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AN $L^P - L^Q$−VERSION OF MORGAN'S THEOREM ASSOCIATED WITH PARTIAL DIFFERENTIAL OPERATORS

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Abstract

In this paper we take the strip $\mathbb{K}_\ell = [0, +\infty[\times[-\ell\pi, \ell\pi]$, where ℓ is a positive integer. We consider, for a nonnegative real number α, two partial differential operators D and D_α on $]0, +\infty[\times] - \ell\pi, \ell\pi[$. We associate a generalized Fourier transform $\mathcal{F}_\alpha$ to the operators D and D_α. For this transform $\mathcal{F}_\alpha$, we establish an $L^p - L^q$−version of the Morgan's theorem under the assumption $1 \le p, q \le +\infty$.

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1. Introduction

A rigourous formulation of the uncertainty principle in the framework of the classical Fourier analysis on $\mathbb{R}$ is to investigate the $L^p - L^q$−sufficient pairs of positive functions in the following meaning. A pair (g, h) of positive functions is called an $L^p - L^q$−sufficient pair if, for every measurable function f, the conditions $g^{-1}f \in L^p(\mathbb{R})$ and $h^{-1}\widehat{f} \in L^q(\mathbb{R})$ imply that $f = 0$ almost everywhere, where $\widehat{f}$ is the Fourier transform of f defined by

$$\widehat{f}(y) = \int_{\mathbb{R}} f(x)\, e^{-ixy} dx \,.$$

Several authors have studied this form of the uncertainty principle in many situations. Let us to indicate some of such works. In 1933, Hardy [9] showed that the pair $(e^{-ax^2}, e^{-b\lambda^2})$ is $L^\infty - L^\infty$−sufficient if and only if $ab > \frac{1}{4}$. After fifty years, M. Cowling and J.F. Price generalized Hardy's theorem to an $L^p - L^q$−version, where $1 \leq p, q \leq +\infty$. In 2001, M. Ebata [5] has given a similar theorem for the group $\mathbf{SU}(1,1)$. In 2003, N.B. Andersen [1] has established an $L^p - L^q$−version of Hardy's theorem for the Jacobi transform. Also, L. Gallardo and K. Trimèche [8], in 2004, have given an $L^p - L^q$−version of Hardy's theorem related to the Dunkl transform. Another famous result is Morgan's theorem. For the classical Fourier transform, this theorem was proved in 1934 by G.W. Morgan [11] and it states that, for $u > 2$ and $v = \frac{u}{u-1}$, the pair $(e^{-a|x|^u}, e^{-b|\lambda|^v})$ is $L^\infty - L^\infty$−sufficient if and only if

$$(au)^{1/u}(bv)^{1/v} > \left(\sin \frac{\pi(v-1)}{2}\right)^{1/v}.$$

Recently, in 2003, S. Ben Farah and K. Mokni [3] have generalized Morgan's theorem to an $L^p - L^q$−version, where $1 \leq p, q \leq +\infty$. Also, they extended this result to the euclidien space $\mathbb{R}^n$, to the Heisenberg group and to noncompact real symmetric spaces. For the Dunkl transform, S. Ayadi [2] in 2004, has given an $L^p - L^q$−version of Morgan's theorem.

In this paper we take the strip $\mathbb{K}_\ell = [0, +\infty[\times[-\ell\pi, \ell\pi]$, where $\ell \in \mathbb{N} \setminus \{0\}$, and for a nonnegative real number α, we consider the following system of partial differential operators

$$\begin{cases} D & = \dfrac{\partial}{\partial \theta} \\ D_\alpha & = \dfrac{\partial^2}{\partial y^2} + [(2\alpha+1)\mathrm{coth} y + \mathrm{th} y]\dfrac{\partial}{\partial y} - \dfrac{1}{\mathrm{ch}^2 y}\dfrac{\partial^2}{\partial \theta^2} + (\alpha+1)^2 \\ \text{with} & (y, \theta) \in]0, +\infty[\times] - \ell\pi, \ell\pi[. \end{cases}$$

For $\alpha = n-1$, n being a positive integer, the operators D and $[D_{n-1} - n^2]$ with the identity generate the algebra $\mathbf{D}(\widetilde{G}/K)$ of left invariant differential operators on $\widetilde{G}/K$, where $\widetilde{G}$ is the universal covering group of $G = \mathbf{U}(n,1)$ and K is the subgroup $\mathbf{U}(n)$ (see [7]).

