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SOBOLEV-MORREY TYPE INEQUALITY FOR RIESZ POTENTIALS, ASSOCIATED WITH THE LAPLACE-BESSEL DIFFERENTIAL OPERATOR *)

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Abstract

We consider the generalized shift operator, generated by the Laplace-Bessel differential operator

$$\Delta_{B_n} = \sum_{k=1}^{n-1} \frac{\partial^2}{\partial x_k^2} + B_n, B_n = \frac{\partial^2}{\partial x_n^2} + \frac{\gamma}{x_n}\frac{\partial}{\partial x_n}, \ \gamma > 0.$$

The B_n-maximal functions and the B_n-Riesz potentials, generated by the Laplace-Bessel differential operator Δ_{B_n} are investigated. We study the B_n-Riesz potentials in the B_n-Morrey spaces and B_n-BMO spaces. An inequality of Sobolev -Morrey type is established for the B_n-Riesz potentials.

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1. Introduction

The classical Riesz potential is an important technical tool in harmonic analysis, theory of functions and partial differential equations. The maximal function, singular integral, potential and related topics associated with the Laplace-Bessel differential operator

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$$\Delta_{B_n} = \sum_{k=1}^{n-1} \frac{\partial^2}{\partial x_k^2} + B_n, \qquad B_n = \frac{\partial^2}{\partial x_n^2} + \frac{\gamma}{x_n}\frac{\partial}{\partial x_n}, \quad \gamma > 0$$

have been investigated by many researchers, see B. Muckenhoupt and E. Stein [14], I. Kipriyanov [13], K. Trimeche [17], L. Lyakhov [12], K. Stempak [15,16], A.D. Gadjiev and I.A. Aliev [1,4], I.A. Aliev and S. Bayrakci [2], I. Ekincioglu and A. Serbetci [10], V.S. Guliyev [5]-[8] and others.

In this paper we consider the generalized shift operator, generated by the Laplace-Bessel differential operator Δ_{B_n} in terms of which the B_n-maximal functions and B_n-Riesz potentials are investigated. We study the B_n-Riesz potential in the B_n-Morrey spaces and B_n-BMO spaces. An inequality of Sobolev-Morrey type is established for the B_n-Riesz potentials.

The structure of the paper is as follows. In Section 1 we present some definitions and auxiliary results. In Section 2 we introduce and study some embeddings into the function spaces, associated with the Laplace-Bessel differential operator. In Section 3 the $L_{p,\gamma}$ boundedness of the B_n-maximal operator is proved. In Section 4 the boundedness of the B_n-maximal operator on B_n-Morrey spaces $L_{p,\lambda,\gamma}$ is proved. The main result of the paper is the inequality of Sobolev-Morrey type for the B_n-Riesz potentials, established in Section 5.

2. Definitions, notation and preliminaries

Suppose that R^n is the n-dimensional Euclidean space, $x = (x_1, \dots, x_n)$ are vectors in R^n. Let $R^n_+ = \{x \in R^n \ ; \ x = (x', x_n), x_n > 0\}$, $|x| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$, $\gamma > 0$, $E_+(x,r) = \{y \in R^n_+ : |x-y| < r\}$. For a measurable set $E \subset R^n_+$ let $|E|_\gamma = \int_E x_n^\gamma dx$, then $|E_+(0,r)|_\gamma = \omega(n,\gamma) r^{n+\gamma}$, where

$$\omega(n,\gamma) = \int_{E_+(0,1)} x_n^\gamma dx = \frac{\pi^{\frac{n-1}{2}} \Gamma\left(\frac{\gamma+1}{2}\right)}{2\Gamma\left(\frac{n+\gamma-2}{2}\right)}.$$

Denote by T^y the generalized shift operator (B_n-shift operator) acting according to the law

$$T^y f(x) = C_\gamma \int_0^\pi f\left(x' - y', (x_n, y_n)_\beta\right) \sin^{\gamma-1} \beta d\beta,$$

where $(x_n, y_n)_\beta = \sqrt{x_n^2 + y_n^2 - 2x_n y_n \cos\beta}$ and $C_\gamma = \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\gamma}{2}\right)} = \frac{2}{\pi}\omega(2,\gamma)$.

We remark that the generalized shift operator T^y is closely connected with the Bessel differential operator B_n (for example, $n = 1$ see [11] and $n > 1$ [13] for details).

For a fixed parameter $\gamma > 0$, let $L_{p,\gamma}(R_+^n)$ be the space of measurable functions on R_+^n with finite norm

$$\|f\|_{L_{p,\gamma}} = \|f\|_{p,\gamma} = \left(\int_{R_+^n} |f(x)|^p x_n^\gamma dx \right)^{1/p}, \quad 1 \le p < \infty.$$

For $p = \infty$ the spaces $L_{\infty,\gamma}(R_+^n)$ are defined by means of the usual modification

$$\|f\|_{L_{\infty,\gamma}} = \|f\|_{L_\infty} = \operatorname*{ess\,sup}_{x \in R_+^n} |f(x)|.$$

The translation operator T^y generates the corresponding B_n-convolution

$$(f \otimes g)(x) = \int_{R_+^n} f(y)[T^y f(x)] y_n^\gamma dy,$$

for which the Young inequality

$$\|f \otimes g\|_{r,\gamma} \le \|f\|_{p,\gamma} \|g\|_{q,\gamma}, \quad 1 \le p, q, r \le \infty, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$$

holds.

