpha > 3$ and

$$\frac{(\alpha + \eta)(\alpha + \eta + 1)(\alpha^2 + 6\alpha\eta + 6\eta^2 + \alpha)}{\alpha(\alpha - 1)(\alpha - 2)(\alpha - 3)} < 2, \tag{33}$$

$\widetilde{I}^{\alpha,1,\eta} : \mathcal{S} \mapsto \mathcal{S}$. The conditions $-1 < \eta \leqq 0,\ \alpha > 2$ and

$$\frac{(\alpha + \eta)(\alpha + \eta + 1)(\alpha + 2\eta)}{\alpha(\alpha - 1)(\alpha - 2)} < 2 \tag{34}$$

imply that $\widetilde{I}^{\alpha,1,\eta} : \mathcal{K} \mapsto \mathcal{S}$.

There are two other cases when the Saigo operators can be simplified and the ${}_3F_2$ functions can be reduced to computable ${}_2F_1(1)$ series.

COROLLARY 6. *Let $\beta = 0$. Then the conditions $-2 < \eta \leqq 0,\ \alpha > 3$ and*

$$\frac{(\alpha + \eta + 1)(\alpha^2 + 3\alpha\eta + 2\eta^2 + \alpha + \eta)}{(\alpha - 1)(\alpha - 2)(\alpha - 3)} < 2 \tag{35}$$

imply that $\widetilde{I}^{\alpha,0,\eta} : \mathcal{S} \mapsto \mathcal{S}$. Similar conditions can be deduced for the case $\mathcal{K} \mapsto \mathcal{S}$.

COROLLARY 7. *Let $\beta = -\alpha$. Then the conditions $\eta \leqq 0,\ \alpha > 3,\ \alpha + \eta \geqq 0$ and*

$$\frac{\alpha(\alpha + 1)^2}{(\alpha - 1)(\alpha - 2)(\alpha - 3)} < 2 \tag{36}$$

imply that $\widetilde{I}^{\alpha,-\alpha,\eta} : \mathcal{S} \mapsto \mathcal{S}$.

The conditions (29), (30) in Theorems 3 and 4 may look, in the general case, somewhat complicated to be solved explicitly. However, for given particular values of the parameters α, β, η, it is easy to check whether these conditions are satisfied or not. This is demonstrated by the above Corollaries 5, 6, 7.

Here we give one more example, with *criteria for univalence of the well-known Erdélyi-Kober (E-K) fractional integral operators* (4) that follows from Theorems 3 and 4 by using the relation mentioned in beginning: $I_1^{\gamma,\alpha} f(z) = z^{-\alpha-\gamma}\, I_{0,z}^{\alpha,-\alpha-\gamma,-\alpha} f(z)$.

Corollary 8. *Let* $\widetilde{I} = \widetilde{I}_1^{\gamma,\alpha} = [\Gamma(\gamma+\alpha+2)/\Gamma(\gamma+2)]I_1^{\gamma,\alpha}$ *be the normalized Erdélyi-Kober fractional integral (4). If the inequalities*

$$\gamma > -2, \quad \alpha > 3,$$
$$\frac{(\gamma+\alpha+1)(2\gamma^2+3\gamma\alpha+\alpha^2+\gamma+\alpha)}{(\alpha-1)(\alpha-2)(\alpha-3)} < 2 \tag{37}$$

hold true, then $\widetilde{I}_1^{\gamma,\alpha}$ *preserves the univalency, i.e.* $\widetilde{I}_1^{\gamma,\alpha} : \mathcal{S} \mapsto \mathcal{S}$. *Analogously, the conditions*

$$\gamma > -2,\ \alpha > 2,\ \gamma^2 + 2\gamma\alpha - \alpha^2 + \gamma + 7\alpha - 4 < 0 \tag{38}$$

imply that $\widetilde{I}_1^{\gamma,\delta} : \mathcal{K} \mapsto \mathcal{S}$.

One can state yet more simplified similar *conditions for the Riemann-Liouville operator* $R^\alpha f(z) = I_{0,z}^{\alpha,-\alpha,-\alpha} f(z)$ by setting $\gamma = 0$ in the above inequalities (37) and (38).

3. Saigo's F_3-operators in the classes $\mathcal{A}, \mathcal{S}, \mathcal{K}$

Results similar to those in Theorems 3 and 4 can be stated also for the F_3-operators (9), involving Appell's third function.

A starting point for these will be the following auxiliary lemma (can be found as Remark, on p. 394, Saigo and Maeda [35].