These operators give rise to generalizations of many two variables structures, like the Fourier transform and the convolution (see [14]), the dispersion and Gaussian distributions (see [13]).

An harmonic analysis related to these operators was introduced, in 1991, by K. Trimèche [14]. In particular, a generalized Fourier transform $\mathcal{F}_\alpha$ associated to the operators D and D_α is defined for a suitable function f as follows

$$\forall(\lambda,\mu) \in \frac{1}{\ell}\mathbb{Z} \times \mathbb{C}\,, \quad \mathcal{F}_\alpha f(\lambda,\mu) = \int_{\mathbb{K}_\ell} f(y,\theta)\varphi_{-\lambda,\mu}(y,\theta)dm_\alpha(y,\theta)\,,$$

where $\varphi_{\lambda,\mu}$ are eigenfunctions of the operators D and D_α, and m_α is a weighted Lebesgue measure on $\mathbb{K}_\ell$ (see section 2).

The main result of this paper is an $L^p - L^q$−version, where $1 \le p,q \le +\infty$, of Morgan's theorem related to the generalized Fourier transform $\mathcal{F}_\alpha$. More precisely, take $u > 2$, $v = \dfrac{u}{u-1}$ and $p,q \in [1,+\infty]$. If a measurable function f on $\mathbb{K}_\ell$ satisfies the conditions $e^{ay^u} f \in L^p(m_\alpha)$ and for all $\lambda \in \frac{1}{\ell}\mathbb{Z}$, $e^{b|\mu|^v}\mathcal{F}_\alpha f(\lambda\,,\,.)_{|\mathbb{R}} \in L^q_\star(|c_{\alpha,\lambda}(\mu)|^{-2}d\mu)$ (see Section 3), where $a,b \in]0,+\infty[$, then, whenever $(au)^{1/u}(bv)^{1/v} > \left(\sin\dfrac{\pi(v-1)}{2}\right)^{1/v}$, the function f is null almost everywhere.

The contents of this paper is as follows: Section 2 is dedicated to some properties and results concerning the eigenfunctions $\varphi_{\lambda,\mu}$ and the generalized Fourier transform $\mathcal{F}_\alpha$. In Section 3 we establish a Phragmen-Lindelöff type result that we need to prove the main statement of this paper. In Section 4 we prove an $L^p - L^q$−version of Morgan's theorem related to the operators D and D_α under the assumption $1 \le p,q \le +\infty$ and $(au)^{1/u}(bv)^{1/v} > \left(\sin\dfrac{\pi(v-1)}{2}\right)^{1/v}$. In the particular case where $\alpha = \dfrac{1}{2}$ and ℓ is even, we show that this last condition is sharp.

2. Generalized Fourier transform associated with the operators D and D_α

This section is organized in the following way. First we introduce the eigenfunctions $\varphi_{\lambda,\mu}$ and recall some of these properties. Next we deal with the generalized Fourier transform $\mathcal{F}_\alpha$.

Proposition 1. (See [14], Théorème I.1) *For* $\lambda \in \frac{1}{\ell}\mathbb{Z}$ *and* $\mu \in \mathbb{C}$, *the initial problem*

$$\begin{cases} D\Phi = i\lambda\Phi \\ D_\alpha\Phi = -\mu^2\Phi \\ \Phi(0,0) = 1, \quad \dfrac{\partial\Phi}{\partial y}(0,\theta) = 0, \quad \theta \in]-\ell\pi, \ell\pi[\end{cases}$$

has a unique solution given by

$$\varphi_{\lambda\,,\,\mu}(y,\theta) = e^{i\lambda\theta}(\mathrm{ch}y)^{\lambda}\varphi_\mu^{(\alpha\,,\,\lambda)}(y),$$

where $\varphi_\mu^{(\alpha\,,\,\lambda)}$ *is the Jacobi function defined by*

$$\varphi_\mu^{(\alpha\,,\,\lambda)}(y) \;=\; {}_2F_1\left(\frac{\alpha+\lambda+1+i\mu}{2}\,,\frac{\alpha+\lambda+1-i\mu}{2}\,;\alpha+1;\,-\mathrm{sh}^2 y\right),$$

