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ASYMPTOTIC PROPERTY OF EIGENVALUES AND EIGENFUNCTIONS OF THE LAPLACE OPERATOR IN DOMAIN WITH A PERTURBED BOUNDARY

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Abstract

In this paper, we consider the variations of eigenvalues and eigenfunctions for the Laplace operator with homogeneous Dirichlet boundary conditions under deformation of the underlying domain of definition. We derive recursive formulas for the Taylor coefficients of the eigenvalues as functions of the shape-perturbation parameter and we establish the existence of a set of eigenfunctions that is jointly holomorphic in the spatial and boundary-variation variables. Using integral equations, we show that these eigenvalues are exactly built with the characteristic values of some meromorphic operator-valued functions.

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1. Introduction

Let Ω be a bounded, open subset of $\mathbf{R}^3$, with smooth boundary $\partial\Omega$. We suppose that this boundary $\partial\Omega$ is parameterized by the function: $\gamma(s,t)$: $[0,\pi] \times [0,2\pi] \to \mathbf{R}^3$ which is analytic, π-periodic in the variable s, and 2π-periodic in the variable t.

Let us consider the following unperturbed eigenvalue problem for the Laplace operator in the domain Ω of the three-dimensional space $\mathbf{R}^3$:

$$-\Delta u_0(x) = \lambda_0^2 u_0(x), \quad x \in \Omega, \quad \text{and} \quad u_0(x) = 0, \quad x \in \partial\Omega. \qquad (1)$$

Throughout this paper, the domain Ω is supposed to be perturbed according to some parameter ϵ and therefore the problem (1) is transformed into the problem (2) as described in Section 2. The properties of eigenvalue problems under shape deformation have been the subject of comprehensive studies [11, 20] and the area continues to carry great importance to this day [3, 6, 8, 9, 13, 14, 15, 21]. A substantial portion of these investigations relate to properties of smoothness and analyticity of eigenvalues and eigenfunctions with respect to perturbations. Bruno and Reitich have presented in [5, Theorem 2, p.172 and Section 3, pp.180-183] some explicit constructions of high-order boundary perturbation expansions for eigenelements in two dimensions. Their algorithm is based on certain properties of joint analytic dependence on the boundary perturbations and spatial variables of the eigenfunctions. But, the main difficulty in solving eigenvalue problems relates to the continuation of multiple eigenvalues of the unperturbed configuration. These eigenvalues may evolve, under shape deformation, as separated, distinct eigenvalues, and this "splitting" may only become apparent at high orders in their Taylor expansion. In [18], Ozawa derived the leading-order term in the asymptotic expansions of simple eigenvalues in domain with a small hole. Nevertheless, in our paper we remove the condition that eigenvalue is simple and provide more accurate asymptotic expansions for eigenvalues and eigenfunction in domain with more general shape. Recently, Lamberti and Lanza de Cristoforis have developed in [14] some preliminary abstract results for the dependence of the eigenvalues upon perturbation. This perturbation is in the form of homeomorphic images $\phi(\Omega)$ of Ω by some homeomorphism ϕ of Ω onto $\phi(\Omega)$. Their applications to the Dirichlet eigenvalue problem for the Laplace operator appear clearly in Section 3 of their paper and in Theorem 3.21 they justify the analyticity result for some symmetric functions of eigenvalues. Our analysis and uniform asymptotic formulas of the eigenvalues and the eigenfunctions, which are represented as sum of a single-layer potential and of a double-layer potential involving the Green function, are considerably different from those in [13, 14]. Next, Our method differ, essentially, from the classical methods used to study the analytic dependence of the eigenfunctions of a real or complex parameter and used to give the asymptotic formulae for the eigenvalues. In this paper we show, by using surface potentials, that if the multiplicity m of λ_0^2 is greater than one, the square roots of eigenvalues are exactly the characteristic values of some meromorphic operator-valued functions that are of Fredholm type with index 0. We then proceed from the results, the definitions and notations found in [10, 16] to construct their asymptotic expressions for ϵ sufficiently small. We would like to find an efficient and a method, different to what we have presented in [2, Section 4, pp.802-817] when we have studied the eigenvalue problem in the presence of a finite number of "imperfections"(the resonant frequencies are exactly the eigenvalues).

The paper is organized as follows. Section 2 provides the formulation of the main problem in this paper and the well known result concerning the analyticity of eigenvalues of the perturbed eigenvalue problem with respect to shape deformation parameter ϵ. Section 3 contains the application of the integral equations method to the Dirichlet eigenvalue problem for the Laplace operator. In particular, we rigorously establish the existence of an operator-valued function L_ϵ and we establish that this operator define complex analytic functions of the spatial variable x and the height parameter ϵ. Finally, in Section 4, we give recursive formula to compute the Taylor coefficients in the expansion of the normal derivatives $q^{(j)}(\epsilon)$ and in the expansion of the square roots of eigenvalues (see Theorem 4.1).

In Theorem 4.2 we show that the eigenfunctions $u_j(\epsilon)$ of problem (2) are jointly analytic in (x, ϵ) and satisfy an asymptotic expansion where its Taylor coefficients are deduced from those of $q^{(j)}(\epsilon)$. As we shall see in a forthcoming paper, the same result can be proved when a finite number of "imperfections" of small diameter ϵ and "nearly touching" the boundary are imbedded in the domain of definition Ω.