Lemma 9. *Let* $\mathrm{Re}(\gamma) > 0, k > \max[0, \mathrm{Re}(\alpha+\alpha'+\beta-\gamma), \mathrm{Re}(\alpha'-\beta')] - 1$, *then*

$$I(\alpha,\alpha',\beta,\beta';\gamma)\, x^k \tag{39}$$

$$= \Gamma \left[\begin{array}{l} k+1, -\alpha-\alpha'-\beta+\gamma+k+1, -\alpha'+\beta'+k+1 \\ -\alpha-\alpha'+\gamma+k+1, -\alpha'-\beta+\gamma+k+1, \beta'+k+1 \end{array} \right] x^{k-\alpha-\alpha'+\gamma}.$$

Then, from the representation (16) of the F_3-operator as a generalized fractional integration operator of form (13) with $m = 3$ and our general results in Kiryakova, Saigo, Srivastava [17], one can obtain

Theorem 10. *The the normalized* F_3*-operator is represented by the Hadamard product:*

$$\widetilde{I} f(z) = \widetilde{I}(\alpha,\alpha',\beta,\beta';\gamma) f(z) := z^{\alpha+\alpha'-\gamma}\ I(\alpha,\alpha',\beta,\beta';\gamma) f(z) = (h * f)(z),$$

where the function $h(z) \in \mathcal{A}$ is the following ${}_4F_3$- generalized hypergeometric function (2):

$$h(z) = z\ {}_4F_3\left(\begin{array}{c} 1, \alpha-\alpha'+2, \beta-\alpha'+2, \gamma-2\alpha'-\beta'+2; \\ \alpha-\alpha'+\beta+2, \gamma-2\alpha'+2, \gamma-\alpha'-\beta'+2; \end{array} z\right). \qquad (40)$$

According to (16), we have to set the following denotations:

$$\gamma_1 := \alpha-\alpha', \gamma_2 := \beta-\alpha', \gamma_3 := \gamma-2\alpha'-\beta';\ \delta_1 := \beta, \delta_2 := \gamma-\alpha'-\beta, \delta_3 := \alpha',$$

leading to $\delta_1 + \delta_2 + \delta_3 = \gamma$, and to require the conditions

$$\alpha-\alpha' > -2, \beta-\alpha' > -2, \gamma-2\alpha'-\beta' > -2;\ \beta > 0, \gamma-\alpha'-\beta > 0, \alpha' > 0 \quad (41)$$

Then the corresponding inequalities from Theorems 3 and 4 in [17] for $\mathcal{S} \mapsto \mathcal{S}$ and $\mathcal{K} \mapsto \mathcal{S}$, in this case involve ${}_4F_3(1)$ series.

THEOREM 11. *Let the conditions (41) be satisfied. Let additionally,*

$$\gamma > 3 \qquad (42)$$

and

$$2\left[\prod_{j=1}^{3} \frac{(\gamma_j+2)(\gamma_j+3)}{(\gamma_j+\delta_j+2)(\gamma_j+\delta_j+3)}\right] {}_4F_3\left(\begin{array}{c} 3, (\gamma_j+4)_1^3; \\ (\gamma_j+\delta_j+4)_1^3; \end{array} 1\right)$$
$$+3\left[\prod_{j=1}^{3} \frac{\gamma_j+2}{\gamma_j+\delta_j+2}\right] {}_4F_3\left(\begin{array}{c} 2, (\gamma_j+3)_1^3; \\ (\gamma_j+\delta_j+3)_1^3; \end{array} 1\right)$$
$$+{}_4F_3\left(\begin{array}{c} 1, (\gamma_j+2)_1^3; \\ (\gamma_j+\delta_j+2)_1^3; \end{array} 1\right) < 2, \qquad (43)$$

with parameters as in (41). Then for each univalent function f in $\mathcal{A}$, the image $\widetilde{I}f$ is also univalent, i.e. $\widetilde{I} : \mathcal{S} \mapsto \mathcal{S}$.

Theorem 12. *Let the conditions (41) be satisfied. Let additionally,*

$$\gamma > 2 \tag{44}$$

and

$$\left[\prod_{j=1}^{3} \frac{\gamma_j + 2}{\gamma_j + \delta_j + 2}\right] \; {}_4F_3\left(\begin{array}{r} 2, (\gamma_j + 3)_1^3; \\ (\gamma_j + \delta_j + 3)_1^3; \end{array} 1 \right) + \; {}_4F_3\left(\begin{array}{r} 1, (\gamma_j + 2)_1^3; \\ (\gamma_j + \delta_j + 2)_1^3; \end{array} 1 \right) < 2. \tag{45}$$

with parameters as in (41). Then $\widetilde{I}$ maps a convex function $f(z)$ into a univalent function, i.e. $\widetilde{I} : \mathcal{K} \mapsto \mathcal{S}$.