${}_2F_1$ *being the Gaussian hypergeometric function (see [6], ChII).*

Properties. (See [14] and also [13])

i) For all $\lambda \in \frac{1}{\ell}\mathbb{Z}$ and $\mu \in \mathbb{C}$, $\varphi_{\lambda\,,\,\mu}$ is even with respect to the first variable and $2\ell\pi-$periodic with respect to the second variable.

ii) For all $\lambda \in \frac{1}{\ell}\mathbb{Z}$, $\mu \in \mathbb{C}$ and $(y,\theta) \in \mathbb{K}_\ell$,

$$\varphi_{\lambda\,,\,\mu}(y,\theta) = e^{i\lambda\theta}(\mathrm{ch}y)^{-\lambda}\varphi_\mu^{(\alpha\,,\,-\lambda)}(y). \tag{1}$$

iii) For all $\lambda \in \frac{1}{\ell}\mathbb{Z}$, $\mu \in \mathbb{C}$ and $(y,\theta) \in \mathbb{K}_\ell$,

$$\overline{\varphi_{\lambda\,,\,\mu}(y,\theta)} = \varphi_{-\lambda\,,\,\mu}(y,\theta) \quad \text{and} \quad \varphi_{\lambda,-\mu}(y,\theta) = \varphi_{\lambda\,,\,\mu}(y,\theta).$$

iv) Consider the following set

$$\Gamma_\ell = \left\{(\lambda,\mu) \in \frac{1}{\ell}\mathbb{Z}\times\mathbb{C} \;\mid\; |\Im\mathrm{m}\mu| \leq \alpha+1\right\} \cup \widetilde{\Omega},$$

where

$$\widetilde{\Omega} = \bigcup_{m\in\mathbb{N}}\left\{(\lambda, i\eta) \in \frac{1}{\ell}\mathbb{Z}\times\mathbb{C} \;\mid\; \eta \geq -(\alpha+1),\; \lambda = \pm(\alpha+2m+1+\eta)\right\}. \tag{2}$$

Then we have

$$\forall(\lambda,\mu)\in\Gamma_\ell\,,\qquad \sup_{(y\,,\,\theta)\in\mathbb{K}_\ell}(y,\theta)|=1. \tag{3}$$

v) According to [10] page 150, we can assert that, for all $(\lambda,\mu)\in\dfrac{1}{\ell}\mathbb{Z}\times\mathbb{C}$ and $(y,\theta)\in\mathbb{K}_\ell$, we have

$$|\varphi_{\lambda\,,\,\mu}(y,\theta)|\le C(1+y)e^{(|\Im\mathrm{m}\mu|-(\alpha+1))y}\,, \tag{4}$$

where C is a positive constant.

Notations.

1) We consider the Lebesgue weighted measure on $\mathbb{K}_\ell$,

$$dm_\alpha(y,\theta)=2^{2(\alpha+1)}(\mathrm{sh}y)^{2\alpha+1}\mathrm{ch}y\,dyd\theta.$$

2) We designate by:

i) $\mathcal{C}(\mathbb{K}_\ell)$ the space of continuous functions on $\mathbb{K}_\ell$.
ii) $\mathcal{C}_c(\mathbb{K}_\ell)$ the space of continuous functions on $\mathbb{K}_\ell$ compactly supported.

3) We denote by $L^p(m_\alpha),\ 1\le p\le+\infty$, the space of measurable functions f on $\mathbb{K}_\ell$ satisfying

$$\|f\|_{p\,,\,\alpha}=\left\{\int_{\mathbb{K}_\ell}|f(y,\theta)|^p dm_\alpha(y,\theta)\right\}^{\frac{1}{p}}<+\infty\qquad\text{if } p<+\infty\,,$$

and

$$\|f\|_{\infty\,,\,\alpha}=\operatorname*{ess\,sup}_{(y\,,\,\theta)\in\mathbb{K}_\ell}|f(y,\theta)|\,.$$

Definition 1. We define the generalized Fourier transform $\mathcal{F}_\alpha$, associated to the operators D and D_α , on $\mathbb{K}_\ell$ by

$$\forall(\lambda,\mu)\in\frac{1}{\ell}\mathbb{Z}\times\mathbb{C}\,,\quad \mathcal{F}_\alpha f(\lambda,\mu)=\int_{\mathbb{K}_\ell}f(y,\theta)\varphi_{-\lambda\,,\,\mu}(y,\theta)dm_\alpha(y,\theta)\,, \tag{5}$$

where $f\in\mathcal{C}_c(\mathbb{K}_\ell)$.