Lemma 1. ([1]) *Let $1 \le p \le \infty$. Then for all $y \in R_+^n$*

$$\|T^y f(\cdot)\|_{L_{p,\gamma}} \le \|f\|_{L_{p,\gamma}}. \tag{2.1}$$

Lemma 2. *For all $x \in R_+^n$ the following equality is valid*

$$\int_{E_+(x,t)} g(y) y_n^\gamma dy = \frac{1}{2C_\gamma} \int_{B((x,0),t)} g\left(z', \sqrt{z_n^2 + z_{n+1}^2}\right) |z_{n+1}|^{\gamma-1} dz dz_{n+1},$$

where $B((x,0),t) = \{(z, z_{n+1}) \in R^{n+1} \ : \ |(x' - z', x_n - \sqrt{z_n^2 + z_{n+1}^2})| < r\}$.

Lemma 3. *For all $x \in R_+^n$ the following equality is valid*

$$\int_{E_+(0,t)} T^y g(x) y_n^\gamma dy = \frac{1}{2} \int_{E((x,0),t)} g\left(z', \sqrt{z_n^2 + z_{n+1}^2}\right) |z_{n+1}|^{\gamma-1} dz dz_{n+1},$$

where $E((x,0),t) = \{(z, z_{n+1}) \in R^{n+1} \ : \ |(x - z, z_{n+1})| < r\}$.

The proof of Lemmas 2, 3 is straightforward via the following substitutions $z' = y'$, $z_n = y_n \cos\beta$, $|z_{n+1}| = y_n \sin\beta$, $0 \le \beta < \pi$, $y \in R_+^n$, $(z, z_{n+1}) \in R^{n+1}$.

3. Function spaces, associated with the Bessel differential operators B_n

Definition 1. Let $1 \le p < \infty$. By $WL_{p,\gamma}(R_+^n)$ we denote the weak $L_{p,\gamma}$ spaces defined as the set of locally integrable functions $f(x)$, $x \in R_+^n$ with the finite norms

$$\|f\|_{WL_{p,\gamma}} = \sup_{r>0} r \left|\left\{x \in R_+^n \ : \ |f(x)| > r\right\}\right|_\gamma^{1/p}.$$

Definition 2. ([6]) Let $1 \le p < \infty$, $0 \le \lambda \le n+\gamma$, $[t]_1 = \min\{1, t\}$. We denote by $L_{p,\lambda,\gamma}(R_+^n)$ Morrey spaces, associated with the Bessel differential operator B_n ($\equiv B_n$-Morrey spaces) and by $\widetilde{L}_{p,\lambda,\gamma}(R_+^n)$ the modified Morrey spaces, associated with B_n ($\equiv$ modified B_n-Morrey spaces) as the set of locally integrable functions $f(x)$, $x \in R_+^n$, with the finite norms

$$\|f\|_{L_{p,\lambda,\gamma}} = \sup_{t>0,\, x\in R_+^n} \left(t^{-\lambda} \int_{E_+(0,t)} T^y \, |f(x)|^p \, y_n^\gamma dy \right)^{1/p},$$

$$\|f\|_{\widetilde{L}_{p,\lambda,\gamma}} = \sup_{t>0,\, x\in R_+^n} \left([t]_1^{-\lambda} \int_{E_+(0,t)} T^y \, |f(x)|^p \, y_n^\gamma dy \right)^{1/p},$$

respectively.

Note that

$$\widetilde{L}_{p,0,\gamma}(R_+^n) = L_{p,0,\gamma}(R_+^n) = L_{p,\gamma}(R_+^n),$$

$$L_{p,n+\gamma,\gamma}\left(R_+^n\right) = L_\infty\left(R_+^n\right).$$

$$\widetilde{L}_{p,\lambda,\gamma}(R_+^n) \subset\!\!\succ L_{p,\gamma}(R_+^n) \quad \text{and} \quad \|f\|_{L_{p,\gamma}} \le \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}. \tag{3.1}$$

Definition 3. ([6]) Let $1 \le p < \infty$, $0 \le \lambda \le n+\gamma$, $[t]_1 = \min\{1, t\}$. We denote by $WL_{p,\lambda,\gamma}(R_+^n)$ the weak B_n-Morrey spaces and by $W\widetilde{L}_{p,\lambda,\gamma}(R_+^n)$ the modified weak B_n-Morrey spaces as the set of locally integrable functions $f(x)$,$x \in R_+^n$ with finite norms

$$\|f\|_{WL_{p,\lambda,\gamma}} = \sup_{r>0} r \sup_{t>0,\, x\in R_+^n} \left(t^{-\lambda} \int_{\{y\in E_+(0,t):\, T^y|f(x)|>r\}} y_n^\gamma dy \right)^{1/p},$$

$$\|f\|_{W\widetilde{L}_{p,\lambda,\gamma}} = \sup_{r>0} r \sup_{t>0,\, x\in R_+^n} \left([t]_1^{-\lambda} \int_{\{y\in E_+(0,t):\, T^y|f(x)|>r\}} y_n^\gamma dy \right)^{1/p},$$

respectively.