2. Problem formulation

We introduce the analytic function $\beta : [0, \pi] \times [0, 2\pi] \to \mathbf{R}^3$, $(s, t) \mapsto \beta(s, t)$ to be π-periodic in the variable s and 2π-periodic in the variable t. Let

$$\gamma_\epsilon(s, t) = \gamma(s, t) + \epsilon\beta(s, t), \quad \epsilon \in \mathbf{R}.$$

With this definition, $(s, t; \epsilon) \mapsto \gamma_\epsilon(s, t)$ is an analytic function on $[0, \pi] \times [0, 2\pi] \times \mathbf{R}$, π-periodic in the variable s, 2π-periodic in the variable t.

Consider the bounded domain Ω_ϵ in $\mathbf{R}^3$ with boundary $\partial\Omega_\epsilon$ parameterized by the function $\gamma_\epsilon(s,t)$:

$$\partial\Omega_\epsilon = \{\gamma_\epsilon(s,t), \quad (s,t) \in [0,\pi] \times [0,2\pi]\}.$$

The outward unit normal to $\partial\Omega_\epsilon$ is denoted by ν_ϵ and for $\epsilon = 0$ we have $\Omega_0 \equiv \Omega$.

In this paper, we deal with the asymptotics of eigenvalues and eigenfunctions associated with the following eigenvalue problems

$$\begin{cases} -\Delta u(\epsilon) = \lambda^2(\epsilon) u(\epsilon) & \text{in } \Omega_\epsilon, \\ u(\epsilon) = 0 & \text{on } \partial\Omega_\epsilon. \end{cases} \tag{2}$$

It is well known that the operator $-\Delta$ on $L^2(\Omega_\epsilon)$ with domain $H^2(\Omega_\epsilon) \cap H_0^1(\Omega_\epsilon)$ is self-adjoint with compact resolvent. Consequently, its spectrum consists entirely of isolated, real and positive eigenvalues with finite multiplicity, and there are corresponding eigenfunctions which make up an orthonormal basis of $L^2(\Omega_\epsilon)$. Throughout this paper, we denote by $\|\cdot\|$ the norm associated to the scalar product $< \cdot, \cdot >$ on $L^2(\Omega)$, and $|||\cdot|||$ the euclidian norm on $\mathbf{R}^3$.

Let $\lambda_0^2 > 0$ denote an eigenvalue of the eigenvalue problem (2) for $\epsilon = 0$ with geometric multiplicity $m \geq 1$. There exists a small constant $r_0 > 0$ such that λ_0^2 is the unique eigenvalue of (2) for $\epsilon = 0$ in the set $\{\lambda^2, \lambda \in D_{r_0}(\lambda_0)\}$, where $D_{r_0}(\lambda_0)$ is a disc of center λ_0 and radius r_0. Let us call the λ_0-group *the totality of the perturbed eigenvalues of* (2) for $\epsilon > 0$ generated by splitting from λ_0. The following analyticity result is well-known [19, §§II.2 and II.6] or [10, §VII.6].

Theorem 2.1. *There exists $\epsilon_0 > 0$ such that for $|\epsilon| < \epsilon_0$, the λ_0-group consists of m eigenvalues, $\lambda_j^2(\epsilon), j = 1, \dots, m$ (repeated according to their multiplicity). Moreover, they are analytic functions with respect to ϵ satisfying $\lambda_j^2(0) = \lambda_0^2, j = 1, \dots, m$. The normalized eigenfunctions associated to the λ_0-group of eigenvalues are analytic with respect to ϵ and their values at $\epsilon = 0$ are m linearly independent solutions of the unperturbed eigenvalue problem.*

The classical regularity results and the previous theorem imply that the eigenfunctions associated to the λ_0-group of eigenvalues are separately analytic in the small parameter ϵ and the spatial variable x. Using an integral equation technique we will also establish, in Section 4, the joint analytic dependence of these functions with respect to (x, ϵ).

3. Integral equation method

The use of integral equations is a convenient tool for a number of investigations. We now develop a boundary integral formulation for solving the eigenvalue problem (2). This method it is used to characterize the eigenvalues (respectively the normal derivatives of the associated eigenfunctions) as characteristic values (respectively as root functions) of some operator-valued function.

Let v be the solution to the following Helmholtz equation:

$$\Delta v + \lambda^2 v = 0, \quad \text{ in } \mathbf{R}^3. \tag{3}$$

We begin by defining the outgoing Green's function $G(x, y)$ as the solution of

$$\Delta_x G(x, y) + \lambda^2 G(x, y) = -\delta_y(x), \quad \text{ in } \mathbf{R}^3,$$

with the radiation condition as $|||x||| \to +\infty$:

$$\Big|\frac{\partial G}{\partial |||x|||} - i\lambda G\Big| = O(\frac{1}{|||x|||^2}).$$

