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AN ABSTRACT SECOND KIND FREDHOLM INTEGRAL EQUATION WITH DEGENERATED KERNEL

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Abstract

The paper presents an abstract linear second kind Fredholm integral equation with degenerated kernel defined by means of the Bittner operational calculus. Fredholm alternative for mutually conjugated integral equations is also shown here. Some examples of solutions of the considered integral equation in various operational calculus models are also given.

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1. Operational calculus

In accordance with the notation used e.g. in Bittner [2], the *Bittner Operational Calculus* is a system

$$\mathrm{CO}(L^0, L^1, S, T_q, s_q, Q), \tag{1}$$

where L^0 and L^1 are linear spaces over the same field $\varGamma$, the linear operation $S : L^1 \longrightarrow L^0$ (written as $S \in L(L^1, L^0)$), called the (abstract) *derivative*, is a surjection. Moreover, Q is a nonempty set of indices q for the operations $T_q \in L(L^0, L^1), s_q \in L(L^1, L^1)$ called *integrals* and *limit conditions*, respectively, and such that $ST_q f = f, f \in L^0$, $s_q x = x - T_q Sx, x \in L^1$. The kernel of S, i.e. $\operatorname{Ker} S := \{c \in L^1 : Sc = 0\}$, is called the *set of constants* for the derivative S.

Limit conditions $s_q, q \in Q$ are projections from L^1 onto the subspace $\operatorname{Ker} S$. Hence

$$s_q c = c, \quad q \in Q, \quad c \in \operatorname{Ker} S. \tag{2}$$

Example 1. Let $L^0 := \mathbb{R}, L^1 := \mathbb{R}^n$ with common number and vector operations. Moreover, let

$$Q := \left\{ q = [q_1, q_2, \ldots, q_n] \in \mathbb{R}^n : \sum_{i=1}^{n} q_i = 1 \right\}$$

and $(\boldsymbol{x}, \boldsymbol{y}) := \sum\limits_{i=1}^{n} x_i y_i,\ \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. It is easy to see that the operations

$$S\boldsymbol{x} := (\boldsymbol{a}, \boldsymbol{x}), \quad \boldsymbol{x} \in L^1,$$

where $\boldsymbol{a} \in L^1, a_i \neq 0, i \in \overline{1, n} := \{1, 2, \ldots, n\}$ is given, and

$$T_q f := \left[\frac{q_1}{a_1} f, \frac{q_2}{a_2} f, \ldots, \frac{q_n}{a_n} f \right],$$

$$s_q \boldsymbol{x} := \left[x_1 - \frac{q_1(\boldsymbol{a}, \boldsymbol{x})}{a_1}, x_2 - \frac{q_2(\boldsymbol{a}, \boldsymbol{x})}{a_2}, \ldots, x_n - \frac{q_n(\boldsymbol{a}, \boldsymbol{x})}{a_n} \right],$$

where $q \in Q, f \in L^0, \boldsymbol{x} \in L^1$, form an operational calculus. We also have $L^1 \not\subset L^0$ and $\operatorname{card} Q = \mathfrak{c}$.

Example 2. Similarly, if

$$L^0 := C^0([a, b], \mathbb{R}), \ \ L^1 := \{x = x(t) \in L^0 : \ x'(a) \ \text{ exists}\},$$

where $Q := \{a\}, [a, b] \subset \mathbb{R}$, then it is not difficult to see that the operations

$$Sx := \begin{cases} \dfrac{x(t) - x(a)}{t - a} & \text{for} \quad t > a \\ \\ x'(a) & \text{for} \quad t = a \end{cases}, \quad x = x(t) \in L^1,$$

$$T_a f := (t - a) f(t), \quad f = f(t) \in L^0,$$

$$s_a x := x(a), \quad x = x(t) \in L^1$$

form an operational calculus (cf. Ex. 5.4 in Przeworska-Rolewicz [7]). Here we have $L^1 \subset L^0$ and $\operatorname{card} Q = 1$.

The assumptions that $L^1 \subset L^0$ and Q has more than one element will be used throughout the paper.

The mapping $I_{q_1}^{q_2} \in L(L^0, \operatorname{Ker} S)$ defined by the formula

$$I_{q_1}^{q_2} f := (T_{q_1} - T_{q_2}) f, \quad q_1, q_2 \in Q, \ f \in L^0$$

is called the *operation of definite integration.*

It is easy to verify that

$$I_{q_1}^{q_2} f = s_{q_2} T_{q_1} f, \quad f \in L^0. \tag{3}$$

Therefore

$$I_{q_1}^{q_2}(L^0) \subset \operatorname{Ker} S. \tag{4}$$

If L^0 is an algebra (a linear ring) and L^1 is its subalgebra, then we say that the derivative S satisfies the *Leibniz condition* if

$$S(x \cdot y) = Sx \cdot y + x \cdot Sy, \quad x, y \in L^1 \tag{5}$$

and the limit condition $s_q, q \in Q$ is *multiplicative* if

$$s_q(x \cdot y) = s_q x \cdot s_q y, \quad x, y \in L^1. \tag{6}$$

2. Abstract Fredholm integral equation

Consider the operational calculus (1) in which

– L^0 is a commutative algebra with unity $e \in L^1$, and L^1 is its subalgebra;

– the derivative S satisfies the Leibniz condition (5);

– the limit conditions s_{q_1}, s_{q_2}, where $q_1, q_2 \in Q$, satisfy the multiplication condition (6).