Note that

$$WL_{p,\gamma}(R_+^n) = WL_{p,0,\gamma}(R_+^n) = W\widetilde{L}_{p,0,\gamma}(R_+^n),$$

$$L_{p,\lambda,\gamma}(R_+^n) \subset WL_{p,\lambda,\gamma}(R_+^n) \text{ and } \|f\|_{WL_{p,\lambda,\gamma}} \le \|f\|_{L_{p,\lambda,\gamma}},$$

$$\widetilde{L}_{p,\lambda,\gamma}(R_+^n) \subset W\widetilde{L}_{p,\lambda,\gamma}(R_+^n) \text{ and } \|f\|_{W\widetilde{L}_{p,\lambda,\gamma}} \le \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}.$$

4. $L_{p,\gamma}$-boundedness of the B_n-maximal operator

In this section we study the $L_{p,\gamma}$-boundedness of the B_n- maximal operator (see [5])

$$M_\gamma f(x) = \sup_{r>0} |E_+(0,r)|_\gamma^{-1} \int_{E_+(0,r)} T^y |f(x)| y_n^\gamma dy.$$

THEOREM 1.
1. If $f \in L_{1,\gamma}\left(R_+^n\right)$, then $M_\gamma f \in WL_{1,\gamma}(R_+^n)$ and

$$\|M_\gamma f\|_{WL_{1,\gamma}} \le C\|f\|_{L_{1,\gamma}},$$

where C is independent of f.
2. If $f \in L_{p,\gamma}(R_+^n)$, $1 < p \le \infty$, then $M_\gamma f \in L_{p,\gamma}(R_+^n)$ and

$$\|M_\gamma f\|_{L_{p,\gamma}} \le C_{p,\gamma}\|f\|_{L_{p,\gamma}},$$

where $C_{p,\gamma}$ depends only on p, γ and n.

P r o o f. The B_n-maximal function may be interpreted as a maximal function defined on a space of homogeneous type. By this we mean a topological space X equipped with a continuous pseudometric ρ and a positive measure μ satisfying

$$\mu(E(x,2r)) \le C_1 \mu(E(x,r)) \tag{4.1}$$

with a constant C_1 independent of x and $r > 0$. Here $E(x,r) = \{y \in X : \rho(x,y) < r\}$, $\rho(x,y) = |x-y|$. Let (X,ρ,μ) be a space of homogeneous type. Define

$$M_\mu f(x) = \sup_{r>0} \mu(E(x,r))^{-1} \int_{E(x,r)} |f(y)| d\mu(y).$$

It is well known that the maximal function M_μ is weak type (1, 1) and is bounded on $L_p(X, d\mu)$ for $1 < p < \infty$ (see [3]). Here we are concerned with the maximal operator defined by $d\mu(x) = x_n^\gamma dx$. It is clear that this measure satisfies the doubling condition (4.1).

We will show that

$$M_\gamma f(x) \leq \frac{2^{n+\gamma} C_2}{\omega(n,\gamma)} M_\mu f(x), \tag{4.2}$$

where $C_2 = \frac{2^{\frac{\gamma}{2}+1} C_\gamma}{\gamma}$.

From the definition of the B_n–shift operator it follows that $T^y \chi_{E_+(0,r)}(x)$ is supported in $E_+(x,r)$.

Moreover, there exists a constant C_2 such that

$$0 \leq T^y \chi_{E_+(0,r)}(x) \leq \min\{1, C_2 r^\gamma x_n^{-\gamma}\}, \qquad \forall y \in E_+(x,r). \tag{4.3}$$

In the case $x_n \leq r$ this follows from the inequality $0 \leq T^y \chi_{E_+(0,r)}(x) \leq 1$.

Also

$$\mu E(x,r) = |E_+(x,r)|_\gamma \leq \prod_{i=1}^{n-1} \int\limits_{|y_i|<r} dy_i \int\limits_{\{y_n>0; |x_n-y_n|<r\}} y_n^\gamma dy_n$$

$$\leq 2^{n+\gamma} r^{n-1} \begin{cases} r x_n^\gamma, & r < x_n \\ r^{1+\gamma}, & r \geq x_n \end{cases} = 2^{n+\gamma} r^{n+\gamma} \begin{cases} (x_n/r)^\gamma, & r < x_n \\ 1, & r \geq x_n. \end{cases}$$

Thus

$$M_\gamma f(x) \leq M_{1,\gamma} f(x) + M_{2,\gamma} f(x) = \sup_{r>x_n} |E_+(0,r)|_\gamma^{-1}$$

$$\times \int\limits_{E_+(0,r)} T^y |f(x)| y_n^\gamma dy + \sup_{0<r\leq x_n} |E_+(0,r)|_\gamma^{-1} \int\limits_{E_+(0,r)} T^y |f(x)| y_n^\gamma dy.$$