In fact G is explicitly given as:

$$G(x, y) = \frac{e^{i\lambda|||x-y|||}}{4\pi|||x-y|||}. \tag{4}$$

3.1. Preliminary results

Consider the equation (3) for the function v in the exterior of Ω_ϵ, multiply by the Green's function G and integrate by parts, we get that for $x \in \mathbf{R}^3 \backslash \overline{\Omega_\epsilon}$,

$$v(x) = \int_{\partial\Omega_\epsilon} \Big(\frac{\partial v}{\partial \nu_\epsilon}\Big|_+(y) G(x, y) - v(y)\frac{\partial G}{\partial \nu_\epsilon(y)}\Big|_+(x, y)\Big) d\sigma(y).$$

The jump condition for $\dfrac{\partial v}{\partial \nu_\epsilon}$ on $\partial\Omega_\epsilon$ yields

$$v(x) = -\int_{\partial\Omega_\epsilon} \frac{\partial G}{\partial \nu_\epsilon(y)}\Big|_+(x, y) v(y) d\sigma(y) + \int_{\partial\Omega_\epsilon} G(x, y)\frac{\partial v}{\partial \nu_\epsilon}\Big|_-(y) d\sigma(y).$$

Of course, the above equations does not hold up to the boundary of Ω_ϵ, but if we take the limit as $x \to \partial\Omega_\epsilon$, we get from for instance [7] that

$$\frac{1}{2} v\big|_{\partial\Omega_\epsilon}(x) = -\int_{\partial\Omega_\epsilon} \frac{\partial G}{\partial \nu_\epsilon(y)}\Big|_+(x, y) v(y) d\sigma(y) \tag{5}$$

$$+\int_{\partial\Omega_\epsilon} G(x,y)\frac{\partial v}{\partial \nu_\epsilon}\big|_{-}(y)d\sigma(y) \text{ for } x\in\partial\Omega_\epsilon.$$

We introduce the following operator, called single-layer potential:

$$Sl(\lambda)g(x) = \int_{\partial\Omega_\epsilon} G(x,y)g(y)d\sigma(y), \quad x \in \mathbf{R}^3 \setminus \partial\Omega_\epsilon.$$

For $g \in \mathcal{C}^\infty(\partial\Omega_\epsilon)$, or even $g \in L^1(\partial\Omega_\epsilon)$, the function $Sl(\lambda)g$ is well-defined and smooth for $x \in \mathbf{R}^3 \setminus \partial\Omega_\epsilon$.

Now define the following operator

$$S(\lambda) : H^{-1/2}(\partial\Omega_\epsilon) \to H^{1/2}(\partial\Omega_\epsilon)$$

where

$$S(\lambda) : g \to \int_{\partial\Omega_\epsilon} G(\cdot,y)g(y)d\sigma(y).$$

For such g and every $x \in \partial\Omega_\epsilon$, we denote by $g_+(x)$ and $g_-(x)$ the limits of $g(y)$ as $y \to x$, from $y \in \Omega_\epsilon$ and $y \in \mathbf{R}^3 \setminus \overline{\Omega_\epsilon}$, respectively, when these limits exist. It is a well-known classical result that, for $x \in \partial\Omega_\epsilon$,

$$(Sl(\lambda)g)_+(x) = (Sl(\lambda)g)_-(x) = S(\lambda)g(x),$$

where $S(\lambda)$ is pseudo-differential operator of order -1.

Throughout this paper, we use for simplicity the notation

$$H^\varrho_\sharp(]0,\pi[\times]0,2\pi[) = H^\varrho(\mathbf{R}^2/]0,\pi[\times]0,2\pi[), \quad \text{for } \varrho \in \mathbf{R},$$

where $H^\varrho(\mathbf{R}^2/]0,\pi[\times]0,2\pi[)$ denotes the classical Sobolev H^ϱ-space on the quotient $\mathbf{R}^2/]0,\pi[\times]0,2\pi[$ (cf. e.g Adams [1]).

Using change of variables and integral equations, the following result immediately holds (see [23]).

Proposition 3.1. *Let*

$$L_\epsilon(\lambda) : H^{-1/2}_\sharp(]0,\pi[\times]0,2\pi[) \to H^{1/2}_\sharp(]0,\pi[\times]0,2\pi[)$$

be defined as follows:

$$\begin{aligned} L_\epsilon(\lambda)f(s,t) =& \Big(S(\lambda)f(\gamma_\epsilon^{-1})\Big)(\gamma_\epsilon(s,t)) \\ =& \int_0^{2\pi}\int_0^\pi G(\gamma_\epsilon(s,t),\gamma_\epsilon(s',t'))|\nabla\gamma_\epsilon(s',t')|f(s',t')ds'dt' \end{aligned}$$

for $f \in H_{\sharp}^{-1/2}(]0,\pi[\times]0,2\pi[)$. *Then the operator-valued function* $L_\epsilon(\lambda)$ *is Fredholm analytic with index* 0 *in* $\mathbf{C} \setminus i\mathbf{R}^-$. *Moreover,* $L_\epsilon^{-1}(\lambda)$ *is a meromorphic function and its poles are in* $\{\Im(z) \le 0\}$, *where* $\Im(z)$ *means the imaginary part of* z *and* $\Re(z)$ *is the real part.*

From now we will focus our attention on solving the eigenvalue problem (2).

3.2. Joint analyticity of kernel

We first show that the kernel of the operator $L_\epsilon(\lambda)$ has the following form.