Remark 1. We notice that for all $f\in L^1(m_\alpha)$ and all $(\lambda,\mu)\in\Gamma_\ell$, $\mathcal{F}_\alpha f$ is well defined.

The following two propositions are proved by K. Trimèche in [14].

Proposition 2. (See [14], Proposition VI.5) *Let p and q be real numbers such that $1 \leq p < 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. We consider the following strip:*

$$S_p = \left\{ \mu \in \mathbb{C} \quad | \quad |\Im\mathrm{m}\mu| < \left(\frac{2}{p} - 1\right)(\alpha + 1) \right\}.$$

Then the function $\varphi_{\lambda,\mu}$ belongs to $L^q(m_\alpha)$ in the following cases:

- $\lambda \in \frac{1}{\ell}\mathbb{Z}$ and $\mu \in S_p$.
- $\mu \in \mathbb{C}$ *such that* $\Re\mathrm{e}\mu = 0$, $\Im\mathrm{m}\mu > 0$ *and* $\lambda = \pm(\alpha + 1 + 2m + \Im\mathrm{m}\mu)$,

$m \in \mathbb{N}$, *with* $\lambda \in \frac{1}{\ell}\mathbb{Z}$.

Proposition 3. (See [14], Proposition VI.7) *We have:*

1) For all $p \in [1, 2[$ and $q \in \mathbb{R}$ such that $\frac{1}{p} + \frac{1}{q} = 1$

i) If $f \in L^p(m_\alpha)$, then

$$|\mathcal{F}_\alpha f(\lambda, \mu)| \leq \|f\|_{p,\alpha} \, \|\varphi_{\lambda,\mu}\|_{q,\alpha}$$

in the two following cases:

- $\lambda \in \frac{1}{\ell}\mathbb{Z}$ and $\mu \in S_p$.
- $\mu \in \mathbb{C}$ *such that* $\Re\mathrm{e}\mu = 0$, $\Im\mathrm{m}\mu > 0$ *and* $\lambda = \mp(\alpha + 1 + 2m + \Im\mathrm{m}\mu)$,

$m \in \mathbb{N}$, *with* $\lambda \in \frac{1}{\ell}\mathbb{Z}$.

ii) If $f \in L^1(m_\alpha)$, then

$$|\mathcal{F}_\alpha f(\lambda, \mu)| \leq \|f\|_{1,\alpha}$$

in the two following cases:

- $\lambda \in \frac{1}{\ell}\mathbb{Z}$ and $\mu \in S_1$.
- $(\lambda, \mu) \in \widetilde{\Omega}$, *where* $\widetilde{\Omega}$ *is given by (2).*

2) For all $p \in [1, 2]$, the generalized Fourier transform $\mathcal{F}_\alpha$ associated to the operators D and D_α is one to one on $L^p(m_\alpha)$.

3. Phragmen-Lindelöff type result

In this section we provide an L^q−version of Phragmen-Lindelöff type principle which we need for the proof of our main result. Firstly we state the following lemma proved in [3].

LEMMA 1. (See [3], Lemma 2.3) *Suppose that* $\rho \in]1,2[\,,\quad q \in [1,+\infty]\,,$ $\sigma > 0$ *and* $B > \sigma \sin\frac{\pi}{2}(\rho - 1)$. *If* g *is an entire function on* $\mathbb{C}$ *satisfying the conditions*

$$|g(x+iy)| \le \text{const}\, e^{\sigma|y|^\rho} \quad \textit{for any} \;\; x,y \in \mathbb{R}$$

and

$$e^{B|x|^\rho}\, g_{|\mathbb{R}} \in L^q(\mathbb{R})\,,$$

then $g = 0$.

