Fractional Calculus
& Applied Analysis
An International Journal for Theory and Applications
VOLUME 8, NUMBER 3 (2005) ISSN 1311-0454

ON THE RIEMANN-LIOUVILLE FRACTIONAL q-INTEGRAL OPERATOR INVOLVING A BASIC ANALOGUE OF FOX H-FUNCTION

S. L. Kalla*, **R. K. Yadav**** and **S. D. Purohit****

Abstract

The present paper envisages the applications of Riemann-Liouville fractional q-integral operator to a basic analogue of Fox H-function. Results involving the basic hypergeometric functions like $G_q(.), J_v(x;q), Y_v(x;q)$, $K_v(x;q), H_v(x;q)$ and various other q-elementary functions associated with the Riemann-Liouville fractional q-integral operator have been deduced as special cases of the main result.

2000 Mathematics Subject Classification: 33D60, 26A33, 33C60
Key Words and Phrases: fractional q-integral, H-function, q-functions

1. Introduction

Agarwal [1] introduced the q-analogue of the Riemann-Liouville fractional integral operator as follows:

$$I_q^\mu f(x) = \frac{1}{\Gamma_q(\mu)} \int_0^x (x - yq)_{\mu-1} \, f(y) d(y;q), \tag{1.1}$$

where 'μ' is an arbitrary order of integration such that $\mathrm{Re}(\mu) > 0$.

Following Jackson [5], Al-Salam [2] and Agarwal [1], we have the basic integration defined as:

$$\int_0^x f(t)d(t;q) = x(1-q)\sum_{k=0}^{\infty} q^k f(xq^k). \tag{1.2}$$

In view of equation (1.2), (1.1) can be expressed as:

$$I_q^{\mu} f(x) = \frac{x^{\mu}(1-q)}{\Gamma_q(\mu)}\sum_{k=0}^{\infty} q^k (1-q^{k+1})_{\mu-1} f(xq^k), \tag{1.3}$$

where $\mathrm{Re}(\mu) > 0$.

Further, for real or complex α and $0 < |q| < 1$, the q-factorial is defined as:

$$(\alpha;q)_n \equiv (q^{\alpha};q)_n = \begin{cases} 1 & ; \quad n = 0 \\ (1-q^{\alpha})(1-q^{\alpha+1})...(1-q^{\alpha+n-1}) & ; \quad n = 1,2,... \end{cases} \tag{1.4}$$

or equivalently,

$$(\alpha;q)_n = \prod_{j=0}^{\infty} \frac{(1-\alpha q^j)}{(1-\alpha q^{n+j})} = \frac{(\alpha;q)_{\infty}}{(\alpha q^n;q)_{\infty}}\,. \tag{1.5}$$

In terms of the basic analogue of the gamma function, we have

$$(\alpha;q)_n = \frac{\Gamma_q(\alpha+n)(1-q)^n}{\Gamma_q(\alpha)}\ , \quad n > 0\,, \tag{1.6}$$

where the q-gamma function, cf. Gasper and Rahman [4], in various form is given by

$$\Gamma_q(\alpha) = \frac{(q;q)_{\infty}}{(q^{\alpha};q)_{\infty}(1-q)^{\alpha-1}} = \frac{(1-q)_{\alpha-1}}{(1-q)^{\alpha-1}} = \frac{(q;q)_{\alpha-1}}{(1-q)^{\alpha-1}}, \tag{1.7}$$

$$(\alpha \neq 0,-1,-2,...).$$

Indeed

$$\underset{q\to 1^-}{lt}\ \frac{(q^{\alpha};q)_n}{(1-q)^n} = (\alpha)_n, \tag{1.8}$$

where

$$(\alpha)_n = \alpha(\alpha+1)...(\alpha+n-1). \tag{1.9}$$

The q-binomial series, cf. Gasper and Rahman [4], is given by

$$ {}_1\phi_0\left[\begin{array}{ll}\alpha\ ; & \\ & q, x \\ -; & \end{array}\right] = \frac{(\alpha x; q)_\infty}{(x; q)_\infty}. \tag{1.10}$$