LEMMA 3.1. *There exist positive numbers* $\epsilon_1 = \epsilon_1(\epsilon_0)(\epsilon_1 \le \epsilon_0)$, ρ, η, *and* r_0 *such that the kernel of the operator* $L_\epsilon(\lambda)$ *has the form:*

$$G(\gamma_\epsilon(s,t),\gamma_\epsilon(s',t'))|\nabla\gamma_\epsilon(s',t')|$$
$$= \sum_{p=-1}^{1}\sum_{k=-1}^{1} \frac{h(s,t,s'+k\pi,t'+2p\pi,\epsilon,\lambda)}{\sqrt{(t-(t'+2p\pi))^2+(s-(s'+k\pi))^2}} + g(s,t,s',t'\epsilon,\lambda),$$

for $(s,t,s',t',\epsilon,\lambda) \in \mathcal{J}$, *where* $h(s,t,s',t',\epsilon,\lambda)$ *and* $g(s,t,s',t',\epsilon,\lambda)$ *are analytic with respect to* $(s,t,s',t',\epsilon,\lambda)$ *in* $\mathcal{J}$. *Here we have put* $\mathcal{J} = \{|\Im(s)| \le \eta, |\Im(t)| \le \eta; |\Im(s')| \le \eta, |\Im(t')| \le \eta; |\epsilon| \le \epsilon_1; \lambda \in D_{r_0}(\lambda_0); -\rho \le \Re(s), \Re(s') \le \pi+\rho; -\rho \le \Re(t), \quad \Re(t') \le 2\pi+\rho\}$.

P r o o f. Upon replacing x by $\gamma_\epsilon(s,t)$ and x' by $\gamma_\epsilon(s',t')$, we immediately obtain the following result for the kernel of L_ϵ, provided ϵ_1, ρ and η are sufficiently small (r_0 is given in Section 2):

$$\frac{1}{4\pi}\frac{e^{i\lambda|||\gamma_\epsilon(s,t)-\gamma_\epsilon(s',t')|||}}{|||\gamma_\epsilon(s,t)-\gamma_\epsilon(s',t')|||} = \frac{h(s,t,s',t',\epsilon,\lambda)}{((s-s')^2+(t-t')^2)^{1/2}},$$

where h is a function defined in the set $\mathcal{J}$. In fact, we have

$$h(s,t,s',t',\epsilon,\lambda) = G(\gamma_\epsilon(s,t),\gamma_\epsilon(s',t'))|\nabla\gamma_\epsilon(s',t')|((s-s')^2+(t-t')^2)^{1/2}.$$

Using classical results, and the fact γ_ϵ is analytic, we see that the function h and its derivatives are analytic in the set $\mathcal{J}$.

To proceed to the proof, we use some idea little close to that found in the proof of Theorem 6.1 in [4, pp.331-333]. The fact that γ_ϵ is $\pi-$periodic in the variable s' and $2\pi-$ periodic in the variable t', there exists a function $g(s,t,s',t',\epsilon,\lambda)$ such that

$$\frac{h(s,t,s',t',\epsilon,\lambda)}{((s-s')^2+(t-t')^2)^{1/2}}$$
$$=\sum_{p=-1}^{1}\sum_{k=-1}^{1}\frac{h(s,t,s'+k\pi,t'+2p\pi,\epsilon,\lambda)}{\sqrt{(t-(t'+2p\pi))^2+(s-(s'+k\pi))^2}}+g(s,t,s',t',\epsilon,\lambda),$$

where this function g_ϵ is given by:

$$g(s,t,s',t',\epsilon,\lambda)=-\Big(\sum_{k=-1}^{1}\frac{h(s,t,s'+k\pi,t'-2\pi,\epsilon,\lambda)}{\sqrt{(t-(t'+2\pi))^2+(s-(s'+k\pi))^2}} \tag{6}$$
$$+\sum_{k=-1}^{1}\frac{h(s,t,s'+k\pi,t'+2\pi,\epsilon,\lambda)}{\sqrt{(t-(t'-2\pi))^2+(s-(s'+k\pi))^2}}\Big).$$

The analyticity of the function g follows, evidently, from that of h. ■

With the result and notation established in Lemma 3.1 we now state the main results in this section.

THEOREM 3.1. *There exists a constant $\eta>0$ and a complex neighborhood $\mathcal{V}$ of* 0 *such that for every function $\phi(s,t;\epsilon)\in H_\sharp^{-1/2}(]0,\pi[\times]0,2\pi[)$ analytic in $(s,t;\epsilon)\in\big\{|\Im(s)|,|\Im(t)|\le\eta\big\}\times\mathcal{V}$, the function $L_\epsilon(\lambda)\phi(s,t;\epsilon)\in H_\sharp^{1/2}(]0,\pi[\times]0,2\pi[)$ is analytic with respect to $(s,t;\epsilon,\lambda)\in\big\{|\Im(s)|,|\Im(t)|\le\eta\big\}\times\mathcal{V}\times D_{r_0}(\lambda_0)$ where $D_{r_0}(\lambda_0)$ is a disc of center λ_0 and radius r_0.*

P r o o f. We find a central difficulty to prove the analytic property of the operator L_ϵ. This difficulty comes from the spatial regularity of its kernel. In order to establish this regularity we may focus, for simplicity, our attention to the change of variables when we integrate by parts as in [4, Lemma 6.2].