It is not difficult to see that

$$I_{q_1}^{q_2}(c \cdot f) = c I_{q_1}^{q_2} f, \quad q_1, q_2 \in Q, \ c \in \operatorname{Ker} S, \ f \in L^0. \tag{7}$$

We also have

$$(\ c, d \in \operatorname{Ker} S\ ) \Longrightarrow (\ cd \in \operatorname{Ker} S\ ).$$

If $\operatorname{Inv}(\operatorname{Ker} S)$ denotes the set of constants $c \in \operatorname{Ker} S$, which are invertible elements in the algebra $\operatorname{Ker} S$, then

$$(\ c \in \operatorname{Inv}(\operatorname{Ker} S)\ ) \Longrightarrow (\ c^{-1} \in \operatorname{Ker} S\ ).$$

The determinant of a matrix $\boldsymbol{C} = [c_{ij}]_{n \times n}$, where $c_{ij} \in \operatorname{Ker} S, \ i, j \in \overline{1, n}, n \in \mathbb{N}$, is defined similarly to the numerical determinant. Namely, it is an element of the algebra $\operatorname{Ker} S$ defined by the formula

$$\det \boldsymbol{C} := \sum_p (-1)^{I_p} c_{1j_1} c_{2j_2} \cdots c_{nj_n},$$

where the summation is extended to all permutations $p = (j_1, j_2, \ldots, j_n)$ of numbers $1, 2, \ldots, n$, whereas I_p denotes the number of inversions in the permutation p. The rules of computing $\det \boldsymbol{C}$ are the same as for a numerical determinant.

In particular, if

$$\boldsymbol{D} = [d_{ij}]_{n\times n} \quad \text{and} \quad W = \det \boldsymbol{D}, \tag{8}$$

where

$$d_{ij} := \delta_{ij} e - \lambda I_{q_1}^{q_2} \alpha_j \beta_i,$$

δ_{ij} is the Kronecker symbol and

$$\lambda \in \operatorname{Ker} S, \quad \alpha_j, \beta_i \in L^0, \quad i, j \in \overline{1, n},$$

then $W \in \operatorname{Ker} S$. If $W \in \operatorname{Inv}(\operatorname{Ker} S)$, then $W^{-1} \in \operatorname{Ker} S$. Moreover,

$$\operatorname{Ker} S \ni c_i := W^{-1} \sum_{j=1}^{n} W_{ji} b_j, \quad i \in \overline{1, n}, \tag{9}$$

where $b_j \in \operatorname{Ker} S, j \in \overline{1, n}$ and W_{ji} are algebraic complements of elements $d_{ji}, i, j \in \overline{1, n}$ of $\boldsymbol{D}$.

The abstract linear integral equation

$$x - \lambda(\alpha_1 I_{q_1}^{q_2} \beta_1 x + \alpha_2 I_{q_1}^{q_2} \beta_2 x + \cdots + \alpha_n I_{q_1}^{q_2} \beta_n x) = f, \tag{10}$$

where

$$\lambda \in \operatorname{Inv}(\operatorname{Ker} S), \quad f, \alpha_i, \beta_i \in L^0, \quad i \in \overline{1, n}$$

are given and $x \in L^0$, will be called the *Fredholm integral equation of the second kind with degenerated kernel.*

The abstract integral equation

$$y - \lambda(\beta_1 I_{q_1}^{q_2} \alpha_1 y + \beta_2 I_{q_1}^{q_2} \alpha_2 y + \cdots + \beta_n I_{q_1}^{q_2} \alpha_n y) = g, \tag{11}$$

where

$$\lambda \in \operatorname{Inv}(\operatorname{Ker} S), \quad g, \alpha_i, \beta_i \in L^0, \quad i \in \overline{1, n}$$

are given and $y \in L^0$, will be called the *conjugate equation* to (10).

The element of the space $L_n^0 := \overset{n}{\underset{i=1}{\oplus}} L^0$ given in the form

$$c_1 \boldsymbol{x}_1 + c_2 \boldsymbol{x}_2 + \cdots + c_m \boldsymbol{x}_m,$$

where $c_1, c_2, \ldots, c_m \in \operatorname{Ker} S$, $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m \in L_n^0$ and

$$c_i \boldsymbol{x}_i := \begin{bmatrix} c_i x_{1i} \\ c_i x_{2i} \\ \vdots \\ c_i x_{ni} \end{bmatrix} \in L_n^0, \quad i \in \overline{1, m},$$

will be called the S-*linear combination* of $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m \in L_n^0$.

The vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m \in L_n^0$ will be called S-*linearly independent* if the condition

$$c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 + \cdots + c_m\boldsymbol{x}_m = \mathbf{0}, \quad c_i \in \operatorname{Ker} S, \quad i \in \overline{1,m}$$

implies

$$c_1 = c_2 = \cdots = c_m = 0.$$

The vectors which are not S-linearly independent will be called S-*linearly dependent* (cf. Przeworska-Rolewicz [6]).

If $\operatorname{Ker} S \simeq \mathbb{R}$, then the S-linear independence of vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m \in L_n^0$ means their linear independence in L_n^0 over $\mathbb{R}$.

The elements $x, y \in L^0$ will be called *orthogonal* if

$$I_{q_1}^{q_2} xy = 0.$$

Theorem 1 (Fredholm alternative). *Let*

$$\alpha_1, \alpha_2, \ldots, \alpha_n \in L^0, \quad \beta_1, \beta_2, \ldots, \beta_n \in L^0$$

state two systems of S-linearly independent elements, respectively.

The integral equations (10), (11) either have the unique solutions x and y for any f and g (in particular, for $f = 0$ and $g = 0$, $x = 0$ and $y = 0$ are the only solutions), or the homogeneous equations

$$x - \lambda(\alpha_1 I_{q_1}^{q_2}\beta_1 x + \alpha_2 I_{q_1}^{q_2}\beta_2 x + \cdots + \alpha_n I_{q_1}^{q_2}\beta_n x) = 0, \tag{12}$$

$$y - \lambda(\beta_1 I_{q_1}^{q_2}\alpha_1 y + \beta_2 I_{q_1}^{q_2}\alpha_2 y + \cdots + \beta_n I_{q_1}^{q_2}\alpha_n y) = 0 \tag{13}$$

corresponding them, have an infinite number of solutions (dependent on the same number of parameters).