Saxena, Modi and Kalla [10], introduced the basic analogue of the Fox H-function in the following manner:

$$H_{A,B}^{m_1,n_1}\left[x; q \,\middle|\, \begin{array}{l}(a,\alpha)\\(b,\beta)\end{array}\right] \tag{1.11}$$

$$= \frac{1}{2\pi\ i}\int_C \frac{\prod_{j=1}^{m_1} G(q^{b_j-\beta_j s}) \prod_{j=1}^{n_1} G(q^{1-\alpha_j+\alpha_j s})\pi x^s}{\prod_{j=m_1+1}^{B} G(q^{1-b_j+\beta_j s}) \prod_{j=n_1+1}^{A} G(q^{\alpha_j-\alpha_j s}) G(q^{1-s}) \sin \pi s}\ ds,$$

where $0 \le m_1 \le B,\ 0 \le n_1 \le A;\ \alpha_i' s$ and $\beta_j' s$ are all positive integers, the contour C is a line parallel to $\mathrm{Re}(ws)$, with indentations if necessary, in such a manner that all poles of $G(q^{b_j-\beta_j s}), 1 \le j \le m_1$ are to the right, and those of $G(q^{1-\alpha_j+\alpha_j s}), 1 \le j \le n_1$, to the left of C. The integral converges if $\mathrm{Re}[s \log(x) - \log \sin \pi s] < 0$ for large values of $|s|$ on the contour, i.e. if $|\{\arg(x) - w_2 w_1^{-1} \log|x|\}| < \pi$, where $0 < |q| < 1$, $\log q = -w = (w_1 + i w_2)$, w, w_1, w_2 are definite quantities w_1 and w_2 being real.

Also

$$G(q^\alpha) = \left\{\prod_{n=0}^{\infty}(1 - q^{\alpha+n})\right\}^{-1} = \frac{1}{(q^\alpha; q)_\infty}. \tag{1.12}$$

If we set $\alpha_j = \beta_i = 1, 1 \le j \le A, 1 \le i \le B$ in (1.11), then it reduces to the basic Meijer's G-function, namely

$$H_{A,B}^{m_1,n_1}\left[x; q \,\middle|\, \begin{array}{l}(a,1)\\(b,1)\end{array}\right] \equiv G_{A,B}^{m_1 n_1}\left[x; q \,\middle|\, \begin{array}{l}a_1, ..., a_A\\b_1, ..., b_B\end{array}\right]$$

$$= \frac{1}{2\pi\ i}\int_C \frac{\prod_{j=1}^{m_1} G(q^{b_j-s}) \prod_{j=1}^{n_1} G(q^{1-a_j+s})\pi x^s}{\prod_{j=m_1+1}^{B} G(q^{1-b_j+s}) \prod_{j=n_1+1}^{A} G(q^{\alpha_j-s}) G(q^{1-s}) \sin \pi s}\ ds, \tag{1.13}$$

where $0 \le m_1 \le B, 0 \le n_1 \le A$; and $\mathrm{Re}[s \log(x) - \log \sin \pi s] < 0$.

Further, if we set $n_1 = 0, m_1 = B$ in (1.13), we get the basic analogue of MacRobert's E-function

$$G_{A,B}^{B,0}\left[x;q \,\middle|\, \begin{matrix} a_1,...,a_A \\ b_1,...,b_B \end{matrix}\right] = E_q[B;b_j:A;a_j:x]. \tag{1.14}$$

Saxena and Kumar [11], introduced the basic analogues of $J_v(x)$, $Y_v(x)$, $K_v(x)$, $H_v(x)$ in terms of $G_q(.)$ function as follows:

$$J_v(x;q) = \{G(q)\}^2 G_{0,3}^{1,0}\left[\frac{x^2(1-q)^2}{4};q \,\middle|\, \begin{matrix} - \\ \frac{v}{2},\frac{-v}{2},1 \end{matrix}\right], \tag{1.15}$$

where $J_v(x;q)$ denotes the q-analogue of Bessel function $J_v(x)$.

$$Y_v(x;q) = \{G(q)\}^2 \; G_{1,4}^{2,0}\left[\frac{x^2(1-q)^2}{4};q \,\middle|\, \begin{matrix} \frac{-v-1}{2} \\ \frac{v}{2},\frac{-v}{2},\frac{-v-1}{2},1 \end{matrix}\right], \tag{1.16}$$

where $Y_v(x;q)$ denotes the q-analogue of the Bessel function $Y_v(x)$.