References

[1] F. G. A v h a d i e v, L. A. A k s e n t'e v, Basic results in sufficient conditions for univalency of analytic functions (In Russian). *Uspehi Matematicheskih Nauk* **30**, No 4 (1975), 3-60.

[2] S. D. B e r n a r d i, Convex and starlike univalent functions. *Trans. Amer. Math. Soc.* **135** (1969), 429-446.

[3] M. B i e r n a c k i, Sur l'intégrale des fonctions univalentes. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **8** (1960), 29-34.

[4] L. d e B r a n g e s, A proof of the Bieberbach conjecture. *Acta Math.* **154** (1985), 137-152.

[5] B. C. C a r l s o n, D. B. Shaffer, Starlike and prestarlike hypergeometric functions. *SIAM J. Math. Anal.* **15** (1984), 737-745.

[6] J a e H o C h o i, M. S a i g o, Starlike and convex functions of complex order involving a certain fractional integral operator. *Fukuoka Univ. Sci. Reports* **28**, No 2 (1998), 29-40.

[7] A. E r d é l y i, W. M a g n u s, F. O b e r h e t t i n g e r, F. G. T r i c o m i, *Higher Transcendental Functions*, Vols. 1, 2, 3. McGraw-Hill, New York (1953).

[8] Yu. E. H o h l o v, Convolutional operators preserving univalent functions (In Russian). *Ukrain. Mat. Zh.* **37** (1985), 220-226.

[9] Yu. E. H o h l o v, Convolutional operators preserving univalent functions. *Pliska Stud. Math. Bulgar.* **10** (1989), 87-92.

[10] S. L. K a l l a, Operators of fractional integration. In: *Proc. Conf. Analytic Functions, Kozubnik 1979*; Publ. as: *Lecture Notes in Math.* **798** (1980), 258-280.

[11] S. L. K a l l a, R. K. S a x e n a, Integral operators involving hypergeometric functions, I: *Math. Z.* **108** (1969), 231-234; II: *Univ. Nac. Tucumán, Rev. Ser.* **A24** (1974), 31-36.

[12] A. A. K i l b a s, O. A. R e p i n, M. S a i g o, Solution in closed form of boundary value problem for degenerate equation of hyperbolic type. *Kyungpook Math. J.* **36**, No 2 (1996), 261-273.

[13] V. S. K i r y a k o v a, Multiple Erdélyi-Kober fractional differintegrals and their uses in univalent, starlike and convex function theory. *Ann. Univ. Sofia Fac. Math. Inform. Livre 1 - Math.* **81** (1987), 261-283.

[14] V. S. K i r y a k o v a, *Generalized Fractional Calculus and Applications* (Pitman Res. Notes in Math. Ser., 301). Longman, Harlow (1994).

[15] V. S. K i r y a k o v a, H. M. S r i v a s t a v a, Generalized multiple Riemann-Liouville fractional differintegrals and their applications in univalent function theory. In: *Analysis, Geometry and Groups: A Riemann Legacy Volume* (H.M. Srivastava and Th.M. Rassias, Ed-s). Hadronic Press, Palm Harbor, Florida (1993), 191-226.

[16] V. S. K i r y a k o v a, M. S a i g o, S. O w a, Distortion and characterization theorems for starlike and convex functions related to generalized fractional calculus. In: *New Development of Convolutions* (Sūrikaisekikenkyūsyo Kōkyūroku, Vol.1012), Kyoto University (1997), 25-46.

[17] V. S. K i r y a k o v a, M. S a i g o, H. M. S r i v a s t a v a, Some criteria for univalence of analytic functions involving generalized fractional calculus operators. *Fract. Calc. Appl. Anal.* **1**, No 1 (1998), 79-104.

[18] J. K r z y z, Z. L e w a n d o w s k i, On the integral of univalent functions. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **11** (1963), 447-448.

[19] E.R. L o v e, Some integral equations involving hypergeometric functions. *Proc. Edinburgh Math. Soc.* **15**, No 3 (1967), 169-198.

[20] R. J. L i b e r a, Some classes of regular univalent functions. *Proc. Amer. Math. Soc.* **16** (1965), 755-758.

