Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 8, NUMBER 4 (2005) ISSN 1311-0454

FRACTIONAL EXTENSIONS OF JACOBI POLYNOMIALS AND GAUSS HYPERGEOMETRIC FUNCTION

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Dedicated to Professor H.M. Srivastava,
on the occasion of his 65th Birth Anniversary

Abstract

This paper refers to a fractional order generalization of the classical Jacobi polynomials. Rodrigues' type representation formula of fractional order is considered. By means of the Riemann–Liouville operator of fractional calculus fractional Jacobi functions are defined, some of their properties are given and compared with the corresponding properties of the classical Jacobi polynomials. These functions appear as a special case of a fractional Gauss function, defined as a solution of the fractional generalization of the Gauss hypergeometric equation.

2000 Mathematics Subject Classification: 26A33, 33C45
Key Words and Phrases: Riemann–Liouville fractional differentiation and integration operators, Jacobi polynomials, Rodrigues' representation, fractional Jacobi functions, Gauss hypergeometric differential equation, fractional Gauss functions

1. Introduction

The fractional calculus becomes one of the most intensively developing areas of mathematical analysis. Its fields of application range from biology

[1] Partially supported by Project MM 1305 - National Science Fund, Bulgarian Ministry of Educ. Sci.

through physics and electrochemistry to economics, probability theory and statistics. On behalf of the nature of their definition, the fractional derivatives provide an excellent instrument for modeling of memory and hereditary properties of various materials and processes. For example, the half-order derivatives and integrals prove to be more useful for the formulation of certain electrochemical problems than the classical methods [3]. Fractional differentiation and integration operators are also used for extensions of the diffusion and wave equations [7] and, recently, of the temperature field problem in oil strata [1].

Generalizing Rodrigues' formula for the classical Jacobi polynomials by means of the Riemann–Liouville fractional differentiation operator, we define the so-called fractional Jacobi functions. We also show that these functions appear as a special case of a solution of the fractional Gauss differential equation, obtained by a modified power series method.

Definition 1. Let $f(t)$ be piecewise continuous on $(0,\infty)$ and integrable on any finite subinterval of $[0,\infty)$. Let $\nu > 0$ and $m \in \mathbb{N}$ such that $m-1 \leq \mu < m$.

(i) The Riemann–Liouville fractional integral of $f(t)$ of order ν is defined by

$$J^{\nu} f(t) \equiv \frac{1}{\Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1} f(\tau) d\tau.$$

(ii) The Riemann–Liouville fractional derivative of $f(t)$ of order μ is defined by

$$D^{\mu} f(t) \equiv D^m \left[J^{m-\mu} f(t) \right].$$

The main properties of the Riemann–Liouville operators for fractional integration and differentiation are described in [4], [5] and [6].

For our later considerations we need just to mention that if $\mu \geq 0$, $t > 0$ and $\alpha > -1$, then the fractional derivative of the power function t^{α} is given [4] by

$$D^{\mu} t^{\alpha} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\mu+1)} t^{\alpha-\mu}. \tag{1}$$

Using the same idea, it can be shown ([4], [5]) that formula (1) holds for negative values of μ as well. In this case, D^{μ} is to be considered as $J^{-\mu}$.

Also, we essentially use the Leibniz rule for fractional differentiation [5, pp. 91-97], that for a continuous on $[0,t]$ function $f(\tau)$ and continuously differentiable on the same interval function $\varphi(\tau)$ takes the form

$$D^{\mu}\left[\varphi(t)f(t)\right]=\sum_{k=0}^{\infty}\binom{\mu}{k}\varphi^{(k)}(t)D^{\mu-k}f(t). \tag{2}$$