$$K_v(x;q) = (1-q) \; G_{0,3}^{2,0}\left[\frac{x^2(1-q)^2}{4};q \,\middle|\, \begin{matrix} - \\ \frac{v}{2},\frac{-v}{2},1 \end{matrix}\right], \tag{1.17}$$

where $K_v(x;q)$ denotes the basic analogue of the Bessel function of the third kind $K_v(x)$.

$$H_v(x;q) = \left(\frac{1-q}{2}\right)^{1-\alpha} \quad G_{1,4}^{3,1}\left[\frac{x^2(1-q)^2}{4};q \,\middle|\, \begin{matrix} \frac{1+\alpha}{2} \\ \frac{v}{2},\frac{-v}{2},\frac{1+\alpha}{2},1 \end{matrix}\right], \tag{1.18}$$

where $H_v(x;q)$ is the basic analogue of Struve's function $H_v(x)$.

In view of the definition (1.11), the following elementary basic (q-) functions are expressible in terms of the basic analogue of Meijer's G-function as:

$$e_q(-x) = G(q)G_{0,2}^{1,0}\left[x(1-q);q \,\middle|\, \begin{matrix} - \\ 0,1 \end{matrix}\right] \tag{1.19}$$

$$\sin_q(x) = \sqrt{\pi}(1-q)^{-1/2}\{G(q)\}^2 \; G_{0,3}^{1,0}\left[\frac{x^2(1-q)^2}{4};q \,\middle|\, \begin{matrix} z- \\ \frac{1}{2},0,1 \end{matrix}\right] \tag{1.20}$$

$$\cos_q(x) = \sqrt{\pi}(1-q)^{-1/2}\{G(q)\}^2 \; G_{0,3}^{1,0}\left[\frac{x^2(1-q)^2}{4};q \,\middle|\, \begin{matrix} - \\ 0,\frac{1}{2},1 \end{matrix}\right] \tag{1.21}$$

$$\sinh_q(x) = \frac{\sqrt{\pi}}{i}(1-q)^{-1/2}\{G(q)\}^2 \; G_{0,3}^{1,0}\left[-\frac{x^2(1-q)^2}{4};q \,\middle|\, \begin{matrix} - \\ \frac{1}{2},0,1 \end{matrix}\right] \tag{1.22}$$

$$\cosh_q(x) = \sqrt{\pi}(1-q)^{-1/2}\{G(q)\}^2\, G^{1,0}_{0,3}\left[-\frac{x^2(1-q)^2}{4};q \,\middle|\, \begin{matrix} - \\ 0,\frac{1}{2},1 \end{matrix}\right]. \quad (1.23)$$

A detailed account of various functions expressible in terms of the Meijer G-function or Fox H-function can be found in research monographs due to Mathai and Saxena [8], [9]. A systematic and unified development of a new generalized fractional calculus, closely related to special functions of a rather general nature is given in [6], [7].

In a recent paper, Yadav and Purohit [12] have investigated some applications of Riemann-Liouville fractional q-integral operator to various basic hypergeometric functions of one variable.

The motive for the present paper is to evaluate the Riemann-Liouville fractional basic integral operator involving basic analogue of the H-function and various other basic hypergeometric functions. The results deduced are believed to find certain applications to the solutions of basic (q-) integral equations.

2. Main results

In this section, we shall evaluate the following q-fractional integral of the Riemann-Liouville type involving a basic analogue of Fox's H-functions:

$$I^{\mu}_q\left\{H^{m_1,n_1}_{A,B}\left[\rho x^{\lambda};q\,\middle|\,\begin{matrix}(a,\alpha)\\(b,\beta)\end{matrix}\right]\right\}$$
$$= \begin{cases} x^{\mu}(1-q)^{\mu}H^{m_1,n_1+1}_{A+1,B+1}\left[\rho x^{\lambda};q\,\middle|\,\begin{matrix}(0,\lambda),(a,\alpha)\\(b,\beta),(-\mu,\lambda)\end{matrix}\right], \lambda\geq 0 \\ \\ x^{\mu}(1-q)^{\mu}H^{m_1+1,n_1}_{A+1,B+1}\left[\rho x^{\lambda};q\,\middle|\,\begin{matrix}(a,\alpha),(1-\lambda)\\(1+\mu,-\lambda),(b,\beta)\end{matrix}\right], \lambda< 0 \end{cases} \quad (2.1)$$

where $0\leq m_1\leq B, 0\leq n_1\leq A$.