If the homogeneous equations (12) and (13) have non-zero solutions, then a fact that the element f is orthogonal to all solutions y of the homogeneous conjugated integral equation (13) is the necessary and sufficient condition of having solution by the non-homogeneous integral equation (10). Analogously, the non-homogeneous conjugated integral (11) has a solution when the element g is orthogonal to all solutions x of the homogeneous integral equation (12).

Proof. Let

$$c_i := I_{q_1}^{q_2}\beta_i x, \quad i \in \overline{1,n}. \tag{14}$$

Hence and from (4) we get

$$c_i \in \operatorname{Ker} S, \quad i \in \overline{1,n}. \tag{15}$$

Now (10) becomes as follows

$$x-\lambda(\alpha_1c_1+\alpha_2c_2+\cdots+\alpha_nc_n)=f. \tag{16}$$

Multiplying (16) by $\beta_1,\beta_2,\ldots,\beta_n$, respectively, we obtain

$$\begin{array}{rcl} \beta_1x-\lambda(c_1\alpha_1\beta_1+c_2\alpha_2\beta_1+\cdots+c_n\alpha_n\beta_1) & = & \beta_1f \\ \beta_2x-\lambda(c_1\alpha_1\beta_2+c_2\alpha_2\beta_2+\cdots+c_n\alpha_n\beta_2) & = & \beta_2f \\ \multicolumn{3}{c}{\dotfill} \\ \beta_nx-\lambda(c_1\alpha_1\beta_n+c_2\alpha_2\beta_n+\cdots+c_n\alpha_n\beta_n) & = & \beta_nf \end{array}. \tag{17}$$

As $\lambda\in\operatorname{Ker}S$ and (14) holds, on the basis of (7) from (17) we get

$$\begin{array}{rcl} I_{q1}^{q2}\beta_1x-\lambda(c_1I_{q1}^{q2}\alpha_1\beta_1+c_2I_{q1}^{q2}\alpha_2\beta_1+\cdots+c_nI_{q1}^{q2}\alpha_n\beta_1) & = & I_{q1}^{q2}\beta_1f \\ I_{q1}^{q2}\beta_2x-\lambda(c_1I_{q1}^{q2}\alpha_1\beta_2+c_2I_{q1}^{q2}\alpha_2\beta_2+\cdots+c_nI_{q1}^{q2}\alpha_n\beta_2) & = & I_{q1}^{q2}\beta_2f \\ \multicolumn{3}{c}{\dotfill} \\ I_{q1}^{q2}\beta_nx-\lambda(c_1I_{q1}^{q2}\alpha_1\beta_n+c_2I_{q1}^{q2}\alpha_2\beta_n+\cdots+c_nI_{q1}^{q2}\alpha_n\beta_n) & = & I_{q1}^{q2}\beta_nf \end{array}.$$

Thus, using (14) and

$$a_{ij}:=I_{q1}^{q2}\alpha_j\beta_i,\quad b_i:=I_{q1}^{q2}\beta_if,\quad i,j\in\overline{1,n}, \tag{18}$$

we obtain the system of n equations with n unknowns $c_1,c_2,\ldots,c_n$

$$\begin{array}{rcl} (e-\lambda a_{11})c_1-\lambda a_{12}c_2-\cdots-\lambda a_{1n}c_n & = & b_1 \\ -\lambda a_{21}c_1+(e-\lambda a_{22})c_2-\cdots-\lambda a_{2n}c_n & = & b_2 \\ \multicolumn{3}{c}{\dotfill} \\ -\lambda a_{n1}c_1-\lambda a_{n2}c_2-\cdots+(e-\lambda a_{nn})c_n & = & b_n \end{array}. \tag{19}$$

Considerations concerning solving the systems of linear equations with real or complex coefficients carry over the systems with coefficients from real or complex commutative linear ring (algebra), here from $\operatorname{Ker}S$ (Bourbaki [4], Przeworska-Rolewicz [6], cf. Wysocki [8]).

Therefore, if the main determinant W of (19), i.e. the determinant (8) is an invertible element, (19) is a Cramer system and its only solution is

$$c_i=W^{-1}W_i,\quad i\in\overline{1,n}, \tag{20}$$

where

$$W_i:=\sum_{j=1}^{n}W_{ji}b_j,\quad i\in\overline{1,n}. \tag{21}$$

Hence and from (9) it follows, that $c_i\in\operatorname{Ker}S, i\in\overline{1,n}$. From (16) and (20) we have

$$x=f+\lambda\sum_{i=1}^{n}\alpha_ic_i=f+\lambda\sum_{i=1}^{n}\alpha_iW^{-1}W_i. \tag{22}$$

Taking (21) into account, finally we obtain

$$x = f + \lambda W^{-1} \sum_{i,j=1}^{n} W_{ji}\alpha_i b_j. \tag{23}$$

Now we are going to verify if the element x of the form (22) states the solution of (10). Substituting (22) to the left side of (10)

$$\begin{array}{rcl} L & := & x - \lambda(\alpha_1 I_{q_1}^{q_2}\beta_1 x + \alpha_2 I_{q_1}^{q_2}\beta_2 x + \cdots + \alpha_n I_{q_1}^{q_2}\beta_n x) \\ & = & f + \lambda \sum\limits_{i=1}^{n} \alpha_i c_i - \lambda\alpha_1 I_{q_1}^{q_2}\beta_1(f + \lambda \sum\limits_{i=1}^{n} \alpha_i c_i) \\ & - & \lambda\alpha_2 I_{q_1}^{q_2}\beta_2(f + \lambda \sum\limits_{i=1}^{n} \alpha_i c_i) - \cdots - \lambda\alpha_n I_{q_1}^{q_2}\beta_n(f + \lambda \sum\limits_{i=1}^{n} \alpha_i c_i) \end{array}$$