NOTATIONS. For $\lambda \in \dfrac{1}{\ell}\mathbb{N}$ we consider the following function defined in $\mathbb{R}$ by

$$c_{\alpha\,,\lambda}(\mu) = \frac{2^{\alpha+\lambda+1-i|\mu|}\,\Gamma(\alpha+1)\,\Gamma(i|\mu|)}{\Gamma\Big(\frac{\alpha+\lambda+1+i|\mu|}{2}\Big)\,\Gamma\Big(\frac{\alpha-\lambda+1+i|\mu|}{2}\Big)}.$$

We denote by $L^p_\star(|c_{\alpha\,,\lambda}(\mu)|^{-2}d\mu)$, $1 \le p \le +\infty$, the space of measurable even functions h on $\mathbb{R}$ satisfying

$$\|h\|_{p\,,c} = \left\{\int_0^{+\infty} |h(\mu)|^p |c_{\alpha\,,\lambda}(\mu)|^{-2} d\mu\right\}^{\frac{1}{p}} < +\infty \qquad \text{if } p < +\infty\,,$$

and

$$\|h\|_{\infty\,,c} = \operatorname*{ess\,sup}_{\mu\in\mathbb{R}_+} |h(\mu)|\,.$$

LEMMA 2. *Let* $\rho \in]1,2[\,,\ q \in [1,+\infty]\,,\ \sigma > 0$ *and* $B > \sigma \sin\frac{\pi}{2}(\rho-1)$. *If* g *is an even entire function on* $\mathbb{C}$ *satisfying the conditions*

$$|g(x+iy)| \le \text{const}\, e^{\sigma|y|^\rho} \quad \textit{for any} \;\; x,y \in \mathbb{R}$$

and

$$e^{B|x|^\rho}\, g_{|\mathbb{R}} \in L^q_\star(|c_{\alpha\,,\lambda}(x)|^{-2}dx)\,,$$

then $g = 0$.

P r o o f. Assume that $1 \le q < +\infty$. According to ([15], p.99) we can assert that the function $x \longmapsto |c_{\alpha\,,\lambda}(x)|^{-2}$ is continuous on $[0,+\infty[$ and there exist a positive constant γ such that $\gamma x^2 \le |c_{\alpha\,,\lambda}(x)|^{-2}$ for all $x \in [0,+\infty[$. Therefore,

$$\gamma \int_1^{+\infty} e^{qB|x|^\rho}\,|g(x)|^q dx \le \int_1^{+\infty} e^{qB|x|^\rho}\,|g(x)|^q |c_{\alpha\,,\lambda}(x)|^{-2} dx < +\infty.$$

This implies that $e^{B|x|^\rho}\, g_{|\mathbb{R}} \in L^q(\mathbb{R})$. Consequently, by using Lemma 1, we get the desired result. ■

4. Morgan's theorem related to the operators D and D_α

Throughout this section ℓ designates a positive integer.

Proposition 4. *Let $p \in [1, +\infty]$, $a \in]0, +\infty[$ and let u be a real number such that $u > 2$. Assume that f is a measurable function on $\mathbb{K}_\ell$ satisfying*

$$e^{ay^u} f \in L^p(m_\alpha).$$

Then we have $f \in L^1(m_\alpha)$. Furthermore, $\mathcal{F}_\alpha f(\lambda, \mu)$ is well defined for every $\lambda \in \frac{1}{\ell}\mathbb{Z}$ and $\mu \in \mathbb{C}$, and the function $\mu \longmapsto \mathcal{F}_\alpha f(\lambda, \mu)$ is analytic on whole $\mathbb{C}$ for every $\lambda \in \frac{1}{\ell}\mathbb{Z}$.

P r o o f.
First case : $p = 1$.

$$\int_{\mathbb{K}_\ell} |f(y,\theta)|\, dm_\alpha(y,\theta) \leq \int_{\mathbb{K}_\ell} |\, e^{ay^u} f(y,\theta)|\, dm_\alpha(y,\theta) < +\infty.$$

Second case : $p = +\infty$.

$$\int_{\mathbb{K}_\ell} |f(y,\theta)|\, dm_\alpha(y,\theta) \leq 2^{2\alpha+3}\pi\, \|e^{ay^u} f\|_{\infty\,,\,\alpha} \int_0^{+\infty} e^{-ay^u + 2(\alpha+1)y} dy < +\infty.$$

Third case : $1 < p < +\infty$. We consider the real number p' satisfying $\frac{1}{p} + \frac{1}{p'} = 1$. Using the Hölder inequality, we get

$$\int_{\mathbb{K}_\ell} |f(y,\theta)|\, dm_\alpha(y,\theta) \leq \|e^{ay^u} f\|_{p\,,\,\alpha} \left\{ \int_{\mathbb{K}_\ell} e^{-ap'y^u} dm_\alpha(y,\theta) \right\}^{\frac{1}{p'}}.$$

On the other hand we have

$$\int_{\mathbb{K}_\ell} e^{-ap'y^u} dm_\alpha(y,\theta) \leq 2^{2\alpha+3}\pi \int_0^{+\infty} e^{-ap'y^u + 2(\alpha+1)y} dy < +\infty.$$

Consequently we have $\quad f \in L^1(m_\alpha)$.