[21] A. E. L i v i n g s t o n, On the radius of univalence of certain analytic functions. *Proc. Amer. Math. Soc.* **17** (1966), 352-357.

[22] O. I. M a r i c h e v, Volterra equation of Mellin convolutional type with a Horn function in the kernel (in Russian). *Izv. AN BSSR Ser. Fiz.-Mat. Nauk*, No.1 (1974), 128-129.

[23] A. M c B r i d e, *Fractional Calculus and Integral Transforms of Generalized Functions* (Research Notes in Math. **31**). Pitman, London (1979).

[24] S. O w a, M. S a i g o, H. M. S r i v a s t a v a, Some characterization theorems for starlike and convex functions involving a certain fractional integral operator. *J. Math. Anal. Appl.* **140** (1989), 419-426.

[25] S. O w a, H. M. S r i v a s t a v a, Some applications of the generalized Libera integral operator. *Proc. Japan Acad. Ser. A Math. Sci.* **62** (1986), 125-128.

[26] S. O w a, H. M. S r i v a s t a v a, Univalent and starlike generalized hypergeometric functions. *Canad. J. Math.* **39** (1987), 1057-1077.

[27] C. P o m m e r e n k e, *Univalent Functions.* Vandenhoeck and Ruprecht, Göttingen (1975).

[28] A. P. P r u d n i k o v, Yu. A. B r y c h k o v, O. I. M a r i c h e v, *Integrals and Series*, Vol. 3: *More Special Functions.* Gordon and Breach Sci. Publ., New York, Philadelphia, London, Paris, Montreux, Tokyo, and Melbourne (1990).

[29] S. R u s c h e w e y h, New criteria for univalent functions. *Proc. Amer. Math. Soc.* **49** (1975), 109-115.

[30] M. S a i g o, A remark on integral operators involving the Gauss hypergeometric functions. *Math. Rep. College General Ed. Kyushu Univ.* **11** (1978), 135-143.

[31] M. S a i g o, A certain boundary value problem for the Euler-Darboux equation. *Math. Japon.* **24** (1979), 377-385; II, *ibid.* **25** (1980), 211-220; III, *ibid.* **26** (1981), 103-119.

[32] M. S a i g o, A generalization of fractional calculus. In: *Fractional Calculus* (*Proc. Internat. Workshop held at Ross Priory, Univ. of Strathclyde*). Pitman, Boston and London (1985), 188-198.

[33] M. S a i g o, On generalized fractional calculus operators. In: *Recent Advances in Applied Mathematics* (*Proc. Internat. Workshop held at Kuwait Univ.*). Kuwait Univ., Kuwait (1996), 441-450.

[34] M. S a i g o, O. A. R e p i n, A. A. K i l b a s, On a non-local boundary value problem for an equation of mixed parabolic-hyperbolic type. *Intern. J. Mathematical and Statistical Sciences* **5**, No 1 (1996), 103-117.

[35] M. S a i g o, N. M a e d a, More generalization of fractional calculus. In: *Transform Methods & Special Functions, Varna'96* (*Proc. Second Internat. Workshop*). Science Culture Technology Publishing, Singapore (1998), 386-400.

[36] I. N. S n e d d o n, The use in mathematical analysis of Erdélyi-Kober operators and of some of their applications. In: *Fractional Calculus and Its Applications* (Lecture Notes in Mathematics, Vol. 457). Springer-Verlag, New York (1975), 37-79.

[37] H. M. S r i v a s t a v a, R. G. B u s c h m a n, *Theory and Applications of Convolution Integral Equations.* Kluwer Acad. Publ. (Ser. Math. and Its Appl., vol. 79), Dordrecht-Boston-London (1992).

[38] H. M. S r i v a s t a v a, S. O w a, A certain one-parameter additive family of operator defined on analytic functions. *J. Math. Anal. Appl.* **118** (1986), 80-87.

[39] H. M. S r i v a s t a v a, M. S a i g o, Multiplication of fractional calculus operators and boundary value problems involving the Euler-Darboux equation. *J. Math. Anal. Appl.* **121** (1987), 325-369.

[40] H. M. S r i v a s t a v a, M. S a i g o, S. O w a, A class of distortion theorems involving certain operators of fractional calculus. *J. Math. Anal. Appl.* **131** (1988), 412-420.

* *Institute of Mathematics and Informatics*
Bulgarian Academy of Sciences
"Acad. G. Bontchev" Str., Block 8
Sofia - 1113, BULGARIA

e-mail: virginia@diogenes.bg

Received: November 13, 2006