yields. Using (7) in the obtained expression we have

$$\begin{array}{rcl} L & = & f + \lambda \sum\limits_{i=1}^{n} \alpha_i c_i - \lambda\alpha_1 I_{q_1}^{q_2}\beta_1 f - \lambda^2\alpha_1 \sum\limits_{i=1}^{n} (I_{q_1}^{q_2}\beta_1\alpha_i)c_i \\ & - & \lambda\alpha_2 I_{q_1}^{q_2}\beta_2 f - \lambda^2\alpha_2 \sum\limits_{i=1}^{n} (I_{q_1}^{q_2}\beta_2\alpha_i)c_i - \cdots \\ & - & \lambda\alpha_n I_{q_1}^{q_2}\beta_n f - \lambda^2\alpha_n \sum\limits_{i=1}^{n} (I_{q_1}^{q_2}\beta_n\alpha_i)c_i. \end{array} \tag{24}$$

Combining the notations (18) in (24) we see that

$$\begin{array}{rcl} L & = & f + \lambda \sum\limits_{i=1}^{n} \alpha_i c_i - \lambda\alpha_1 b_1 - \lambda^2\alpha_1 \sum\limits_{i=1}^{n} a_{1i}c_i \\ & - & \lambda\alpha_2 b_2 - \lambda^2\alpha_2 \sum\limits_{i=1}^{n} a_{2i}c_i - \cdots - \lambda\alpha_n b_n - \lambda^2\alpha_n \sum\limits_{i=1}^{n} a_{ni}c_i \\ & = & f + \lambda\alpha_1(c_1 - \lambda a_{11}c_1 - \lambda a_{12}c_2 - \cdots - \lambda a_{1n}c_n) - \lambda\alpha_1 b_1 \\ & + & \lambda\alpha_2(-\lambda a_{21}c_1 + c_2 - \lambda a_{22}c_2 - \cdots - \lambda a_{2n}c_n) - \lambda\alpha_2 b_2 + \cdots \\ & + & \lambda\alpha_n(-\lambda a_{n1}c_1 - \lambda a_{n2}c_2 - \cdots + c_n - \lambda a_{nn}c_n) - \lambda\alpha_n b_n. \end{array} \tag{25}$$

Since the constants $c_1, c_2, \ldots, c_n$, determined by (20) state the solutions of (19), the expressions in parentheses in (25) are equal to $b_1, b_2, \ldots, b_n$, respectively. Therefore, from (25) $L = f$.

We have proved, that in case when $W \in \mathrm{Inv}(\mathrm{Ker}\, S)$ the only solution of non-homogeneous integral equation (10) has been expressed by (23).

Applying the following notations

$$c_i^* := I_{q_1}^{q_2}\alpha_i y, \quad i \in \overline{1, n},$$

we rewrite (11) to the form of

$$y - \lambda(\beta_1 c_1^* + \beta_2 c_2^* + \cdots + \beta_n c_n^*) = g. \tag{26}$$

Multiplying the last equality by $\alpha_1, \alpha_2, \ldots, \alpha_n$, respectively and acting member by member on the given equation by the operation of definite integration $I_{q_1}^{q_2}$, we come to a system

$$\begin{array}{rcl} (e-\lambda a_{11})c_1^* - \lambda a_{21}c_2^* - \cdots - \lambda a_{n1}c_n^* & = & b_1^* \\ -\lambda a_{12}c_1^* + (e-\lambda a_{22})c_2^* - \cdots - \lambda a_{n2}c_n^* & = & b_2^* \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & \\ -\lambda a_{1n}c_1^* - \lambda a_{2n}c_2^* - \cdots + (e-\lambda a_{nn})c_n^* & = & b_n^* \end{array}, \tag{27}$$

where

$$a_{ij} := I_{q_1}^{q_2}\alpha_j\beta_i, \quad b_i^* := I_{q_1}^{q_2}\alpha_i g, \quad i \in \overline{1,n}.$$

Let $\boldsymbol{D}^*$ be the matrix of that system. It is easy to notice, that $\boldsymbol{D}^* = \boldsymbol{D}^t$, where '$t$' denotes the transposition of a matrix. Taking $W^* = \det \boldsymbol{D}^*$, we obtain $W^* = W$.

Therefore, if (10) has only one solution, then (11) has also a unique solution.

If $W \in \operatorname{Inv}(\operatorname{Ker} S)$, the homogeneous integral equations (12) and (13) have only zero solutions, i.e. $x = 0$ i $y = 0$. When $W \notin \operatorname{Inv}(\operatorname{Ker} S)$, the equations have an infinite number of solutions dependent on the same number of parameters.

From (22) and (26) it follows, that (10) and (13) have solutions

$$x = f + \lambda \sum_{i=1}^{n} \alpha_i c_i, \tag{28}$$

$$y = \lambda \sum_{i=1}^{n} \beta_i c_i^*, \tag{29}$$

respectively.

Let $W \notin \operatorname{Inv}(\operatorname{Ker} S)$. Then the homogeneous system

$$\sum_{j=1}^{n} (\delta_{ji} e - \lambda a_{ji}) c_j^* = 0, \quad i \in \overline{1,n}, \tag{30}$$

which corresponds to (27), has the infinite number of non-zero solutions. As $\beta_1, \beta_2, \ldots, \beta_n$ are by assumption, S-linearly independent, the homogeneous conjugated integral equation (13) corresponding to (30), has also an infinite number of non-zero solutions of (29) form. We prove that in this case the system (19) has a solution

$$\boldsymbol{c} := \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

(whereas the solution of the non-homogeneous integral equation (10), which corresponds to it, is (28)) if and only if for any solution

$$\boldsymbol{c}^* := \begin{bmatrix} c_1^* \\ c_2^* \\ \vdots \\ c_n^* \end{bmatrix}$$

of the homogeneous conjugated system (30)

$$(\boldsymbol{b}, \boldsymbol{c}^*) := \sum_{i=1}^{n} b_i c_i^* = 0, \tag{31}$$

holds, where

$$\boldsymbol{b} := \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Notice, that we can rewrite (19) in the form of

$$\sum_{j=1}^{n} c_j \boldsymbol{d}_j = \boldsymbol{b}, \tag{32}$$

where

$$\boldsymbol{d}_1 := \begin{bmatrix} e - \lambda a_{11} \\ -\lambda a_{21} \\ \vdots \\ -\lambda a_{n1} \end{bmatrix}, \boldsymbol{d}_2 := \begin{bmatrix} -\lambda a_{12} \\ e - \lambda a_{22} \\ \vdots \\ -\lambda a_{n2} \end{bmatrix}, \dots, \boldsymbol{d}_n := \begin{bmatrix} -\lambda a_{1n} \\ -\lambda a_{2n} \\ \vdots \\ e - \lambda a_{nn} \end{bmatrix}.$$

It follows that (19) has the solutions $c_1, c_2, \dots, c_n$ if and only if $\boldsymbol{b}$ is an S-linear combination of vectors $\boldsymbol{d}_1, \boldsymbol{d}_2, \dots, \boldsymbol{d}_n \in \bigoplus_{i=1}^{n} \operatorname{Ker} S \subset L_n^1$. The constants $c_1, c_2, \dots, c_n$ are coefficients of that combination.

The fact, that $\boldsymbol{c}^*$ is a solution of (30) can be written as

$$(\boldsymbol{c}^*, \boldsymbol{d}_j) = 0, \quad j \in \overline{1, n}. \tag{33}$$

Therefore, if $\boldsymbol{c}$ and $\boldsymbol{c}^*$ state the solutions of (19) and (30), respectively, then due to (32) and (33) we get

$$(\boldsymbol{b}, \boldsymbol{c}^*) = (\sum_{j=1}^{n} c_j \boldsymbol{d}_j, \boldsymbol{c}^*) = \sum_{j=1}^{n} c_j (\boldsymbol{c}^*, \boldsymbol{d}_j) = 0,$$

and so (31) holds.

Suppose that (31) and (33) hold.

On the basis of (33) for any $c_1, c_2, \dots, c_n \in \operatorname{Ker} S$ we have

$$c_1(\boldsymbol{c}^*, \boldsymbol{d}_1) + c_2(\boldsymbol{c}^*, \boldsymbol{d}_2) + \cdots + c_n(\boldsymbol{c}^*, \boldsymbol{d}_n) = 0,$$

and then, after some operations, we get

$$\begin{aligned} &[c_1(e-\lambda a_{11}) + c_2(-\lambda a_{12}) + \cdots + c_n(-\lambda a_{1n})]c_1^* \\ +&[c_1(-\lambda a_{21}) + c_2(e-\lambda a_{22}) + \cdots + c_n(-\lambda a_{2n})]c_2^* + \cdots \\ +&[c_1(-\lambda a_{n1}) + c_2(-\lambda a_{n2}) + \cdots + c_n(e-\lambda a_{nn})]c_n^* = 0. \end{aligned}$$

Hence and from the assumption (31) it follows that we can admit that

$$\begin{aligned} b_1 &= c_1(e-\lambda a_{11}) + c_2(-\lambda a_{12}) + \cdots + c_n(-\lambda a_{1n}), \\ b_2 &= c_1(-\lambda a_{21}) + c_2(e-\lambda a_{22}) + \cdots + c_n(-\lambda a_{2n}), \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ b_n &= c_1(-\lambda a_{n1}) + c_2(-\lambda a_{n2}) + \cdots + c_n(e-\lambda a_{nn}), \end{aligned}$$

which assures us that $\boldsymbol{b}$ is the S-linear combination (32).

Notice, that the condition (31) denotes the orthogonality of f and y. Indeed, using (29) and (7), we have

$$I_{q_1}^{q_2} fy = I_{q_1}^{q_2}(f \cdot \lambda \sum_{i=1}^{n} \beta_i c_i^*) = \lambda \sum_{i=1}^{n} I_{q_1}^{q_2}(\beta_i f) c_i^* = \lambda \sum_{i=1}^{n} b_i c_i^* = \lambda(\boldsymbol{b}, \boldsymbol{c}^*) = 0$$

if and only if $(\boldsymbol{b}, \boldsymbol{c}^*) = 0$, as $\lambda \in \operatorname{Inv}(\operatorname{Ker} S)$. It gives us the second part of the Fredholm alternative concerning the integral equations (10) and (13). The proof of that part of the theorem for (11) and (12) runs as before. ■

The following equation

$$x - \lambda \alpha I_{q_1}^{q_2} \beta x = f, \tag{34}$$

where $\lambda \in \operatorname{Inv}(\operatorname{Ker} S)$ and $f, \alpha, \beta, x \in L^0$, states the particular case of (10). If $e - \lambda I_{q_1}^{q_2} \alpha\beta \in \operatorname{Inv}(\operatorname{Ker} S)$, then from (23) it follows that

$$x = f + \lambda(e - \lambda I_{q_1}^{q_2} \alpha\beta)^{-1} \alpha I_{q_1}^{q_2} \beta f \tag{35}$$

is the only solution of (34).

3. Examples

A. Let be given a classical model of the operational calculus (Bittner [2]), in which

$$L^0 := C^0([a,b], \mathbb{R}), \quad L^1 := C^1([a,b], \mathbb{R}), \quad Q := [a,b] \subset \mathbb{R}$$

and

$$S := \frac{d}{dt}, \quad T_q := \int_q^t, \quad s_q := |_{t=q}, \quad q \in Q.$$

In that case, for $q_1 = a, q_2 = b$, the abstract integral equation (34) takes the form of

$$x(t) - \lambda\alpha(t) \int_a^b \beta(\tau)x(\tau)d\tau = f(t), \tag{36}$$

where

$$\lambda \in \mathbb{R} \simeq \operatorname{Ker} S, \quad f(t), \alpha(t), \beta(t), x(t) \in C^0([a,b], \mathbb{R}).$$

With a common multiplication of functions, for the derivative S and the limit conditions $s_q, q \in Q$ the formulas (5) and (6) take place, respectively. Therefore, from (35) we obtain the solution of (36):

$$x(t) = f(t) + \frac{\lambda\alpha(t) \int_a^b \beta(\tau)f(\tau)d\tau}{1 - \lambda \int_a^b \alpha(\tau)\beta(\tau)d\tau},$$

if only $1 - \lambda \int_a^b \alpha(\tau)\beta(\tau)d\tau \neq 0$ (cf. Piskorek [5]).

B. Let $E : L^0 \longrightarrow L^0$ and $E_{|_{L^1}} : L^1 \longrightarrow L^1$ be automorphisms of algebras L^0 and L^1, respectively.

It is a simple matter to verify that the operations

$$\begin{aligned} \overline{S}x &:= E^{-1}SEx, \quad x \in L^1, \\ \overline{T}_q f &:= E^{-1}T_q E f, \quad q \in Q, f \in L^0, \\ \overline{s}_q x &:= E^{-1}s_q E x, \quad q \in Q, x \in L^1 \end{aligned}$$

form the operational calculus

$$\overline{CO}(L^0, L^1, \overline{S}, \overline{T}_q, \overline{s}_q, Q)^1, \tag{37}$$

in which $\overline{S}$ satisfies the Leibniz condition and $\overline{s}_q$, where $q \in Q$, are multiplicative.

If we define isomorphisms

$$\varphi := E^{-1}_{|_{L^1}}, \quad \psi := E^{-1},$$

we get

$$\overline{S} = \psi S \varphi^{-1}, \quad \overline{T}_q = \varphi T_q \psi^{-1}, \quad \overline{s}_q = \varphi s_q \varphi^{-1}$$

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[1] $\widetilde{S} := ESE^{-1}, \widetilde{T}_q := ET_qE^{-1}, \widetilde{s}_q := Es_qE^{-1}$ also form the operational calculus.

and we say, that operational calculi (1) and (37) are *equivalent* (Bittner [1,2]).

Let

$$L^0 := C^0(\mathbb{R},\mathbb{R}), \quad L^1 := C^1(\mathbb{R},\mathbb{R}), \quad Q := \mathbb{R}.$$

Moreover, let $S, T_q, s_q, q \in Q$ be defined as in Ex. A.

Taking

$$Ex := x(g(t)), \quad x = x(t) \in L^0,$$

where $g = g(t) \in C^1(\mathbb{R},\mathbb{R}), g'(t) > 0, t \in \mathbb{R}$ is a given function, we obtain the model of the operational calculus in which

$$\overline{S}x = \left[\frac{dx(g(\overline{t}))}{dg} \cdot \frac{dg(\overline{t})}{d\overline{t}}\right]\Big|_{\overline{t}=g^{-1}(t)}, \quad x = x(t) \in L^1,$$

$$\overline{T}_q f = \int_q^{g^{-1}(t)} f(g(\tau))d\tau, \quad q \in Q, f = f(t) \in L^0,$$

$$\overline{s}_q x = x(g(q)), \quad q \in Q, x = x(t) \in L^1$$

(cf. Ex. 5.2.3 in Przeworska-Rolewicz [6]).

In the considering model the abstract integral equation (34) takes the form of

$$x(t) - \lambda\alpha(t)\int_a^b \beta(g(\tau))x(g(\tau))d\tau = f(t),$$

for $q_1 = a, q_2 = b$, where

$$\lambda \in \mathbb{R} \simeq \operatorname{Ker} S, \quad f(t), \alpha(t), \beta(t), x(t) \in C^0(\mathbb{R},\mathbb{R}).$$

From (35) we get its solutions

$$x(t) = f(t) + \frac{\lambda\alpha(t)\int_a^b \beta(g(\tau))f(g(\tau))d\tau}{1 - \lambda\int_a^b \alpha(g(\tau))\beta(g(\tau))d\tau},$$

if only $1 - \lambda\int_a^b \alpha(g(\tau))\beta(g(\tau))d\tau \neq 0$.

C. Let be given an operational calculus in which

$$L^0 := C^1(\mathbb{R}\times\mathbb{R},\mathbb{R}), \quad S := \frac{\partial}{\partial\xi} + v\frac{\partial}{\partial\eta}, \; v \in \mathbb{R}\setminus\{0\},$$

$$L^1 := \{x \in L^0 : \; Sx \in L^0\}, \quad T_q f := \int_{\xi_0}^{\xi} f(\tau, \eta - v(\xi - \tau))d\tau,$$

$$s_q x := x(\xi_0, \eta - v(\xi - \xi_0)), \quad q = \xi_0 \in Q := \mathbb{R},$$

where $f = f(\xi, \eta) \in L^0, x = x(\xi, \eta) \in L^1$ (see Bittner, Mieloszyk [3]).

With a common multiplication of functions, for the derivative S and the limit conditions $s_q, q \in Q$ the formulas (5), (6) hold, respectively.

Since

$$\operatorname{Ker} S = \{x(\xi, \eta) : \ x(\xi, \eta) = \varphi(\eta - v\xi), \varphi \in C^1(\mathbb{R}, \mathbb{R})\},$$

so for $q_1 = a, q_2 = b$ the equation (34) takes the form

$$x(\xi, \eta) - \lambda(\eta - v\xi)\alpha(\xi, \eta) \int_a^b \beta(\tau, \eta - v(\xi - \tau))x(\tau, \eta - v(\xi - \tau))d\tau = f(\xi, \eta),$$

where

$$\lambda(\eta - v\xi) \in \operatorname{Ker} S, \quad f(\xi, \eta), \alpha(\xi, \eta)\beta(\xi, \eta), x(\xi, \eta) \in C^1(\mathbb{R} \times \mathbb{R}, \mathbb{R}).$$

From (35) we obtain the solution of the equation

$$x(\xi, \eta) = f(\xi, \eta) + \frac{\lambda(\eta - v\xi)\alpha(\xi, \eta) \int_a^b \beta(\tau, \eta - v(\xi - \tau))f(\tau, \eta - v(\xi - \tau))d\tau}{1 - \lambda(\eta - v\xi) \int_a^b \alpha(\tau, \eta - v(\xi - \tau))\beta(\tau, \eta - v(\xi - \tau))d\tau}$$

determined in

$$\Omega = \left\{(\xi, \eta) \in \mathbb{R}^2 : 1 - \lambda(\eta - v\xi) \int_a^b \alpha(\tau, \eta - v(\xi - \tau))\beta(\tau, \eta - v(\xi - \tau))d\tau \neq 0\right\}.$$

D. Let us examine the solvability of an equation

$$x(t) - \frac{16}{3} \int_0^1 (-13t\tau^9 + 19t^2\tau^{12})x(\tau^3)d\tau = f(t). \tag{38}$$

For this purpose we use the model of the operational calculus from Ex. B, accepting $g(t) = t^3, a = 0, b = 1$ and

$$\alpha_1(t) = t, \quad \alpha_2(t) = 19t^2, \quad \beta_1(t) = -13t^3, \quad \beta_2(t) = t^4.$$

Thus (38) has the form of (10) of the integral equation with degenerated kernel

$$x(t) - \frac{16}{3} \left(t \int_0^1 (-13\tau^9)x(\tau^3)d\tau + 19t^2 \int_0^1 \tau^{12}x(\tau^3)d\tau\right) = f(t), \tag{39}$$

and the system (19) corresponding to (39) is of the form

$$\begin{array}{rcl} \frac{19}{3}c_1 + \frac{247}{3}c_2 & = & b_1 \\ -\frac{1}{3}c_1 - \frac{13}{3}c_2 & = & b_2 \end{array}. \tag{40}$$

Since its main determinant $W = 0$, there exist non-zero solutions of the homogeneous equation

$$x(t) - \frac{16}{3}\left(t\int_0^1 (-13\tau^9)x(\tau^3)d\tau + 19t^2\int_0^1 \tau^{12}x(\tau^3)d\tau\right) = 0$$

and of the conjugated homogeneous one

$$y(t) - \frac{16}{3}\left(-13t^3\int_0^1 \tau^3 y(\tau^3)d\tau + t^4\int_0^1 19\tau^6 y(\tau^3)d\tau\right) = 0. \tag{41}$$

The system (30) corresponding to (41) is of the form

$$\begin{array}{rcl} \frac{19}{3}c_1^* - \frac{1}{3}c_2^* & = & 0 \\ \frac{247}{3}c_1^* - \frac{13}{3}c_2^* & = & 0 \end{array}.$$

Hence

$$c_1^* = \frac{1}{19}C, \quad c_2^* = C, \quad C \in \mathbb{R}.$$

So, from (29) it follows that the solutions of the conjugated homogeneous equation (41) are expressed as

$$y(t) = \frac{16}{3}C(t^4 - \frac{13}{19}t^3), \quad C \in \mathbb{R}. \tag{42}$$

On account of the second part of the Fredholm alternative, the non-homogeneous equation (39) has a solution if and only if a function $f(t)$ is orthogonal to all solutions (42) of (41), i.e. if

$$I_0^1 fy = \frac{16}{3}C\int_0^1 f(\tau^3)(\tau^{12} - \frac{13}{19}\tau^9)d\tau = 0.$$

For example, the function

$$f(t) = \frac{49}{13}\sqrt[3]{t} - \frac{45}{11} \tag{43}$$

satisfies the equation. In that case

$$b_1 = \frac{19}{22}, \quad b_2 = -\frac{1}{22}$$

and the system (40) has solutions

$$c_1 = \frac{3}{22} - 13C, \quad c_2 = C, \quad C \in \mathbb{R}.$$

From (28) it follows, that the solutions of the non-homogeneous equation (39) with its right member given by formula (43), have the form of

$$x(t) = \frac{49}{13}\sqrt[3]{t} - \frac{45}{11} + \frac{16}{3}[(\frac{3}{22} - 13C)t + 19Ct^2], \quad C \in \mathbb{R}.$$

E. Now we determine the solution of

$$x(\xi,\eta) + \xi\int_{-1}^{1}(\eta-\xi+\tau)x(\tau,\eta-\xi+\tau)d\tau + \eta\int_{-1}^{1}\tau x(\tau,\eta-\xi+\tau)d\tau = 5\xi + 2\eta. \tag{44}$$