By virtue of relation (4) and the fact that $(1+y)e^{-(\alpha+1)y} \leq 1$, for all $y > 0$, we can write

$$\int_{\mathbb{K}_\ell} |f(y,\theta)\varphi_{-\lambda\,,\,\mu}(y,\theta)| dm_\alpha(y,\theta) \leq C \int_{\mathbb{K}_\ell} |f(y,\theta)|\, e^{|\Im \mathrm{m}\mu| y} dm_\alpha(y,\theta)\,,$$

where C is a positive constant.

By the same manner as above we show, for all $\lambda \in \dfrac{1}{\ell}\mathbb{Z}$ and $\mu \in \mathbb{C}$, that we have

$$\int_{\mathbb{K}_\ell} |f(y,\theta)\varphi_{-\lambda\,,\mu}(y,\theta)|dm_\alpha(y,\theta) < +\infty\,.$$

Let us now to prove the analyticity of the function $\mu \longmapsto \mathcal{F}_\alpha f(\lambda,\mu)$ on $\mathbb{C}$. We have, for all $(y,\theta) \in \mathbb{K}_\ell$ and $\lambda \in \dfrac{1}{\ell}\mathbb{Z}$, the function $\mu \longmapsto \varphi_{-\lambda\,,\mu}(y,\theta)$ is analytic on $\mathbb{C}$ (see [13], Corollary 1.1). Again by a same manner as above, we prove that, for all $\mu_0 > 0$, the function $\mu \longmapsto \mathcal{F}_\alpha f(\lambda,\mu)$ is analytic on the strip $\{\mu \in \mathbb{C} \quad | \quad |\Im\mathrm{m}\mu| < \mu_0\}$.

This completes the proof of the proposition. ∎

Theorem 1. *Let $p,q \in [1,+\infty]$, $a,b \in]0,+\infty[$ and let u,v be two real numbers such that $u > 2$ and $\dfrac{1}{u}+\dfrac{1}{v} = 1$. Assume that f is a measurable function on $\mathbb{K}_\ell$ satisfying:*

i) $e^{ay^u} f \in L^p(m_\alpha)$,

ii) for all $\lambda \in \dfrac{1}{\ell}\mathbb{Z}$, $e^{b|\mu|^v}\mathcal{F}_\alpha f(\lambda\,,.)_{|\mathbb{R}} \in L^q_\star(|c_{\alpha\,,\lambda}(\mu)|^{-2}d\mu)$.

If $(au)^{1/u}(bv)^{1/v} > \left(\sin\dfrac{\pi(v-1)}{2}\right)^{1/v}$, then f is null almost everywhere.

P r o o f. As in the proof of Proposition 4, we have for all $(\lambda,\mu) \in \dfrac{1}{\ell}\mathbb{Z}\times\mathbb{C}$,

$$|\mathcal{F}_\alpha f(\lambda,\mu)| \leq C\int_{\mathbb{K}_\ell} |f(y,\theta)|\,e^{|\Im\mathrm{m}\mu|y}dm_\alpha(y,\theta)\,, \tag{6}$$

where C is a positive constant. Choose

$$\delta \in \left](bv)^{-1/v}\left(\sin\frac{\pi(v-1)}{2}\right)^{1/v}\;,\;(au)^{1/u}\right[.$$

Applying the convex inequality $|\xi\tau| \leq \dfrac{1}{u}|\xi|^u + \dfrac{1}{v}|\tau|^v$ to the real numbers δy and $\dfrac{\Im\mathrm{m}\mu}{\delta}$, we get

$$|\Im\mathrm{m}\mu|\,y \leq \frac{\delta^u y^u}{u} + \frac{|\Im\mathrm{m}\mu|^v}{v\delta^v}. \tag{7}$$

Next, by combining the relations (6) and (7) we obtain

$$|\mathcal{F}_\alpha f(\lambda,\mu)| \le C e^{\frac{|\Im\mathrm{m}\mu|^v}{v\delta^v}} \int_{\mathbb{K}_\ell} |f(y,\theta)|\, e^{\frac{\delta^u y^u}{u}} dm_\alpha(y,\theta).$$

Put

$$I = \int_{\mathbb{K}_\ell} |f(y,\theta)|\, e^{\frac{\delta^u y^u}{u}} dm_\alpha(y,\theta).$$

Thus we have

$$I = \int_{\mathbb{K}_\ell} |e^{ay^u} f(y,\theta)|\, e^{(\frac{\delta^u}{u}-a)y^u} dm_\alpha(y,\theta).$$

Consider the function ψ_δ defined on $\mathbb{K}_\ell$ by $\psi_\delta(y,\theta) = e^{(\frac{\delta^u}{u}-a)y^u}$. Taking account that $\frac{\delta^u}{u} < a$, we can assert that $\psi_\delta(y,\theta) \in L^p(m_\alpha)$ for all $p \in [1,+\infty]$. Take $p' \in [1,+\infty]$ such that $\frac{1}{p} + \frac{1}{p'} = 1$. It is easy to see that

$$I \le \|e^{ay^u} f\|_{p\,,\alpha}\, \|\psi_\delta\|_{p'\,,\alpha}.$$

Using this last inequality we can assert that we have

$$\forall\lambda \in \frac{1}{\ell}\mathbb{Z}\,,\ \forall\mu\in\mathbb{C}\,, \qquad |\mathcal{F}_\alpha f(\lambda,\mu)| \le k\, e^{\frac{|\Im\mathrm{m}\mu|^v}{v\delta^v}}\,, \tag{8}$$

where k is a positive constant.

We have $1 < v < 2$ and $b > \frac{1}{v\delta^v}\sin\frac{\pi(v-1)}{2}$. Moreover, for all $\lambda \in \frac{1}{\ell}\mathbb{Z}$, the function $\mu \longmapsto \mathcal{F}_\alpha f(\lambda,\mu)$ is analytic on $\mathbb{C}$. The condition ii) and the relation (8) allow us to assert that $\mathcal{F}_\alpha f = 0$, by using Lemma 2. Finally, by applying 2) of Proposition 3, we find that $f = 0$ almost everywhere. ∎

In the end of this section we shall prove, in the particular case where $\alpha = \frac{1}{2}$ and ℓ is even, that the condition

$$(au)^{1/u}(bv)^{1/v} > \left(\sin\frac{\pi(v-1)}{2}\right)^{1/v}$$

in Theorem 1 is sharp.

For this goal we need the following proposition proved in [3].

PROPOSITION 5. (See [3], Proposition 3.1.) *Let* $p, q \in [1, +\infty]$, $a > 0$, $b > 0$, *and let* u *and* v *be positive real numbers satisfying* $u > 2$ *and* $\frac{1}{u} + \frac{1}{v} = 1$. *If*

$$(au)^{1/u}(bv)^{1/v} \leq \left(\sin \frac{\pi(v-1)}{2}\right)^{1/v},$$

then there are infinity many even measurable functions on $\mathbb{R}$ *satisfying the conditions*

$$e^{a|y|^u} f \in L^p(\mathbb{R}) \qquad \text{and} \qquad e^{b|\mu|^v} \widehat{f} \in L^q(\mathbb{R}),$$

where $\widehat{f}$ *is the classical Fourier transform on* $\mathbb{R}$.

THEOREM 2. *Let* $p, q \in [1, +\infty]$, $a, b \in]0, +\infty[$ *and let* u, v *be two real numbers such that* $u > 2$ *and* $\frac{1}{u} + \frac{1}{v} = 1$. *Assume that*

$$(au)^{1/u}(bv)^{1/v} < \left(\sin \frac{\pi(v-1)}{2}\right)^{1/v}.$$

If ℓ *is even, then there exists a nonzero measurable function* f *on* $\mathbb{K}_\ell$ *satisfying the conditions:*

i) $e^{ay^u} f \in L^p(m_{1/2})$,

ii) $e^{b|\mu|^v} \mathcal{F}_{1/2} f(\lambda, .)_{|\mathbb{R}} \in L^q_\star(|c_{1/2,\lambda}(\mu)|^{-2} d\mu)$, *for all* $\lambda \in \frac{1}{\ell}\mathbb{Z}$.