If $r \geq x_n$, then $\mu E(x,r) \leq 2^{n+\gamma} r^{n+\gamma}$, $|E_+(0,r)|_\gamma = \omega(n,\gamma) r^{n+\gamma}$ and $T^y \chi_{E_+(0,r)}(x) \leq 1$. Thus yields

$$M_{1,\gamma} f(x) \leq \sup_{r>x_n} |E_+(0,r)|_\gamma^{-1} \int_{E_+(x,r)} |f(y)| T^y \chi_{E_+(0,r)}(x) y_n^\gamma dy$$

$$\leq \frac{2^{n+\gamma}}{\omega(n,\gamma)} \sup_{r>0} \frac{1}{\mu E(x,r)} \int_{E(x,r)} |f(y)| d\mu(y) = \frac{2^{n+\gamma}}{\omega(n,\gamma)} M_\mu f(x).$$

If $r < x_n$, then $\mu E(x,r) \leq 2^{n+\gamma} r^n x_n^\gamma$, $|E_+(0,r)|_\gamma = \omega(n,\gamma) r^{n+\gamma}$ and $T^y \chi_{E_+(0,r)}(x) \leq C_2 r^\gamma x_n^{-\gamma}$. Thus yields

$$M_{2,\gamma} f(x) \leq \sup_{r \leq x_n} |E_+(0,r)|_\gamma^{-1} \int_{E_+(x,r)} |f(y)| T^y \chi_{E_+(0,r)}(x) y_n^\gamma dy$$

$$\leq \frac{2^{n+\gamma} C_2}{\omega(n,\gamma)} \sup_{r>0} \frac{1}{\mu E(x,r)} \int_{E(x,r)} |f(y)| d\mu(y) = \frac{2^{n+\gamma} C_2}{\omega(n,\gamma)} M_\mu f(x).$$

Therefore we get (4.2), which completes the proof. ∎

Remark 1. For the one-dimensional case Theorem 1 was proved earlier by K. Stempak [16].

5. $L_{p,\lambda,\gamma}$-boundedness of the B_n-maximal functions

In this section we study the $L_{p,\lambda,\gamma}$-boundedness of the B_n-maximal operators.

Theorem 2.

1. If $f \in L_{1,\lambda,\gamma}\left(R_+^n\right)$, $0 \leq \lambda < n+\gamma$, then $M_\gamma f \in WL_{1,\lambda,\gamma}\left(R_+^n\right)$ and

$$\|M_\gamma f\|_{WL_{1,\lambda,\gamma}} \leq C \|f\|_{L_{1,\lambda,\gamma}}, \tag{5.1}$$

where C is independent of f.

2. If $f \in L_{p,\lambda,\gamma}\left(R_+^n\right)$, $1 < p < \infty$, $0 \leq \lambda < n+\gamma$, then $M_\gamma f \in L_{p,\lambda,\gamma}\left(R_+^n\right)$ and

$$\|M_\gamma f\|_{L_{p,\lambda,\gamma}} \leq C_{p,\gamma} \|f\|_{L_{p,\lambda,\gamma}}, \tag{5.2}$$

where $C_{p,\gamma}$ depends only on p,γ and n.

P r o o f. We need to introduce another maximal function defined on a space of homogeneous type (Y, d, ν). By this we mean a topological space $Y = R^{n+1}$ equipped with a continuous pseudometric d and a positive measure ν satisfying

$$\nu(E((x,x_{n+1}),2r)) \leq C_1 \nu(E((x,x_{n+1}),r)) \tag{5.3}$$

with a constant C_1 independent of (x,x_{n+1}) and $r>0$. Here $E((x,x_{n+1}),r) = \{(y,y_{n+1}) \in Y : d(((x,x_{n+1}),(y,y_{n+1})) < r\}$, $d\nu(y,y_{n+1}) = |y_{n+1}|^{\gamma-1} dy dy_{n+1}$, $d((x,x_{n+1}),(y,y_{n+1})) = |(x,x_{n+1}) - (y,y_{n+1})| \equiv (|x-y|^2 + (x_{n+1}-y_{n+1})^2)^{\frac{1}{2}}$.

Let (Y,d,ν) be a space of homogeneous type. Define

$$M_\nu \overline{f}(x,x_{n+1}) = \sup_{r>0} \nu(E((x,x_{n+1}),r))^{-1} \int_{E((x,x_{n+1}),r)} \left|\overline{f}(y,y_{n+1})\right| d\nu(y),$$

where $\overline{f}(x,x_{n+1}) = f\left(x', \sqrt{x_n^2+x_{n+1}^2}\right)$.

It is well known that the maximal function M_ν is of weak type (1, 1) and is bounded on $L_p(Y,d\nu)$ for $1<p<\infty$ (see [3]). Here we are concerned with the maximal operator defined by $d\nu(y,y_{n+1}) = |y_{n+1}|^{\gamma-1}dydy_{n+1}$. It is clear that this measure satisfies the doubling condition (5.3).

It can be proved that

$$M_\gamma f\left(z', \sqrt{z_n^2+z_{n+1}^2}\right) = M_\nu \overline{f}\left(z', \sqrt{z_n^2+z_{n+1}^2}, 0\right), \tag{5.4}$$

$$M_\gamma f(x) = M_\nu \overline{f}(x,0). \tag{5.5}$$

Indeed, Lemma 3,

$$\int_{E_+(0,t)} T^y \left| f\left(z', \sqrt{z_n^2+z_{n+1}^2}\right)\right| y_n^\gamma dy$$

$$= \frac{1}{2}\int_{E\left(\left(z', \sqrt{z_n^2+z_{n+1}^2}, 0\right), r\right)} \left|\overline{f}(y,y_{n+1})\right| d\nu(y,y_{n+1})$$

and

$$|E_+(0,t)|_\gamma = \nu E\left(\left(z', \sqrt{z_n^2+z_{n+1}^2}, 0\right), r\right)$$

imply (5.4). Furthermore, taking $z_{n+1}=0$ in (5.4) we get (5.5).