From Lemma 3.1 we have,

$$L_\epsilon(\lambda)f(s,t)=\sum_{p=-1}^{1}\int_0^{2\pi}\int_0^{\pi}\sum_{k=-1}^{1}\frac{h(s,t,s'+k\pi,t'+2p\pi,\epsilon,\lambda)}{\sqrt{(t-(t'+2p\pi))^2+(s-(s'+k\pi))^2}}$$
$$\times f(s',t')ds'dt'+\int_0^{2\pi}\int_0^{\pi}g(s,t,s',t',\epsilon,\lambda)f(s',t')ds'dt'.$$

We define

$$F(s,t,\epsilon,\lambda)$$

$$= \sum_{p=-1}^{1} \int_0^{2\pi} \int_0^{\pi} \sum_{k=-1}^{1} \frac{h(t,s,t'+2p\pi,s'+k\pi,\epsilon,\lambda)}{\sqrt{(t-(t'+2p\pi))^2+(s-(s'+k\pi))^2}} f(s',t')ds'dt'$$

and

$$\mathcal{G}(s,t,\epsilon,\lambda) = \int_0^{2\pi} \int_0^{\pi} g(s,t,s',t',\epsilon,\lambda) f(s',t')ds'dt'.$$

Obviously the relation (6) implies that the analyticity of $\mathcal{G}(s,t,\epsilon,\lambda)$ is deduced from that of the following functions:

$$(s,t,\epsilon,\lambda) \mapsto \sum_{k=-1}^{1} \int_0^{2\pi} \int_0^{\pi} \frac{h(s,t,s'+k\pi,t'+2\pi,\epsilon,\lambda)}{\sqrt{(t-(t'+2\pi))^2+(s-(s'+k\pi))^2}} ds'dt',$$

$$\sum_{k=-1}^{1} \int_0^{2\pi} \int_0^{\pi} \frac{h(s,t,s'+k\pi,t'+2\pi,\epsilon,\lambda)}{\sqrt{(t-(t'-2\pi))^2+(s-(s'+k\pi))^2}} ds'dt'.$$

Now, it suffices to verify the analyticity for the first function in the last line. But for the second function, we deduce the result by the same manner. To do this, we introduce the function

$$G_1(s,t,t',\epsilon,\lambda) = \sum_{k=-1}^{1} \int_0^{\pi} \frac{h(s,t,s'+k\pi,t'+2\pi,\epsilon,\lambda)}{\sqrt{(t-(t'+2\pi))^2+(s-(s'+k\pi))^2}} ds',$$

and by a change of variables, we get

$$G_1(s,t,t',\epsilon,\lambda) = \int_{-\pi}^{2\pi} \frac{h(s,t,s',t'+2\pi,\epsilon,\lambda)}{\sqrt{(t-(t'+2\pi))^2+(s-s')^2}} ds'.$$

Further, if we define the function

$$K_1(s,t,s',t',\epsilon,\lambda) = \int_s^{s'} h(s,t,z,t'+2\pi,\epsilon,\lambda)dz,$$

integration by parts yields

$$\begin{aligned} G_1(s,t,t',\epsilon,\lambda) &= \big((t-t'-2\pi)^2+(s-2\pi)^2\big)^{-1/2} K_1(s,t,2\pi,t',\epsilon,\lambda) \\ &-\big((t-t'-2\pi)^2+(s+\pi)^2\big)^{-1/2} K_1(s,t,-\pi,t',\epsilon,\lambda) \\ &+\int_{-\pi}^{2\pi} \big((t-t'-2\pi)^2+(s-s')^2\big)^{-3/2} K_1(s,t,s',t',\epsilon,\lambda)ds'. \end{aligned}$$

Clearly, the function $(s,t,\epsilon,\lambda) \to \int_0^{2\pi} G_1(s,t,t',\epsilon,\lambda)dt'$ can be extended to a complex analytic function in $\mathbf{C} \times \mathbf{C} \times \mathcal{V} \times D_{r_0}(\lambda_0)$ and so the analyticity of $\mathcal{G}(s,t,\epsilon,\lambda)$ holds. We now prove the result for the function $F(s,t,\epsilon,\lambda)$. As was done for the proof of $\mathcal{G}$, we first define

$$F_1(s,t,t',\epsilon,\lambda) = \sum_{k=-1}^{1} \int_0^{\pi} \frac{h(s,t,s'+k\pi,t'+2p\pi,\epsilon,\lambda)}{\sqrt{(t-(t'+2p\pi))^2+(s-(s'+k\pi))^2}} ds'.$$

By a change of variables we get

$$F_1(s,t,t',\epsilon,\lambda) = \sum_{k=-1}^{1} \int_{-\pi}^{2\pi} \frac{h(s,t,s',t'+2p\pi,\epsilon,\lambda)}{\sqrt{(t-(t'+2p\pi))^2+(s-s')^2}} ds'.$$

Therefore,

$$F(s,t,\epsilon,\lambda) = \sum_{p=-1}^{1} \int_0^{2\pi} \int_{-\pi}^{2\pi} \frac{h(s,t,s',t'+2p\pi,\epsilon,\lambda)}{\sqrt{(t-(t'+2p\pi))^2+(s-s')^2}} ds'dt'.$$