P r o o f. Let a', a'' and b' be real numbers such that $a' > a'' > a$, $b' > b$, and

$$(a'u)^{1/u}(b'v)^{1/v} < \left(\sin \frac{\pi(v-1)}{2}\right)^{1/v}.$$

From Proposition 5, there exists a nonzero even measurable function h on $\mathbb{R}$ such that

$$e^{2a'|y|^u} h \in L^p(\mathbb{R}) \qquad \text{and} \qquad e^{b'|\mu|^v} \widehat{h} \in L^q(\mathbb{R}).$$

Choose k an infinitely differentiable function compactly supported and odd on $\mathbb{R}$. Let $g = h \star k$ be the classical convolution product of h and k. g is an odd function on $\mathbb{R}$. Since $\widehat{k}$ is bounded on $\mathbb{R}$ we have

$$e^{b'|\mu|^v} \widehat{g} \in L^q(\mathbb{R}). \tag{9}$$

Suppose that $\mathrm{Supp} k \subset [-A, A]$, $A > 0$. For $p = +\infty$, by using the fact that the function $\xi \longmapsto e^{2a''(\xi + A)^u} e^{-2a'\xi^u}$ is bounded on $[0, +\infty[$ we conclude that

$e^{2a''|y|^u}g \in L^\infty(\mathbb{R})$. For $1 \leq p < +\infty$, the generalized Minkowski inequality (see [12], page 21) and the fact that the function $\xi \longmapsto e^{2pa''(\xi+A)^u}e^{-2pa'\xi^u}$ is bounded on $[0,+\infty[$ allow us to conclude again that $e^{2a''|y|^u}g \in L^p(\mathbb{R})$. In all cases we have

$$e^{2a|y|^u}g \in L^p(\mathbb{R}). \tag{10}$$

Take the function f defined on $\mathbb{K}_\ell$ by

$$f(y,\theta) = \frac{e^{i\theta/2}g(y)(\mathrm{ch}y)^{1/2}}{\mathrm{sh}2y}.$$

It is easy to check, by using (10), that we have

$$e^{ay^u}f \in L^p(m_{1/2}).$$

According to (5) and (1) we can write, for all $\lambda \in \frac{1}{\ell}\mathbb{Z}$ and all $\mu \in \mathbb{R}$,

$$\mathcal{F}_{1/2}f(\lambda,\mu) = 4\left(\int_{-\ell\pi}^{\ell\pi} e^{i(1/2-\lambda)\theta}d\theta\right)\left(\int_0^{+\infty} g(y)\varphi_\mu^{(1/2\,,\,\lambda)}(y)\mathrm{sh}y(\mathrm{ch}y)^{\lambda+1/2}dy\right).$$

Thus it follows that, for all $\lambda \neq \frac{1}{2}$ and all $\mu \in \mathbb{R}$, $\mathcal{F}_{1/2}f(\lambda,\mu) = 0$.

On the other hand we have

$$\mathcal{F}_{1/2}f(1/2,\mu) = 4\ell\pi\int_0^{+\infty} g(y)\varphi_\mu^{(1/2\,,\,1/2)}(y)\mathrm{sh}2y dy\,,$$

where $\varphi_\mu^{(1/2\,,\,1/2)}$ is the Jacobi function which is the unique solution of the following initial problem

$$\begin{cases} \dfrac{d^2\psi}{dy^2} + 4\dfrac{\mathrm{ch}2y}{\mathrm{sh}2y}\dfrac{d\psi}{dy} \quad = \quad -(\mu^2+4)\psi \\ \\ \psi(0) = 1 \qquad \text{and} \qquad \psi'(0) = 0 \end{cases}.$$

Hence we have

$$\forall y > 0\,, \qquad \varphi_\mu^{(1/2\,,\,1/2)}(y) = \frac{2\sin\mu y}{\mu\mathrm{sh}2y}\,,$$

then, since g is odd, we get

$$\mathcal{F}_{1/2}f(1/2,\mu) = \frac{4\ell\pi}{\mu}\widehat{g}(\mu).$$

Furthermore, a straightforward calculation, using well known formulas of gamma function, gives us

$$|c_{1/2\,,\,1/2}|^{-2} = \frac{\mu^2}{4}.$$

Thus, by using the relation (9), we obtain

$$\forall \lambda \in \frac{1}{\ell}\mathbb{Z}\,, \quad e^{b|\mu|^v}\mathcal{F}_{1/2}f(\lambda\,,\,.)_{|\mathbb{R}} \in L^q_\star(|c_{1/2\,,\,\lambda}(\mu)|^{-2}d\mu).$$

■

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