Using Lemma 3 and equality (5.4) we have

$$\int_{E_+(0,t)} T^y \left(M_\gamma f(x)\right)^p y_n^\gamma dy$$

$$= \frac{1}{2}\int_{E((x,0),t)} \left(M_\gamma f\left(z', \sqrt{z_n^2+z_{n+1}^2}\right)\right)^p |z_{n+1}|^{\gamma-1} dz dz_{n+1}$$

$$= \frac{1}{2}\int_{E((x,0),t)} \left(M_\nu \overline{f}\left(z', \sqrt{z_n^2+z_{n+1}^2}, 0\right)\right)^p d\nu(z,z_{n+1}).$$

In [9] there was proved that the analogue of the Fefferman-Stein theorem for maximal functions defined on a space of homogeneous type is valid, if condition (5.3) is satisfied. Therefore

$$\int_{E((x,x_{n+1}),t)} (M_\nu \varphi(y,y_{n+1}))^p \, \psi(y,y_{n+1}) d\nu(y,y_{n+1})$$

$$\leq C_3 \int_{E((x,x_{n+1}),t)} |\varphi(y,y_{n+1})|^p M_\nu \psi(y,y_{n+1}) d\nu(y,y_{n+1}). \qquad (5.6)$$

Then taking $\varphi(y,y_{n+1}) = \overline{f}\left(y', \sqrt{y_n^2 + y_{n+1}^2}, 0\right)$ and $\psi = 1$ we obtain from equality (5.6) and Lemma 3 that

$$\int_{E_+(0,t)} T^y \left(M_\gamma f(x)\right)^p y_n^\gamma dy$$

$$= \frac{1}{2} \int_{E((x,0),t)} \left(M_\nu \overline{f}\left(z', \sqrt{z_n^2 + z_{n+1}^2}, 0\right)\right)^p \, d\nu(z, z_{n+1})$$

$$\leq C_4 \int_{E((x,0),t)} \left| \overline{f}\left(z', \sqrt{z_n^2 + z_{n+1}^2}, 0\right)\right|^p \, d\nu(z, z_{n+1})$$

$$= C_4 \int_{E((x,0),t)} \left| f\left(z', \sqrt{z_n^2 + z_{n+1}^2}\right)\right|^p \, d\nu(z, z_{n+1})$$

$$= 2 \, C_4 \int_{E_+(0,t)} T^y |f(x)|^p y_n^\gamma dy \leq 2 \, C_4 \, r^\lambda \, \|f\|^p_{L_{p,\lambda,\gamma}}.$$

■

Remark 2. Theorem 1 is obtained from Theorem 2 under the choice $\lambda = 0$.

Similarly, we prove the following

Theorem 3.

1. If $f \in \widetilde{L}_{1,\lambda,\gamma}\left(R_+^n\right)$, $0 \leq \lambda < n+\gamma$, then $M_\gamma f \in W\widetilde{L}_{1,\lambda,\gamma}\left(R_+^n\right)$ and

$$\|M_\gamma f\|_{W\widetilde{L}_{1,\lambda,\gamma}} \leq C \|f\|_{\widetilde{L}_{1,\lambda,\gamma}},$$

where C is independent of f.

2. If $f \in \widetilde{L}_{p,\lambda,\gamma}\left(R_+^n\right)$, $1 < p < \infty$, $0 \leq \lambda < n+\gamma$, then $M_\gamma f \in \widetilde{L}_{p,\lambda,\gamma}\left(R_+^n\right)$ and

$$\|M_\gamma f\|_{\widetilde{L}_{p,\lambda,\gamma}} \leq C_{p,\gamma} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}},$$

where $C_{p,\gamma}$ depends only on p, γ and n.

6. Sobolev theorem for the B_n-Riesz potentials

Consider the B_n-Riesz potentials

$$I_\gamma^\alpha f(x) = \int_{R_+^n} T^y |x|^{\alpha-n-\gamma} f(y) y_n^\gamma dy, \quad 0 < \alpha < n+\gamma.$$

For the B_n-Riesz potentials the following generalized Hardy–Littlewood–Sobolev theorem is valid.

THEOREM 4. *Let $0 < \alpha < n+\gamma$, $0 \le \lambda < n+\gamma$.*
If $f \in \widetilde{L}_{p,\lambda,\gamma}(R_+^n)$, $1 < p < \frac{n+\gamma}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n+\gamma}$, then $I_\gamma^\alpha f \in \widetilde{L}_{q,\lambda,\gamma}(R_+^n)$ and

$$\left\| I_\gamma^\alpha f \right\|_{\widetilde{L}_{q,\lambda,\gamma}} \le C_{p,\lambda} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}, \tag{6.1}$$

where $C_{p,\lambda}$ is independent of f.
If $f \in \widetilde{L}_{1,\lambda,\gamma}(R_+^n)$, $1 - \frac{1}{q} = \frac{\alpha}{n+\gamma}$, then $I_\gamma^\alpha f \in W\widetilde{L}_{q,\lambda,\gamma}(R_+^n)$ and

$$\left\| I_\gamma^\alpha f \right\|_{W\widetilde{L}_{q,\lambda,\gamma}} \le C_\lambda \|f\|_{\widetilde{L}_{1,\lambda,\gamma}}, \tag{6.2}$$

where C_λ is independent of f.