Proof of (2.1). To prove (2.1), we apply equations (1.3) and (1.11) to the left hand side to obtain

$$x^{\mu}(1-q)^{\mu}\sum_{k=0}^{\infty}\frac{q^k(q^{\mu};q)_k}{(q;q)_k}$$

$$\times\ \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m_1}G(q^{b_j-\beta_j s})\prod_{j=1}^{n_1}G(q^{1-a_j+\alpha_j s})\pi(px^{\lambda}q^{\lambda k})^s}{\prod_{j=m_1+1}^{B}G(q^{1-b_j+\beta_j s})\prod_{j=n_1+1}^{A}G(q^{a_j-\alpha_j s})G(q^{1-s})\sin\pi s}\,ds.$$

Interchanging the order of summation and integration, which is valid under conditions given with (1.11), the above expression reduces to

$$\frac{x^{\mu}(1-q)^{\mu}}{2\pi i}\int_C \frac{\prod_{j=1}^{m_1} G(q^{b_j-\beta_j s})\prod_{j=1}^{n_1} G(q^{1-a_j+\alpha_j s})\pi(px^{\lambda})^s}{\prod_{j=m_1+1}^{B} G(q^{1-b_j+\beta_j s})\prod_{j=n_1+1}^{A} G(q^{a_j-\alpha_j s})G(q^{1-s})\sin\pi s}$$

$$\times\sum_{k=0}^{\infty}\frac{q^{k(1+\lambda s)}(q^{\mu};q)_k}{(q;q)_k}\,ds,$$

on summing inner ${}_1\phi_0(.)$ series with the help of (1.10), it yields after some simplifications

$$\frac{x^{\mu}(1-q)^{\mu}}{2\pi i}$$

$$\times\int_C \frac{\prod_{j=1}^{m_1} G(q^{b_j-\beta_j s})\prod_{j=1}^{n_1} G(q^{1-a_j+\alpha_j s})G(q^{1+\lambda s})\pi(px^{\lambda})^s}{\prod_{j=m_1+1}^{B} G(q^{1-b_j+\beta_j s})G(q^{1+\mu+\lambda s})\prod_{j=n_1+1}^{A} G(q^{a_j-\alpha_j s})G(q^{1-s})\sin\pi s}ds,$$

which on interpretation in view of (1.11), leads us to the right hand side of the result (2.1). The second part follows similarly.

3. Applications

In this section, we shall derive certain basic integrals of the Riemann-Liouville type, involving various basic functions expressible in terms of the basic analogue of Fox's H-function or Meijer's G-function, as the application of the main result (2.1). These results are presented in tabular form, see Table 1.

Proofs of the results (3.1) and (3.2) follows directly from (2.1) with $\lambda=\rho=1$ and on using the definitions (1.13)-(1.14) respectively.

While if we assign $m_1=1, n_1=A=0, B=3, b_1=v/2, b_2=-v/2, b_3=1,\ \lambda=2, \rho=\frac{(1-q)^2}{4}$ in equation (2.1), we obtain

$$I_q^{\mu}\left\{H_{0,3}^{1,0}\left[\frac{x^2(1-q)^2}{4};q\,\middle|\,\begin{matrix}-\\(\frac{v}{2},1),(\frac{-v}{2},1),(1,1)\end{matrix}\right]\right\}=x^{\mu}(1-q)^{\mu}$$

$$\times H_{1,4}^{1,1}\left[\frac{x^2(1-q)^2}{4};q\,\middle|\,\begin{matrix}(0,2),\\(\frac{v}{2},1),(\frac{-v}{2},1)(1,1),(-\mu,2)\end{matrix}\right], \tag{3.12}$$

in view of of the definitions (1.15), the left hand side of (3.12) reduces to

$$I_q^{\mu}\{J_v(x;q)\} = x^{\mu}(1-q)^{\mu}\{G(q)\}^2$$

$$\times H_{1,4}^{1,1}\left[\frac{x^2(1-q)^2}{4};q\,\middle|\,\begin{matrix}(0,2)\\(\frac{v}{2},1),(\frac{-v}{2},1)(1,1),(-\mu,2)\end{matrix}\right]. \tag{3.13}$$

This completes the proof of (3.3).