Consider also the Gauss hypergeometric differential equation

$$x(1-x)y''+[c-(a+b+1)x]y'-aby=0 \tag{3}$$

which has as a solution the Gauss hypergeometric function defined as

$${}_2F_1(a,b;c;x)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{x^k}{k!}, \tag{4}$$

where the power series converges for $|x|<1$ ([8, p. 63], [10, p. 283]) and $(x)_n$ is the Pochhammer symbol

$$(x)_n\equiv\frac{\Gamma(x+n)}{\Gamma(x)}=x(x+1)\dots(x+n-1).$$

2. Fractional Jacobi functions

The classical Jacobi polynomials are usually defined by means of Rodrigues' formula

$$P_n^{(\alpha,\beta)}(x)\equiv(-2)^n(n!)^{-1}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right], \tag{5}$$

where $\alpha>-1$, $\beta>-1$ [2]. Their basic properties ([2], [8]) are given in Table 1. We generalize the Jacobi polynomials by setting in (5) the Riemann–Liouville fractional derivative D^{ν}.

Definition 2. The fractional Jacobi functions are defined by the formula

$$P_{\nu}^{\alpha,\beta}(t)\equiv(-2)^{-\nu}\Gamma(\nu+1)^{-1}(1-t)^{-\alpha}(1+t)^{-\beta}D^{\nu}\left[(1-t)^{\nu+\alpha}(1+t)^{\nu+\beta}\right], \tag{6}$$

where $\nu>0$, $\alpha>-1$, $\beta>-1$.

Taking into account that the binomial coefficients with real arguments are defined [6] as

$$\binom{\alpha}{\beta}\equiv\frac{\Gamma(1+\alpha)}{\Gamma(1+\beta)\Gamma(1+\alpha-\beta)}, \tag{7}$$

it is possible to derive the following properties of the fractional Jacobi functions, similar to the properties of the classical Jacobi polynomials.

Theorem 3. *For the fractional Jacobi functions the following representation holds:*

$$P_\nu^{(\alpha,\beta)}(t) = 2^{-\nu} \sum_{k=0}^{\infty} \binom{\nu+\alpha}{\nu-k} \binom{\nu+\beta}{k} (t-1)^k (t+1)^{\nu-k}. \tag{8}$$

P r o o f. Since $(1+\tau)^{\nu+\beta}$ is continuously differentiable on $[0,t]$, the Leibniz rule (2) applied to the fractional derivative in (6) yields

$$P_\nu^{(\alpha,\beta)}(t) = \frac{(-2)^{-\nu}}{\Gamma(\nu+1)} (1-t)^{-\alpha}(1+t)^{-\beta} \sum_{k=0}^{\infty} \binom{\nu}{k} \left\{ D^k \left[(1+t)^{\nu+\beta} \right] \right\} \times \left\{ D^{\nu-k} \left[(1-t)^{\nu+\alpha} \right] \right\}.$$

Then from the generalized binomial theorem it follows

$$D^{\nu-k}\left[(1-t)^{\nu+\alpha}\right] = \sum_{r=0}^{\infty} \binom{\nu+\alpha}{r} (-1)^{\nu+\alpha-r} D^{\nu-k}\left[t^{\nu+\alpha-r}\right].$$

Further, formulas (1) and (7) yield

$$\begin{aligned} P_\nu^{(\alpha,\beta)}(t) &= 2^{-\nu}(t-1)^{-\alpha} \sum_{k=0}^{\infty} \left\{ \binom{\nu+\beta}{k} (t+1)^{\nu-k} \binom{\nu+\alpha}{\nu-k} \times \sum_{r=0}^{\infty} \binom{\alpha+k}{r} t^{\alpha+k-r}(-1)^r \right\} \\ &= 2^{-\nu}(t-1)^{-\alpha} \sum_{k=0}^{\infty} \binom{\nu+\beta}{k} \binom{\nu+\alpha}{\nu-k} (t+1)^{\nu-k}(t-1)^{\alpha+k}, \end{aligned}$$

that proves the desired result. ∎

Applying (4) and (7) it is possible to prove the following statement.

Theorem 4. *The fractional Jacobi functions can be represented as*

$$P_\nu^{(\alpha,\beta)}(t) = \binom{\nu+\alpha}{\nu} {}_2F_1\left(-\nu, \nu+\alpha+\beta+1; \alpha+1; \frac{1-t}{2}\right)$$

$$= \frac{1}{\Gamma(1+\nu)} \sum_{k=0}^{\infty} \binom{\nu}{k} \frac{\Gamma(1+\nu+\alpha+\beta+k)}{\Gamma(1+\nu+\alpha+\beta)} \frac{\Gamma(1+\alpha+\nu)}{\Gamma(1+\alpha+k)} \left(\frac{t-1}{2}\right)^k.$$

From the fact that the Gauss hypergeometric function (4) satisfies the Gauss hypergeometric differential equation (3) it follows the validity of the following assertion.

Theorem 5. *The fractional Jacobi functions satisfy the linear homogeneous differential equation of the second order*

$$(1-t^2)y'' + [\beta - \alpha - (\alpha+\beta+2)t]y' + \nu(\nu+\alpha+\beta+1)y = 0,$$

or

$$\frac{d}{dt}\left\{(1-t)^{\alpha+1}(1+t)^{\beta+1}y'\right\} + \nu(\nu+\alpha+\beta+1)(1-t)^{\alpha}(1+t)^{\beta}y = 0.$$

Theorems 3 and 4, together with some properties [9] of the Gauss hypergeometric function (4), imply further interesting properties of the fractional Jacobi functions, namely:

Theorem 6. *For* $n-1 \le \nu < n$ $(n \in \mathbb{N})$ *the fractional Jacobi functions satisfy the following properties:*

(i) $\lim_{\nu\to n} P_\nu^{(\alpha,\beta)}(t) = P_n^{(\alpha,\beta)}(t);$

(ii) $P_\nu^{(\alpha,\beta)}(-t) = (-1)^\nu P_\nu^{(\beta,\alpha)}(t);$

(iii) $P_\nu^{(\alpha,\beta)}(1) = \binom{\nu+\alpha}{\nu};$

(iv) $P_\nu^{(\alpha,\beta)}(-1) = \binom{\nu+\beta}{\nu};$

(v) $\frac{d}{dt}P_\nu^{(\alpha,\beta)}(t) = \frac{1}{2}(\nu+\alpha+\beta+1)P_{\nu-1}^{(\alpha+1,\beta+1)}(t).$

To compare the classical Jacobi polynomials and the fractional Jacobi functions, a summary of their properties is provided in Table 1. If ν is approaching to a natural number, the fractional Jacobi functions become the classical Jacobi polynomials and their properties remain unchanged. Therefore, the fractional Jacobi functions may be considered as fractional indices generalizations of the classical Jacobi polynomials.

3. Fractional Gauss functions

In this section we generalize the Gauss hypergeometric function (4) by solving the fractional generalization of the differential equation (3).

Definition 7. The linear homogeneous fractional differential equation

$$t^{\mu}(1-t^{\mu})D^{2\mu}y(t)+[c-(a+b+1)t^{\mu}]\,D^{\mu}y(t)-aby=0,\ 0<\mu\leq 1 \quad (9)$$

is called the fractional Gauss hypergeometric equation.

Definition 8. The fractional Gauss function is defined as the series

$$ {}^{\mu}_{2}F_1(a,b;c;t)=y_0t^{\rho}\sum_{k=0}^{\infty}\prod_{j=0}^{k}\frac{g_j(\rho)}{f_{j+1}(\rho)}t^{k\mu},\ 0<\mu\leq 1, \quad (10)$$

where

$$f_k(\rho)\equiv\frac{\Gamma(1+\rho+k\mu)}{\Gamma(1+\rho+(k-2)\mu)}+c\frac{\Gamma(1+\rho+k\mu)}{\Gamma(1+\rho+(k-1)\mu)}, \quad (11)$$

$$g_k(\rho)\equiv\frac{\Gamma(1+\rho+k\mu)}{\Gamma(1+\rho+(k-2)\mu)}+(a+b+1)\frac{\Gamma(1+\rho+k\mu)}{\Gamma(1+\rho+(k-1)\mu)}+ab, \quad (12)$$

and $\rho>-1$ satisfies the equation

$$f_0(\rho)=\frac{\Gamma(1+\rho)}{\Gamma(1+\rho-2\mu)}+c\frac{\Gamma(1+\rho)}{\Gamma(1+\rho-\mu)}=0. \quad (13)$$

By means of a modified power series method we establish the validity of the following assertion.

Theorem 9. *The fractional Gauss function (10) is a solution of the fractional Gauss hypergeometric equation (9).*

The relation between the fractional Jacobi functions and the fractional Gauss functions is a consequence of Theorem 4, and is given by the formula

$$P_{\nu}^{(\alpha,\beta)}(t)=\binom{\nu+\alpha}{\nu}{}^{1}_{2}F_1\left(-\nu,\nu+\alpha+\beta+1;\alpha+1;\frac{1-t}{2}\right).$$

Property	Jacobi polynomials	Fractional Jacobi functions
Definition	$P_n^{\alpha,\beta}(t) = (-2)^{-n}(n!)^{-1}(1-t)^{-\alpha}(1+t)^{-\beta} \times \frac{d^n}{dt^n}\left[(1-t)^{n+\alpha}(1+t)^{n+\beta}\right]$	$P_\nu^{\alpha,\beta}(t) = (-2)^{-\nu}\Gamma(\nu+1)^{-1}(1-t)^{-\alpha}(1+t)^{-\beta} \times D^\nu\left[(1-t)^{\nu+\alpha}(1+t)^{\nu+\beta}\right]$
Explicitly	$P_n^{(\alpha,\beta)}(t) = 2^{-n}\sum_{k=0}^{\infty}\binom{n+\alpha}{n-k}\binom{n+\beta}{k}(t-1)^k(t+1)^{n-k}$	$P_\nu^{(\alpha,\beta)}(t) = 2^{-\nu}\sum_{k=0}^{\infty}\binom{\nu+\alpha}{\nu-k}\binom{\nu+\beta}{k}(t-1)^k(t+1)^{\nu-k}$
Explicitly	$P_n^{(\alpha,\beta)}(t) = \binom{n+\alpha}{n}{}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-t}{2}\right)$	$P_\nu^{(\alpha,\beta)}(t) = \binom{\nu+\alpha}{\nu}{}_2F_1\left(-\nu, \nu+\alpha+\beta+1; \alpha+1; \frac{1-t}{2}\right)$
Value at 1	$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$	$P_\nu^{(\alpha,\beta)}(1) = \binom{\nu+\alpha}{\nu}$
Value at -1	$P_n^{(\alpha,\beta)}(1) = \binom{n+\beta}{n}$	$P_\nu^{(\alpha,\beta)}(1) = \binom{\nu+\beta}{\nu}$
Equation	$(1-t^2)y'' + [\beta-\alpha-(\alpha+\beta+2)t]y' + n(n+\alpha+\beta+1)y = 0$	$(1-t^2)y'' + [\beta-\alpha-(\alpha+\beta+2)t]y' + \nu(\nu+\alpha+\beta+1)y = 0$
Derivative	$\frac{d}{dt}P_n^{(\alpha,\beta)}(t) = \frac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(t)$	$\frac{d}{dt}P_\nu^{(\alpha,\beta)}(t) = \frac{1}{2}(\nu+\alpha+\beta+1)P_{\nu-1}^{(\alpha+1,\beta+1)}(t)$

Table 1: Properties of the classical Jacobi polynomials and the fractional Jacobi functions ($n \in \mathbb{N}$, $n-1 \le \nu < n$).

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Received: November 10, 2005