P r o o f. Let $f \in \widetilde{L}_{p,\lambda,\gamma}\left(R_+^n\right)$. Then

$$I_\gamma^\alpha f(x) = \left(\int_{E_+(0,t)} + \int_{R_+^n \setminus E_+(0,t)} \right) T^y f(x) |y|^{\alpha-n-\gamma} y_n^\gamma dy \equiv A(x,t) + C(x,t). \tag{6.3}$$

For $A(x,t)$ we have

$$|A(x,t)| \le \int_{E_+(0,t)} T^y |f(x)| \, |y|^{\alpha-n-\gamma} y_n^\gamma dy$$

$$\le \sum_{k=-\infty}^{-1} \left(2^k t\right)^{\alpha-n-\gamma} \int_{E_+(0,2^{k+1}t) \setminus E_+(0,2^k t)} T^y |f(x)| \, y_n^\gamma dy.$$

Hence

$$|A(x,t)| \le C_5 t^\alpha M_\gamma f(x) \qquad \text{with} \qquad C_5 = \frac{\omega(n,\gamma) 2^{n+\gamma}}{2^\alpha - 1}. \tag{6.4}$$

In the second integral by the Hölder inequality and inequality (2.1) we have

$$|C(x,t)| \le \|T^y f(\cdot)\|_{L_{p,\gamma}} \left\| |y|^{\alpha-n-\gamma} \right\|_{L_{p',\gamma}(R_+^n \setminus E_+(0,t))} \le C_6 t^{\alpha - \frac{n+\gamma}{p}} \|f\|_{L_{p,\gamma}}.$$

Consequently, we use inequality (3.1) and the relation $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n+\gamma}$ and get

$$|C(x,t)|\leq C_6\|f\|_{L_{p,\gamma}}t^{-(n+\gamma)/q}\leq C_6 t^{-\frac{n+\gamma}{q}}\,\|f\|_{\widetilde{L}_{p,\lambda,\gamma}}\,. \tag{6.5}$$

Thus, from (6.4) and (6.5), we have

$$\left|I_\gamma^\alpha f(x)\right|\leq C_5 t^\alpha M_\gamma f(x)+C_6 t^{-\frac{n+\gamma}{q}}\,\|f\|_{\widetilde{L}_{p,\lambda,\gamma}}\,.$$

Minimizing with respect to t, at $t=\left[(M_\gamma f(x))^{-1}\,\|f\|_{\widetilde{L}_{p,\lambda,\gamma}}\right]^{p/(n+\gamma)}$ we have

$$\left|I_\gamma^\alpha f\,(x)\right|\leq (C_5+C_6)\,(M_\gamma f(x))^{p/q}\,\|f\|_{\widetilde{L}_{p,\lambda,\gamma}}^{1-p/q}\,.$$

Hence, by Theorem 3, we have

$$\int_{E_+(0,t)}T^y\left|I_\gamma^\alpha f(x)\right|^q y_n^\gamma dy\leq C_7\,\|f\|_{\widetilde{L}_{p,\lambda,\gamma}}^{q-p}\int_{E_+(0,t)}T^y\,(M_\gamma f(y))^p\,y_n^\gamma dy$$

$$\leq C_8[t]_1^\lambda\,\|f\|_{\widetilde{L}_{p,\lambda,\gamma}}^{q-p}\,\|f\|_{\widetilde{L}_{p,\lambda,\gamma}}^{p}\leq C_8[t]_1^\lambda\,\|f\|_{\widetilde{L}_{p,\lambda,\gamma}}^{q}\,,$$

which yields (6.1).

Let $f\in\widetilde{L}_{1,\lambda,\gamma}(R_+^n)$. It suffices to prove inequality (6.1) with 2β instead of β on the left–hand of the inequality. So,

$$\left|\left\{y\in E_+(0,t)\ :\ T^y\left|I_\gamma^\alpha f(x)\right|>2\beta\right\}\right|_\gamma$$

$$\leq|\{y\in E_+(0,t)\ :\ T^y|A(x,t)|>\beta\}|_\gamma+|\{y\in E_+(0,t)\ :\ T^y|C(x,t)|>\beta\}|_\gamma\,.$$

Taking into account inequality (6.4) and Theorem 3, we have

$$|\{y\in E_+(0,t)\ :\ T^y|A(x,t)|>\beta\}|_\gamma$$

$$\leq\left|\left\{y\in E_+(0,t)\ :\ T^y\,(M_\gamma f(x))>\frac{\beta}{C_5t^\alpha}\right\}\right|_\gamma\leq\frac{C_9t^\alpha}{\beta}\cdot[t]_1^\lambda\,\|f\|_{\widetilde{L}_{1,\lambda,\gamma}}$$

and thus if $C_6t^{-\frac{n+\gamma}{q}}\,\|f\|_{\widetilde{L}_{1,\lambda,\gamma}}=\beta$, then $|C\,(x,t)|\leq\beta$ and consequently, $|\,\{y\in E_+(0,t)\ :\ T^y\,|C(x,t)|>\beta\}\,|_\gamma=0$.

Finally

$$\left|\left\{y\in E_+(0,t)\ :\ T^y\left|I_\gamma^\alpha f(x)\right|>2\beta\right\}\right|_\gamma\leq C_9[t]_1^\lambda\left(\frac{\|f\|_{\widetilde{L}_{1,\lambda,\gamma}}}{\beta}\right)^q,$$

which proves the theorem. ∎

Theorem 5. *Let $0 < \alpha < n + \gamma$ and $0 \le \lambda < n + \gamma - \alpha p$.*

If $f \in L_{p,\lambda,\gamma}(R_+^n)$, where $1 < p < \frac{n+\gamma}{\alpha}$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n+\gamma-\lambda}$, then $I_\gamma^\alpha f \in L_{q,\lambda,\gamma}(R_+^n)$ and

$$\left\| I_\gamma^\alpha f \right\|_{L_{q,\lambda,\gamma}} \le C_{p,\lambda} \left\| f \right\|_{L_{p,\lambda,\gamma}}, \tag{6.6}$$

where $C_{p,\lambda}$ is independent of f.

If $f \in L_{1,\lambda,\gamma}(R_+^n)$, $1 - \frac{1}{q} = \frac{\alpha}{n+\gamma-\lambda}$, then $I_\gamma^\alpha f \in WL_{q,\lambda,\gamma}(R_+^n)$ and

$$\left\| I_\gamma^\alpha f \right\|_{WL_{q,\lambda,\gamma}} \le C_\lambda \left\| f \right\|_{L_{1,\lambda,\gamma}}, \tag{6.7}$$

where C_λ is independent of f.

P r o o f. Let $f \in L_{p,\lambda,\gamma}\left(R_+^n\right)$. From (6.3), for $C(x,t)$ by the Hölder's inequality we have

$$|C(x,t)| \le \left(\int\limits_{R_+^n \setminus E_+(0,t)} |y|^{-\beta} T^y \left| f(x) \right|^p y_n^\gamma dy \right)^{\frac{1}{p}}$$

$$\times \left(\int\limits_{R_+^n \setminus E_+(0,t)} |y|^{\left(\frac{\beta}{p} + \alpha - n - \gamma\right)p'} y_n^\gamma dy \right)^{\frac{1}{p'}} \le C_{13} t^{\frac{\lambda - n - \gamma}{p} + \alpha} \left\| f \right\|_{L_{p,\lambda,\gamma}}. \tag{6.8}$$

Thus, from (6.4) and (6.8) we have

$$\left| I_\gamma^\alpha f(x) \right| \le C_{14} \left(t^\alpha M_\gamma f(x) + t^{\frac{\lambda - n - \gamma}{q}} \left\| f \right\|_{L_{p,\lambda,\gamma}} \right).$$

Minimizing with respect to t, at $t = \left[(M_\gamma f(x))^{-1} \left\| f \right\|_{L_{p,\lambda,\gamma}} \right]^{p/(\lambda - n - \gamma)}$ we arrive at

$$\left| I_\gamma^\alpha f(x) \right| \le C_{14} \left(M_\gamma f(x) \right)^{p/q} \left\| f \right\|_{L_{p,\lambda,\gamma}}^{1-p/q}.$$

Hence, by Theorem 2, we have

$$\int_{E_+(0,t)} T^y \left| I_\gamma^\alpha f(x) \right|^q y_n^\gamma dy \le C_{14} \left\| f \right\|_{L_{p,\lambda,\gamma}}^{q-p} \int_{E_+(0,t)} T^y \left(M_\gamma f(x) \right)^p y_n^\gamma dy$$

$$\le C_{15} t^\lambda \left\| f \right\|_{L_{p,\lambda,\gamma}}^{q-p} \left\| f \right\|_{L_{p,\lambda,\gamma}}^{p} \le C_{15} t^\lambda \left\| f \right\|_{L_{p,\lambda,\gamma}}^{q}.$$

Let $f \in L_{1,\lambda,\gamma}(R_+^n)$. It suffices to prove the inequality (6.7) with 2β instead of β on the left-hand side of the inequality. So

$$\left| \left\{ y \in E_+(0,t) \ : \ T^y \left| I_\gamma^\alpha f(x) \right| > 2\beta \right\} \right|_\gamma$$

$\leq |\{y \in E_+(0,t) \ : \ T^y|A(x,t)| > \beta\}|_\gamma + |\{y \in E_+(0,t) \ : \ T^y|C(x,t)| > \beta\}|_\gamma.$

Taking into account inequality (6.4) and Theorem 2 we have

$$|\{y \in E_+(0,t) \ : \ T^y|A(x,t)| > \beta\}|_\gamma$$

$$\leq \left|\left\{y \in E_+(0,t) \ : \ T^y(M_\gamma f(x)) > \frac{\beta}{C_5 t^\alpha}\right\}\right|_\gamma \leq \frac{C_{16} t^\alpha}{\beta} \cdot t^\lambda \, \|f\|_{L_{1,\lambda,\gamma}}$$

and thus if $C_{13} t^{\frac{\lambda-n-\gamma}{q}} \|f\|_{L_{1,\lambda,\gamma}} = \beta$, then $|C(x,t)| \leq \beta$ and consequently, $|\{y \in E_+(0,t) \ : \ T^y|C(x,t)| > \beta\}|_\gamma = 0$.

Finally,

$$\left|\left\{y \in E_+(0,t) \ : \ T^y|I_\gamma^\alpha f(x)| > 2\beta\right\}\right|_\gamma \leq \frac{C_{16}}{\beta} t^\lambda t^\alpha \, \|f\|_{L_{1,\lambda,\gamma}}$$
$$= C_{16} t^\lambda \left(\frac{\|f\|_{L_{1,\lambda,\gamma}}}{\beta}\right)^q.$$

The theorem is proved. ■

COROLLARY. ([4]) *Let* $0 < \alpha < n+\gamma$.

If $1 < p < \frac{n+\gamma}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n+\gamma}$, $f \in L_{p,\gamma}(R_+^n)$, *then* $I_\gamma^\alpha f \in L_{q,\gamma}(R_+^n)$ *and*

$$\left\|I_\gamma^\alpha f\right\|_{L_{q,\gamma}} \leq C_{p,\gamma} \|f\|_{L_{p,\gamma}}, \tag{6.9}$$

where C_p *is independent of* f.

If $f \in L_{1,\gamma}(R_+^n)$, $\frac{1}{q} = 1 - \frac{\alpha}{n+\gamma}$, *then* $I_\gamma^\alpha f \in WL_{q,\gamma}(R_+^n)$ *and*

$$\left\|I_\gamma^\alpha f\right\|_{WL_{q,\gamma}} \leq C_\lambda \, \|f\|_{L_{1,\gamma}}, \tag{6.10}$$

where C_λ *is independent of* f.

THEOREM 6. *Let* $0 < \alpha < n+\gamma$.

If $1 < p < \frac{n+\gamma}{\alpha}$, *then the condition* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n+\gamma}$ *is necessary for inequality (6.9) to be valid.*

If $p = 1$, *then the condition* $1 - \frac{1}{q} = \frac{\alpha}{n+\gamma}$ *is necessary for inequality (6.10) to hold.*

P r o o f. Let $1 < p < \frac{n+\gamma}{\alpha}$, $f \in L_{p,\gamma}\left(R_+^n\right)$ and inequality (6.9) hold.

Define $f_t(x) =: f(tx)$. Then

$$\|f_t\|_{L_{p,\gamma}} = t^{-\frac{n+\gamma}{p}} \, \|f\|_{L_{p,\gamma}}$$

and

$$\left\|I_\gamma^\alpha f_t\right\|_{L_{q,\gamma}} = t^{-\alpha-\frac{n+\gamma}{q}} \left\|I_\gamma^\alpha f\right\|_{L_q^\gamma(R_+^n)}.$$

By the inequality (6.9),

$$\left\|I_\gamma^\alpha f\right\|_{L_{q,\gamma}} \leq C_{p,q} t^{\alpha+\frac{n+\gamma}{q}-\frac{n+\gamma}{p}} \|f\|_{L_{p,\gamma}}.$$

If $\frac{1}{p} > \frac{1}{q} + \frac{\alpha}{n+\gamma}$, then in the case $t \to 0$ we have $\left\|I_\gamma^\alpha f\right\|_{L_{q,\gamma}} = 0$ for all $f \in L_{p,\gamma}(R_+^n)$.

As well as if $\frac{1}{p} < \frac{1}{q} + \frac{\alpha}{n+\gamma}$, then at $t \to \infty$ we obtain $\left\|I_\gamma^\alpha f\right\|_{L_{q,\gamma}} = 0$ for all $f \in L_{p,\gamma}(R_+^n)$.

Therefore $\frac{1}{p} = \frac{1}{q} + \frac{\alpha}{n+\gamma}$.

Now, let $f \in L_{1,\gamma}\left(R_+^n\right)$ and inequality (6.10) hold. We have

$$\left\|I_\gamma^\alpha f_t\right\|_{WL_{q,\gamma}} = t^{-\alpha-\frac{n+\gamma}{q}} \left\|I_\gamma^\alpha f\right\|_{WL_{q,\gamma}}.$$

By inequality (6.10)

$$\left\|I_\gamma^\alpha f\right\|_{WL_{q,\gamma}} \leq C_q t^{\alpha+\frac{n+\gamma}{q}-(n+\gamma)} \|f\|_{L_{1,\gamma}}.$$

If $1 > \frac{1}{q} + \frac{\alpha}{n+\gamma}$, then in the case $t \to 0$ we have $\left\|I_\gamma^\alpha f\right\|_{WL_{q,\gamma}} = 0$ for all $f \in L_{1,\gamma}(R_+^n)$.

Similarly, if $1 < \frac{1}{q} + \frac{\alpha}{n+\gamma}$, then for $t \to \infty$ we obtain $\left\|I_\gamma^\alpha f\right\|_{WL_{q,\gamma}} = 0$ for all $f \in L_{1,\gamma}(R_+^n)$.

Therefore $1 = \frac{1}{q} + \frac{\alpha}{n+\gamma}$. ■

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