This allows us to introduce the following function

$$F_2(s,t,s',\epsilon,\lambda) = \sum_{p=-1}^{1} \int_0^{2\pi} \frac{h(s,t,s',t'+2p\pi,\epsilon,\lambda)}{\sqrt{(t-(t'+2p\pi))^2+(s-s')^2}} dt',$$

which, by a change of variables, becomes

$$F_2(s,t,s',\epsilon,\lambda) = \int_{-2\pi}^{4\pi} \frac{h(s,t,s',t',\epsilon,\lambda)}{\sqrt{(t-t')^2+(s-s')^2}} dt'.$$

Hence the following relation is valid

$$F(s,t,\epsilon,\lambda) = \int_{-2\pi}^{4\pi} \int_{-\pi}^{2\pi} \frac{h(s,t,s',t',\epsilon,\lambda)}{\sqrt{(t-t')^2+(s-s')^2}} ds'dt',$$

and can be extended to a complex analytic function in $\mathbf{C} \times \mathbf{C} \times \mathcal{V} \times D_{r_0}(\lambda_0)$. ∎

We use some notations and definitions given by Gohberg and Sigal [10] and by Reinhard and Möller [16]. The fact that λ_0 is a characteristic value of $L_0(\lambda)$ implies that from Keldys's theorem, which is simplified in [15, p.462], there exist $\{\phi_0^i : 1 \leq i \leq m\}$ C.S.E.A.V of L_0 at λ_0 and $\{\psi_0^i : 1 \leq i \leq m\}$ C.S.E.A.V of L_0^* such that the following operator

$$A_0 = \frac{1}{2i\pi} \int_{|\lambda-\lambda_0|=\rho} (L_0(\lambda))^{-1} d\lambda = \sum_{i=1}^{m} \phi_0^i \otimes \psi_0^i$$

is well defined on $Ker(L_0(\lambda_0))$.

Analogously, the result of Reinhard and Möller which is due to Keldyš [12], implies that for each characteristic value $\lambda_i(\epsilon)$ $(1 \leq i \leq m)$ there exists $\{\phi_{i,j}(\epsilon) : 1 \leq i \leq m, \quad 1 \leq j \leq m_i\}$ C.S.E.A.V of L_ϵ at $\lambda_i(\epsilon)$ and $\{\psi_{i,j}(\epsilon) : 1 \leq i \leq m, \quad 1 \leq j \leq m_i\}$ C.S.E.A.V of L_ϵ^* such that the following operator

$$A_i(\epsilon) = \frac{1}{2i\pi} \int_{|\lambda-\lambda_i(\epsilon)|=\rho} (L_\epsilon(\lambda))^{-1} d\lambda = \sum_{j=1}^{m_i} \phi_{i,j}(\epsilon) \otimes \psi_{i,j}(\epsilon) \tag{7}$$

is well defined on $Ker(L_\epsilon(\lambda_i(\epsilon)))$. Next, the following operator

$$A(\epsilon) = \sum_{i=1}^{m} A_i(\epsilon), \text{ for } |\epsilon| < \epsilon_1 \tag{8}$$

is well defined.

Based on [4, 22] and on the relation (8) can one see that the operator $A(\epsilon)$ is self-adjoint and holomorphic function with respect to $\epsilon \in]-\epsilon_1, \epsilon_1[$. It is quite easy to see that $A_0 = A(\epsilon = 0)$. In order to prove the results in Section 4, we investigate the properties of the eigenelements corresponding to the operators A_0 and A_ϵ. Then, let $(\mu_0^j)_{1\leq j\leq h}$ (h denotes, here, the number of eigenvalues for the operator A_0) be the family of eigenvalues of the operator A_0 with multiplicity m_j each one. Using the generalization of Theorem 2.1 and [11, 20], we know that there exist $\epsilon_2 = \epsilon_2(\epsilon_1) > 0$ such that for $|\epsilon| < \epsilon_2$ and for $j \in \{1, \cdots, h\}$ the μ_0^j-group consists of m_j eigenvalues of $A(\epsilon)$, $\mu_{j,l}(\epsilon)$, $\quad l = 1, \cdots, m_j$ (repeated according to their multiplicity) and we have $\mu_{j,l}(\epsilon) \to \mu_0^j$ as $\epsilon \to 0$. Let $\epsilon_3 = \inf(\epsilon_1, \epsilon_2)$. For $|\epsilon| < \epsilon_3$, the following projector is well defined:

$$P_j(\epsilon) = \frac{1}{2i\pi} \int_{|\mu-\mu_0^j|=\rho_1} (\mu - A(\epsilon))^{-1} d\mu = \sum_{l=1}^{m_j} \sum_{r=1}^{m_{jl}} q_{l,r}^{(j)}(\epsilon) \otimes q_{l,r}^{(j)}(\epsilon), \tag{9}$$

where for $1 \leq j \leq h$ and for $1 \leq l \leq m_j$, the family $(q_{l,r}^{(j)}(\epsilon))_{1\leq r\leq m_{jl}}$ denotes the orthogonal family of eigenfunctions corresponding to the eigenvalues

$\mu_{j,l}(\epsilon)$. For $\epsilon = 0$, the unperturbed projector is given by:

$$P_j(0) = \frac{1}{2i\pi} \int_{|\mu-\mu_0^j|=\rho_2} (\mu - A(0))^{-1} d\mu = \sum_{l=1}^{m_j} q_l^{(j)}(0) \otimes q_l^{(j)}(0), \qquad (10)$$

where the family $(q_l^{(j)}(0))_{1\leq l\leq m_j}$ is the orthogonal family of eigenfunctions corresponding to the eigenvalue μ_0^j. Now it seems natural, from the previous results, that for all $j = 1, \cdots, h$ the family $(q_l^{(j)}(0))_{1\leq l\leq m_j}$ are m_j-root functions of $L_0(\lambda_0)$ and for all $l = 1, \cdots, m_j$, the family $(q_{l,r}^{(j)}(\epsilon))_{1\leq r\leq m_{jl}}$ are m_{jl}-characteristic functions of $L_\epsilon(\lambda_j(\epsilon))$ and $\sum_{j=1}^h m_j = m$.

4. Analyticity and asymptotic expansion

This section is devoted to study the asymptotic of the eigenvalues and eigenfunctions of (2) when the parameter ϵ goes to zero. We will give a method different from those found in literature [6, 9, 13, 15, 18] to calculate the Taylor coefficients in the expansions of the eigenelements in a neighborhood of zero when the eigenvalue λ_0^2 of $-\Delta$ is not simple. The part (ii) of Theorem 4.1 gives the expansions of the characteristic values $\lambda_j(\epsilon)$, but to deduce the result for the eigenvalues it is enough to take the square of that of the characteristic values.

On the other hand, it is no difficult to see that the following operators are well defined,

$$P(\epsilon) = \sum_{j=1}^{h} P_j(\epsilon), \text{ for } |\epsilon| < \epsilon_3 \text{ and } P(0) = \sum_{j=1}^{h} P_j(0). \qquad (11)$$

The following holds.

PROPOSITION 4.1. *For $|\epsilon| < \epsilon_3$ we have:*

(i) The operator $P(\epsilon)$ is holomorphic for $\epsilon \in]-\epsilon_3, \epsilon_3[$ and $P(\epsilon) = P(0) + R(\epsilon)$, where $R(\epsilon)$ is holomorphic with respect to ϵ.

(ii) $P(\epsilon) = \sum_{j=1}^m q^{(j)}(\epsilon) \otimes q^{(j)}(\epsilon)$ where $(q^{(j)}(\epsilon))_{1\leq j\leq m}$ denotes an orthonormal basis of $KerL_\epsilon(\lambda_j(\epsilon))$. Also, $P(0) = \sum_{j=1}^m q^{(j)}(0) \otimes q^{(j)}(0)$ where $(q^{(j)}(0))_{1\leq j\leq m}$ is an orthonormal basis of $KerL_0(\lambda_0)$.

P r o o f.

(i) This property is clear if we remember that the operator $A(\epsilon)$ is holomorphic and then, can be expressed as: $A(\epsilon) = A(0) + \tilde{A}(\epsilon)$, where the operator $\tilde{A}(\epsilon)$ is holomorphic with respect to ϵ and goes to 0 as $\epsilon \to 0$. Next, the operator $R(\epsilon)$ is well defined if we consider, for $\epsilon \in]-\epsilon_3, \epsilon_3[$, the following Neumann series:

$$(\mu - A(\epsilon))^{-1} = (\mu - A(0))^{-1} + \sum_{p=1}^{\infty} (\mu - A(0))^{-1} [\tilde{A}(\epsilon)(\mu - A(0))^{-1}]^p,$$

which converges uniformly with respect to μ in a neighborhood of μ_0^j.

(ii) Since the elements of the family $(q_l^{(j)}(0))_{1\le j\le h, 1\le l\le m_j}$ are root functions of $L_0(\lambda_0)$ we may organize this family as follows:

$$q_1^{(1)}(0) = q_0^{(1)}, \quad q_2^{(1)}(0) = q_0^{(2)}, \cdots, \quad q_{m_h}^{(h)}(0) = q_0^{(m)}.$$

Thus, we obtain the family $(q_0^{(j)})_{1\le j\le m}$ which defines an orthogonal basis. By the same arguments the family $\left(q_{l,r}^{(j)}(\epsilon)\right)_{1\le l\le m_j, 1\le r\le m_{jl}, 1\le j\le h}$ given by (9), can be organized to obtain:

$$q_{1,1}^{(1)}(\epsilon) = q^{(1)}(\epsilon), \quad q_{1,2}^{(1)}(0) = q^{(2)}(\epsilon), \cdots, \quad q_{m_h, m_{hm_h}}^{(h)}(\epsilon) = q^{(m)}(\epsilon)$$

where for all $j = 1, \cdots, m$, $q^{(j)}(\epsilon) \to q_0^{(j)}$ as $\epsilon \to 0$. The family $(q^{(j)}(\epsilon))_{1\le j\le m}$ defines an orthogonal basis in $Ker(L_\epsilon(\lambda_j(\epsilon)))$ and so the order of organization of its elements directly depends on the order of organization of the basis $(q_0^{(j)})_{1\le j\le m}$. The proof is achieved if we substitute the relation (9) and (10) into (11).

■

Let us denote by $B_\epsilon \equiv (a_{lr})_{1\le l,r\le m}$ the $m \times m$ matrix, where for $l = 1, \cdots, m$ and for $r = 1, \cdots, m$, the coefficients a_{lr} are given by:

$$a_{lr} =< q_0^{(l)}, q_\epsilon^{(r)} > .$$

If I_m denotes the identity matrix, then for $\epsilon = 0$, $B_0 = I_m$. Proposition 4.1 implies that:

$$\sum_{j=1}^{m} < \cdot, q^{(j)}(\epsilon) > q^{(j)}(\epsilon) = \sum_{j=1}^{m} < \cdot, q_0^{(j)} > q_0^{(j)} + R(\epsilon). \tag{12}$$

From the last relation (12) we write

$$(B_\epsilon \phi_\epsilon)_l = q_0^{(l)} + R(\epsilon) q_0^{(l)},$$

where

$$q_0 = (q_0^{(1)}, \cdots, q_0^{(m)})^T \quad \text{and } \phi_\epsilon = (q^{(1)}(\epsilon), \cdots, q^{(m)}(\epsilon))^T. \tag{13}$$

With this notation established we now state the main results.

THEOREM 4.1. *There exists some constant $\epsilon_4 = \epsilon_4(\epsilon_3) > 0$, $(\epsilon_4 \leq \epsilon_3)$, such that for all $j \in \{1, \cdots, m\}$:*

(i) The functions $q^{(j)}(\epsilon)(s,t)$ are holomorphic in $(s,t;\epsilon)$ and satisfy the following asymptotic formulae uniformly for $(s,t) \in [0,\pi] \times [0,2\pi]$:

$$q^{(j)}(\epsilon) = q_0^{(j)} + \sum_{n \geq 1} q_n^{(j)} \epsilon^n, \quad \textit{for } |\epsilon| < \epsilon_4, \tag{14}$$

where the first coefficient satisfies $q_1^{(j)} = R_1 q_0^{(j)}$ and for $n \geq 2$, the coefficients $q_n^{(j)}$ are given by

$$q_n^{(j)} = R_n q_0^{(j)} - \sum_{k=1}^{n-1} \sum_{i=1}^{m} < q_0^{(j)}, q_k^{(i)} > q_{n-k}^{(i)}. \tag{15}$$

$R_0 = 0$ and R_n $(n \geq 1)$ being the Taylor coefficients of $R(\epsilon)$.

(ii) The characteristic values $\lambda_j(\epsilon)$ satisfy

$$\lambda_j(\epsilon) = \lambda_0 + \sum_{n \geq 1} \lambda_n^{(j)} \epsilon^n, \quad \textit{for } |\epsilon| < \epsilon_4. \tag{16}$$

The first coefficient satisfies:

$$\lambda_1^{(j)} = -1 - \frac{< L_0(\lambda_0) q_1^{(j)}, l_1^{(0)} q_0^{(j)} >}{\| l_1^{(0)} q_0^{(j)} \|^2},$$

and for $n \geq 2$:

$$\lambda_n^{(j)} = -\frac{1}{\| l_1^{(0)} q_0^{(j)} \|^2} \Big[\sum_{i=2}^{n} (\sum_{r_1 + \cdots + r_i = n} \lambda_{r_1}^{(j)} \lambda_{r_2}^{(j)} \cdots \lambda_{r_i}^{(j)}) < l_i^{(0)} q_0^{(j)}, l_1^{(0)} q_0^{(j)} >$$

$$+\sum_{l=0}^{n-1}\sum_{k=0}^{l} < F_{k,l-k}q_{n-l}^{(j)}, l_1^{(0)}q_0^{(j)} > + \sum_{k=1}^{n} < F_{k,n-k}q_0^{(j)}, l_1^{(0)}q_0^{(j)} > \Big].$$

Here for all integers k and r, the expressions $l_k^{(r)}$ and $F_{k,r}$ are two operator-valued functions with simple forms.

P r o o f.

(i) Define the matrix $\mathcal{D}_\epsilon = (d_{rp}^\epsilon)_{rp}$; the coefficients d_{rp}^ϵ are given by

$d_{rp}^\epsilon = \delta_r^p + < R(\epsilon)q_0^{(r)}, q_0^{(p)} >$, where δ_r^p denotes the Kronecker symbol.

Obviously, $\mathcal{D}_\epsilon$ is analytic because the operator-valued function $R(\epsilon)$ is analytic with respect to $\epsilon \in]-\epsilon_3, \epsilon_3[$ (see Proposition 4.1). In other words, if we take the inner product of (12) by $q_0^{(p)}$ we deduce that

$$\sum_{l=1}^{m} < q_0^{(r)}, q^{(l)}(\epsilon) >< q^{(l)}(\epsilon), q_0^{(p)} >= \delta_r^p + < R(\epsilon)q_0^{(r)}, q_0^{(p)} >$$

which implies that

$$B_\epsilon^2 = \mathcal{D}_\epsilon. \tag{17}$$

Next, relation (12) implies that

$$B_\epsilon \phi_\epsilon = \phi_0 + R(\epsilon)\phi_0, \tag{18}$$

and therefore,

$$\phi_\epsilon = (B_\epsilon)^{-1}[\phi_0 + R(\epsilon)\phi_0].$$

On the other hand we will justify if the function ