Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 9, NUMBER 1 (2006) ISSN 1311-0454

A FRACTIONAL $LC - RC$ CIRCUIT

A.A. Rousan [1], N.Y. Ayoub [2], F.Y. Alzoubi [1], H. Khateeb [1], M. Al-Qadi [1], M.K. Hasan(Qaseer)[1], B.A. Albiss [1]

Abstract

We suggest a fractional differential equation that combines the simple harmonic oscillations of an LC circuit with the discharging of an RC circuit. A series solution is obtained for the suggested fractional differential equation. When the fractional order $\alpha = 0$, we get the solution for the RC circuit, and when $\alpha = 1$, we get the solution for the LC circuit. For arbitrary α we get a general solution which shows how the oscillatory behavior (LC circuit) go over to a decay behavior (RC circuit) as grows from 0 to 1, and vice versa. An explanation of the behavior is proposed based on the idea of the evolution of a resistive property in the inductor giving a new value to the inductance that affects the frequency of the oscillator.

Mathematics Subject Classification: 26A33, 30B10, 33B15, 44A10, 47N70, 94C05

Key Words and Phrases: fractional calculus, differintegration, fractional differential equation, simple harmonic oscillator, damping, series solution, LCR circuit, intermediate stages

1. Introduction

Fractional calculus is gaining more grounds and is enabling scientists to see the evolution of one kind of phenomenon into another kind of phenomenon through the evolution of the fractional order of differentiation, α.

Engheta [1]-[3], and Rousan et al [4], have studied such evolutions in electromagnetic multipoles and gravity respectively. Kobelev and Romanov [5] showed that fractional equations give the most complete possible description of the system memory. On the problem of dynamic systems, there has been a great deal of work. Yuan and Agrawal [6] presented a numerical scheme to solve a fractionally damped single degree-of-freedom spring-mass-damper forced system using Laguerre integral formula and considering derivative of the order 0.5 in their example. A comparison of numerical methods is found in Shokooh and Suarez [7]. A numerical method was used by Koh and Kelly [8] in applying fractional derivatives of order 0.5 in the formulation of stress-strain relationship of elastomers. Suarez and Shokooh [9] proposed a closed form solution to the problem using an eigenvector expansion method with fractional derivative of order 0.5 only. Laplace transform was used by Bagley and Trovik [10] and[11]. Fourier transform was used by Gaul et al [12] and [13] with fractional order of 0.5 only. Hartley and Lorenzo [14] employed the Mittag-Leffler function to study the initialization problem for a system of linear fractional order differential equations for discrete values of the fractional order of derivation. Koeller [15] studied creep and relaxation functions taking the order of derivation to vary from 0.05 to 0.35 to show that a continuous transition from solid state to the fluid state occurs when the memory parameter(order of derivation) varies from zero to one expressing the results in terms of Mittag-liffler function. Diethelm et al [16] recently offer a useful discussion and comparison of different numerical solutions for fractional equations. Many useful books on fractional differential equations [17] applications of fractional calculus [18] and fractional calculus [19] and [20] have been published.

In this paper the simple harmonic oscillations of the charge can be generated through an LC circuit (inductor-capacitor circuit). We shall refer to the angular frequency of this circuit as $\omega_1 = 1/(LC)^{1/2}$. The RC circuit (resistor-capacitor circuit) is used to show the discharging of a capacitor (decay) with the time constant $\tau = RC$, which we shall take its inverse and refer to it as $\omega_2 = 1/\tau$. These two situations are represented by a well-known second order differential equation for the oscillations and by a first order differential equation for the decay. We will combine the two equations in one fractional differential equation of arbitrary order α and we will present a series solution to this fractional differential equation to get a better approximation, which can be achieved by increasing the number of terms in the summation. The discussion of the fractional problem will not be thus limited to the fractional order of 0.5, but all values of the fractional order from zero to one will be considered. Our aim in this work is to unify both these situations (LC and RC) into a single fractional differential equation and to see the behavior of a new hybrid circuit that depends on ω_1 and ω_2, when the order of differentiation is fractional. Fractional driven LC and RC circuits will be discussed in future work.

2. The fractional $LC - RC$ circuit

The LC circuit is governed by the equation:

$$\frac{d^2Q}{dt^2} + \omega_1^2 Q = 0, \tag{1}$$

while the RC circuit is governed by the equation:

$$\frac{dQ}{dt} + \omega_2 Q = 0, \tag{2}$$

where ω_1 and ω_2 are as defined in Introduction. We suggest a fractional differential equation that is dimensionally consistent and that depend on ω_1 and ω_2 such that:

$$\frac{d^{1+\alpha}Q}{dt^{1+\alpha}} + \omega(\alpha)^{1+\alpha} Q = 0, \tag{3}$$

where $0 < \alpha < 1$ and $\omega(\alpha) = \omega_1^{\alpha}\omega_2^{1-\alpha}$, and where $\omega(\alpha)^{1+\alpha}$ will make the two terms of equation (3) of the same dimension. We immediately notice that when $\alpha = 0$, eq. (3) goes over to eq. (2) and when $\alpha = 1$, eq. (3) goes over to eq. (1). Using Laplace transform applied to eq.(3), we get:

$$s^{1+\alpha}q(s) - \left(\frac{d^{\alpha}Q(t)}{dt^{\alpha}}\right)_{t=0} - s\left(\frac{d^{\alpha-1}Q(t)}{dt^{\alpha-1}}\right)_{t=0} + \omega(\alpha)^{1+\alpha}q(s) = 0, \tag{4}$$

where $q(s) = L[Q(t)]$, i.e. $q(s)$ is the Laplace transform of $Q(t)$. In eq. (4) we have the two initial conditions which can take a variety of initial values, we shall discuss only one case and leave the other cases for further study:

$$\left(\frac{d^{\alpha}Q(t)}{dt^{\alpha}}\right)_{t=0} = \omega(\alpha)Q_0\,, \quad \text{and} \quad \left(\frac{d^{\alpha-1}Q(t)}{dt^{\alpha-1}}\right)_{t=0} = 0\,, \tag{5}$$

where Q_0 is the initial charge at $t = 0$ in the case of the RC discharge circuit ($\alpha = 0$) and $\omega_1 Q_0$ the initial current at $t = 0$ in the case of the LC

oscillatory circuit ($\alpha = 1$). In this case, taking the highest power of s as a common factor from the denominator and then expanding the denominator in an alternating geometric series:

$$s^{1+\alpha}q(s) + \omega(\alpha)^{1+\alpha}q(s) - \omega(\alpha)^{\alpha}Q_0 = 0 \quad \text{and} \quad q(s) = Q_0\frac{\omega(\alpha)^{\alpha}}{s^{1+\alpha} + \omega\alpha^{1+\alpha}}\,,$$

using the inverse Laplace transform of

$$1/s^{\alpha+\alpha n+n+1} = L\left[t^{\alpha+\alpha n+n}/\Gamma[\alpha+\alpha n+n+1]\right],$$

we get

$$Q(t) = Q_0\sum_{n=0}^{\infty}(-1)^n\frac{(\omega t)^{\alpha+\alpha n+n}}{\Gamma[\alpha+\alpha n+n+1]}\,. \tag{6}$$

We notice that when $\alpha = 0$, eq. (6) becomes the solution of eq. (2), and when $\alpha = 1$, eq. (6) becomes the solution of eq. (1).

3. Results and discussion

With the initial values specified in eq.(5), we have plotted $Q(t)/Q_0$ versus $\theta = \omega t$ (*theta* is in radians throughout the paper) for various α's from eq. (6). Fig. (1) shows the variation of $Q(t)/Q_0$ versus θ for α=0, 0.3, 0.5, 0.7, and 1. From **Figure 1** we notice that at $\alpha = 1$ the behavior is purely oscillatory with a fixed period in of value 2π and a first zero appearing at half period of π, with a peak value of 1, which represents the solution to eq. (1). As α decreases, the oscillations persist but they do not repeat themselves with a fixed period, and the peak decays as increases. If we take the case of $\alpha = 0.5$, we notice that the first zero appears at $\theta = 2.95335$ as opposed to $\theta = 3.14159$ for $\alpha = 1$. Also the higher zeroes (second, third, fourth) are not multiples of the first zero as in the case of $\alpha = 1$. Also the first peak value is no longer 1 as in $\alpha = 1$, but 0.726 in $\alpha = 0.5$. Also the first peak position is at $\theta = 0.79$ rather than $\theta = \pi/2$ for $\alpha = 1$. However, in **Figure 2** we have plotted the θ position of the first zero versus α, and we see that when α decreases from the value of 1 the first zero position decreases slightly until a value around $\alpha = 0.6$ it reaches its minimum which is 2.93821 not very different from π. However as α gets smaller beyond 0.6, the first zero position gets larger and towards $\alpha = 0$ it will tend to infinity. A general behavior is noticed from **Figure 1** as α decreases the first peak position shifts towards $\theta = 0$. Actually the dependence of the θ position of the first peak position versus α is shown in **Figure 3**, where the dependence seem to be linear.

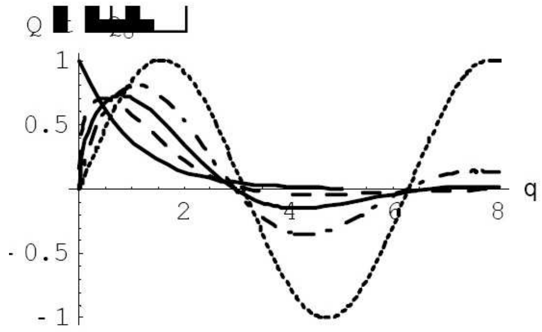

Figure 1: $Q(t)/Q_0$ vs. θ for various fractional orders α(0,0.3,0.5,0.7,1), oscillations start to damp as alpha decreases

The first peak value is plotted versus α in **Figure 4**, we notice at first its value decreases as α decreases and it reaches its minimum value of 0.7 around $\alpha = 0.3$. However the first peak value then increases as α further decreases until we reach a value of 1 at $\alpha = 0$. This variation of the first peak value is quite expected because of the initial condition given in eq. (5) above.

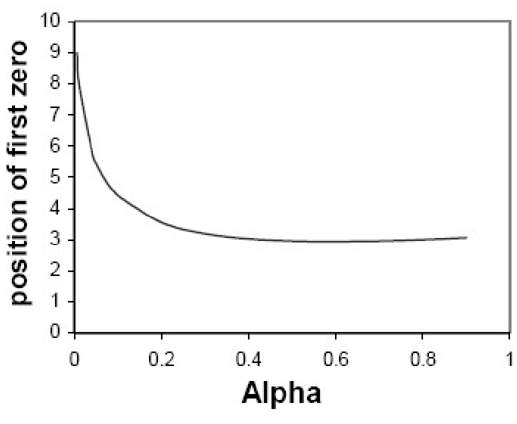

Figure 2: Position of first zero versus the fractional order alpha (α)

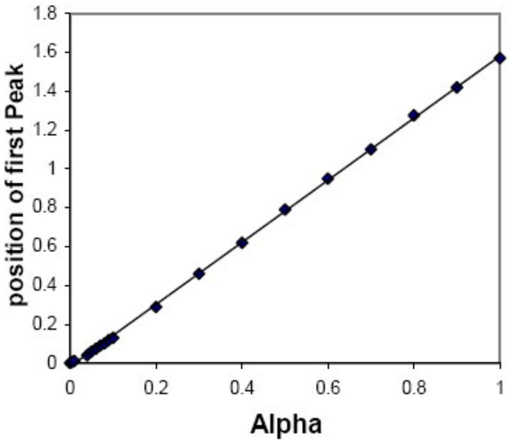

Figure 3: Position of first peak versus the fractional order alpha (α)

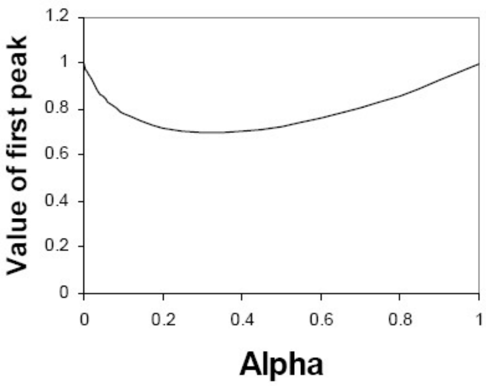

Figure 4: Value of the height of the first peak versus the fractional order alpha (α)

The result of eq. (6), which is plotted in **Figure 1**, is of no surprise. Suppose we have an LC circuit with a time-dependent inductance $L(t)$, while keeping the capacitance C independent of time. The LC circuit in this case will resemble an RLC circuit where the time derivative of $L(t)$ will take the place of R and the solution of the ordinary differential equation obtained from applying Kirchhoff's loop rule will be both oscillatory and decaying. However our hybrid $LC - RC$ circuit analyzed using fractional calculus may solve a certain class of problems in which the inductance L is dependent on $Q(t)$, [i.e. $L(Q(t))$]. Inductors of this sort may involve inductors that are made of reasonably soft loops that may change shape as $Q(t)$ changes with time.

In view of the discussion above, we notice that the inductor, as α departs from the value of 1, starts to develop a resistive behavior and a pseudo-LCR circuit starts to develop until α becomes zero the circuit becomes an RC circuit. In this intermediate case, i.e the case of $0 < \alpha < 1$, we suppose that the inductor starts to loose some of its character as an inductor and has a partial "memory loss" which grows as α decreases. This loss of memory becomes stronger until the inductor is no more an inductor and becomes just a resistor of the wire making it, and the circuit is then an RC circuit (a capacitor discharges in a resistor).

In this course of deteriorating inductance the inductance in the hybrid circuit for a particular α behaves like a time dependent inductance in a regular LC circuit. It is suggested that the above fractional treatment will fit the analysis of LC circuits that may contain inductance that depends on the charge flowing through it.

The concept of intermediate stages has been introduced by many authors ([1]-[4], and [21]).

Equation (6), the solution of the fractional differential equation representing the case discussed above, is assumed to describe the case of the inductor that develops a resistive character by loosing part of its nature as an ideal inductor. The LC circuit is then some kind of an LCR circuit which becomes an RC circuit at the end of the evolution process ($\alpha = 0$).

4. Conclusion

We were able to show in this work how an LC circuit can evolve into an RC circuit and correspondingly how a regular oscillatory behavior can also evolve into a decay behavior where the inductor starts to loose some of its character as an inductor and develop a resistive behavior in a manner of a memory loss. Furthermore, as we mentioned in the results and discussion above, eq. (6) has the potential of being used to describe the behavior of complex LC circuits in which the inductance depends on the charge Q, which in turn depends on time. It is worth noticing that the phase (ωt) rather than t appears as the variable in this equation, which makes it applicable to various oscillating systems such as mechanical oscillations among others.

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[1] *Department of Physical Sciences*
Jordan University of Science and Technology
P.O.Box 3030, Irbid - 21100, JORDAN
e-mail: akram@just.edu.jo

Received: December 21, 2005

[2] *Department of Physics, Yarmouk University*
Irbid - 21163, JORDAN