The proof of (3.4)-(3.6) follows similarly. To prove (3.7), we take $m_1 = 1, n_1 = A = 0, B = 2, \alpha_i = \beta_j = 1, b_1 = 0, b_2 = 1, \lambda = 1, \rho = (1-q)$ in the main result (2.1) and then on making use of the definitions(1.13) and (1.19) we arrive at (3.7).

The results (3.8)-(3.11) can be proved similarly by assigning particular values to the parameters m_1, n_1, A, B, λ and ρ, keeping in view the definitions (1.20)-(1.23).

Finally, it is interesting to observe that in view of a limit formula for $G_q(.)$ function due to Saxena and Kumar [11], if we let $q \to 1^{-1}$, in (3.1), we obtain a known result mentioned in Erdélyi [3] [eq. no. 96, table (13.1), p. 200].

Acknowledgment

S. D. Purohit is grateful to CSIR, Govt. of INDIA, for providing a Senior Research Fellowship (No. 9/98(51) 2003. EMR.I) to enable him to carry out the present investigations.

Eq. No.	$f(x)$	$I_q^{\mu} f(x) = \frac{1}{\Gamma_q(\mu)} \int_0^x (x - yq)_{\mu-1} f(y) d(y;q)$, $\mathrm{Re}(\mu) > 0$
3.1	$G_{A,B}^{m_1,n_1}\left[x;q \,\middle\vert\, \begin{matrix} a_1,...,a_A \\ b_1,...,b_B \end{matrix}\right]$	$x^{\mu}(1-q)^{\mu} G_{A+1,B+1}^{m_1,n_1+1}\left[x;q \,\middle\vert\, \begin{matrix} 0,a_1,...,a_A \\ b_1,...,b_B,-\mu \end{matrix}\right]$ $0 \le m_1 \le B, 0 \le n_1 \le A$
3.2	$E_q[B;b_j:A;a_j:x]$	$x^{\mu}(1-q)^{\mu} G_{A+1,B+1}^{B,1}\left[x;q \,\middle\vert\, \begin{matrix} 0,a_1,...,a_A \\ b_1,...,b_B,-\mu \end{matrix}\right]$
3.3	$J_v(x;q)$	$x^{\mu}(1-q)^{\mu}\{G(q)\}^2$ $.H_{1,4}^{1,1}\left[\frac{x^2(1-q)^2}{4};q \,\middle\vert\, \begin{matrix} (0,2) \\ (\frac{v}{2},1),(\frac{-v}{2},1),(1,1),(-\mu,2) \end{matrix}\right]$
3.4	$Y_v(x;q)$	$x^{\mu}(1-q)^{\mu}\{G(q)\}^2$ $.H_{2,5}^{2,1}\left[\frac{x^2(1-q)^2}{4};q \,\middle\vert\, \begin{matrix} (0,2),(\frac{-v-1}{2},1) \\ (\frac{v}{2},1),(\frac{-v}{2},1),(\frac{-v-1}{2},1),(1,1),(-\mu,2) \end{matrix}\right]$
3.5	$K_v(x;q)$	$x^{\mu}(1-q)^{\mu+1}$ $.H_{1,4}^{2,1}\left[\frac{x^2(1-q)^2}{4};q \,\middle\vert\, \begin{matrix} (0,2) \\ (\frac{v}{2},1),(\frac{-v}{2},1),(1,1),(-\mu,2) \end{matrix}\right]$