To this end, consider the operational calculus model from Ex. C. If we take $v = 1, \lambda = 1, a = -1, b = 1$ and

$$\begin{aligned} \alpha_1(\xi,\eta) = \xi, &\quad \alpha_2(\xi,\eta) = -\eta, \\ \beta_1(\xi,\eta) = -\eta, &\quad \beta_2(\xi,\eta) = \xi, \end{aligned}$$

then (44) admits the form of (10) – the Fredholm integral equation with degenerated kernel. Functions ξ and $-\eta$ are $\left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta}\right)$ -linearly independent. Indeed, if for all $(\xi,\eta) \in \mathbb{R}^2$

$$\xi\varphi_1(\eta-\xi) - \eta\varphi_2(\eta-\xi) = 0, \quad \text{where} \quad \varphi_1, \varphi_2 \in C^1(\mathbb{R},\mathbb{R}) \tag{45}$$

yields, then

$$\begin{array}{rcl} \varphi_1(\eta-\xi) - \xi\varphi_1'(\eta-\xi) + \eta\varphi_2'(\eta-\xi) & \equiv & 0 \\ \xi\varphi_1'(\eta-\xi) - \varphi_2(\eta-\xi) - \eta\varphi_2'(\eta-\xi) & \equiv & 0 \end{array},$$

and hence $\varphi_1(\eta-\xi) \equiv \varphi_2(\eta-\xi)$. Finally, from (45) we get

$$(\eta-\xi)\varphi_1(\eta-\xi) = 0 \quad \text{for each} \quad (\xi,\eta) \in \mathbb{R}^2,$$

which, due to $\varphi_1 \in C^0$, means that $\varphi_1(\eta - \xi) \equiv 0$.
The system (19) corresponding to (44) is of the form

$$\begin{array}{rcl} \frac{5}{3}c_1 - [2(\eta-\xi)^2 + \frac{2}{5}]c_2 & = & -4(\eta-\xi)^2 - \frac{14}{3} \\ -\frac{2}{3}c_1 + \frac{5}{3}c_2 & = & \frac{14}{3} \end{array}. \tag{46}$$

Its determinant

$$W = W(\xi, \eta) = \frac{7}{3} - \frac{4}{3}(\eta - \xi)^2$$

is not an invertible element, because $W = 0$ for $\eta = \xi \pm \frac{\sqrt{7}}{2}$.

For all $(\xi, \eta) \in \mathbb{R}^2$ which lie on the straight line $\eta = \xi + \frac{\sqrt{7}}{2}$, the equation (44) reduces itself to

$$x(\xi) + \xi \int_{-1}^{1} (\tau + \frac{\sqrt{7}}{2}) x(\tau) d\tau + (\xi + \frac{\sqrt{7}}{2}) \int_{-1}^{1} \tau x(\tau) d\tau = 7\xi + \sqrt{7}. \tag{47}$$

In that case the system (30) takes the form of

$$\begin{array}{rcl} \frac{5}{3}c_1^* - \frac{2}{3}c_2^* & = & 0 \\ -\frac{25}{6}c_1^* + \frac{5}{3}c_2^* & = & 0 \end{array}.$$

Hence

$$c_1^* = \frac{2}{5}C, \quad c_2^* = C, \quad C \in \mathbb{R}.$$

Due to that, on the basis of (29), we get solutions of the conjugated homogeneous equation corresponding to (47)

$$y(\xi) = \frac{3}{5}C\xi - \frac{\sqrt{7}}{5}C, \quad C \in \mathbb{R}. \tag{48}$$

Functions $f(\xi) = 7\xi + \sqrt{7}$ and (48) are orthogonal, since

$$\int_{-1}^{1} (7\tau + \sqrt{7})(\frac{3}{5}\tau - \frac{\sqrt{7}}{5}) d\tau = 0.$$

Therefore there exist solutions of (47). For $\eta = \xi + \frac{\sqrt{7}}{2}$ the system (46) takes the form of

$$\begin{aligned} \frac{5}{3}c_1 - \frac{25}{6}c_2 &= -\frac{35}{3} \\ -\frac{2}{3}c_1 + \frac{5}{3}c_2 &= \frac{14}{3} \end{aligned} \tag{49}$$

and it has solutions

$$c_1 = \frac{5}{2}C - 7, \quad c_2 = C, \quad C \in \mathbb{R}.$$

Hence and from (28) it follows, that the functions

$$x(\xi) = \frac{3}{2}C\xi - \frac{\sqrt{7}}{2}C + \sqrt{7}, \quad C \in \mathbb{R}$$

are solutions of the non-homogeneous equation (47).

For all $(\xi, \eta) \in \mathbb{R}^2$ which lie on the straight line $\eta = \xi - \frac{\sqrt{7}}{2}$, the equation (44) reduces itself to

$$x(\xi) + \xi \int_{-1}^{1} (\tau - \frac{\sqrt{7}}{2}) x(\tau) d\tau + (\xi - \frac{\sqrt{7}}{2}) \int_{-1}^{1} \tau x(\tau) d\tau = 7\xi - \sqrt{7}. \tag{50}$$

In that case solutions of the conjugated homogeneous equation corresponding to (50) are given by the formula

$$y(\xi) = \frac{3}{5}C\xi + \frac{\sqrt{7}}{5}C, \quad C \in \mathbb{R}. \tag{51}$$

Functions $f(\xi) = 7\xi - \sqrt{7}$ and (51) also satisfy the condition of orthogonality. Therefore (50) have solutions. For $\eta = \xi - \frac{\sqrt{7}}{2}$ the system (46) also takes the form of (49). Hence and from (28) it follows that functions

$$x(\xi) = \frac{3}{2}C\xi + \frac{\sqrt{7}}{2}C - \sqrt{7}, \quad C \in \mathbb{R}$$

are solutions of the non-homogeneous equation (50).

For those $(\xi, \eta) \in \mathbb{R}^2$ which satisfy the condition $\eta \neq \xi \pm \frac{\sqrt{7}}{2}$, (46) is a Cramer system and its solution is

$$c_1 = -2, \quad c_2 = 2.$$

Hence and from (22) it follows that the function

$$x(\xi, \eta) = 3\xi$$

is the only solution of (44) for $\xi \neq \eta \pm \frac{\sqrt{7}}{2}$, where $\xi, \eta \in \mathbb{R}$.

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