Table 1

3.6	$H_v(x;q)$	$x^{\mu}(1-q)^{\mu-1-\alpha}2^{\alpha-1} .H^{3,2}_{2,5}\left[\frac{x^2(1-q)^2}{4};q \,\middle\vert\, \begin{matrix}(0,2),(\frac{\alpha+1}{2},1)\\(\frac{v}{2},1),(\frac{-v}{2},1),(\frac{\alpha+1}{2},1),(1,1),(-\mu,2)\end{matrix}\right]$
3.7	$e_q(-x)$	$x^{\mu}(1-q)^{\mu}G(q)G^{1,1}_{1,3}\left[x(1-q);q \,\middle\vert\, \begin{matrix}0\\0,1,-\mu\end{matrix}\right]$
3.8	$\sin_q(x)$	$x^{\mu}\sqrt{\pi}(1-q)^{\mu-1/2}\{G(q)\}^2 .H^{1,1}_{1,4}\left[\frac{x^2(1-q)^2}{4};q \,\middle\vert\, \begin{matrix}(0,2)\\(\frac{1}{2},1),(0,1),(1,1),(-\mu,2)\end{matrix}\right]$
3.9	$\cos_q(x)$	$x^{\mu}\sqrt{\pi}(1-q)^{\mu-1/2}\{G(q)\}^2 .H^{1,1}_{1,4}\left[\frac{x^2(1-q)^2}{4};q \,\middle\vert\, \begin{matrix}(0,2),\\(0,1)(\frac{1}{2},1),(1,1),(-\mu,2)\end{matrix}\right]$
3.10	$\sinh_q(x)$	$x^{\mu}\frac{\sqrt{\pi}}{i}(1-q)^{\mu-1/2}\{G(q)\}^2 .H^{1,1}_{1,4}\left[\frac{-x^2(1-q)^2}{4};q \,\middle\vert\, \begin{matrix}(0,2),\\(\frac{1}{2},1),(0,1),(1,1),(-\mu,2)\end{matrix}\right]$
3.11	$\cosh_q(x)$	$x^{\mu}\sqrt{\pi}(1-q)^{\mu-1/2}\{G(q)\}^2 .H^{1,1}_{1,4}\left[\frac{-x^2(1-q)^2}{4};q \,\middle\vert\, \begin{matrix}(0,2),\\(0,1),(\frac{1}{2},1),(1,1),(-\mu,2)\end{matrix}\right]$

Table 1, cont'd

References

1. R. P. Agarwal, Certain fractional q-integrals and q-derivatives. *Proc. Camb. Phil. Soc.* **66** (1969), 365-370.
2. W. A. Al-Salam, Some fractional q-integrals and q-derivatives. *Proc. Edin. Math. Soc.* **15** (1966), 135-140.
3. A. Erdélyi (Editor), *Tables of Integral Transforms.* Vol. II, McGraw-Hill Book Co. Inc., New York (1954).
4. G. Gasper, G. and M. Rahman, *Basic Hypergeometric Series.* Cambridge University Press, Cambridge (1990).
5. F. H. Jackson, Basic integration. *Quart. J. Math. (Oxford)*, **2** (1951), 1-16.
6. S. L. Kalla and V. S. Kiryakova, An H-function generalized fractional calculus based upon composition of Erdélyi-Kober operators in Lp. *Math. Japonica* **35** (1990), 1151-1171.
7. V. S. Kiryakova, *Generalized Fractional Calculus and Applications.* Longman Scientific & Tech., Essex (1994).
8. A. Mathai and R. K. Saxena, *Generalized Hypergeometric Functions With Applications in Statistics and Physical Sciences.* Springer-Verlag, Berlin (1973).
9. Mathai, A. M and Saxena, R. K.: The H-function With Application in Statistics and Other Disciplines. John Wiley & Sons. Inc. New York. (1978).
10. R. K. Saxena, G. C. Modi and S. L. Kalla, A basic analogue of Fox's H-function. *Rev. Tec. Ing., Univ. Zulia* **6** (1983), 139-143.
11. R. K. Saxena and R. Kumar, Recurrence relations for the basic analogue of the H-function. *J. Nat. Acad. Math.* **8** (1990), 48-54.
12. R. K. Yadav and S. D. Purohit, Application of Riemann-Liouville fractional q-integral operator to basic hypergeometric functions. To appear in: *Acta Ciencia Indica* **30**, No. 3 (2004).

* *Department of Mathematics and Computer Science*
P.O. Box-5969, Safat 13060, KUWAIT
e-mail: shyamkalla@yahoo.com
Received: 24 April 2005

** *Department of Mathematics & Statistics*
J. N. Vyas University, Jodhpur-342001 (INDIA)
e-mail: rkmdyadav@yahoo.co.in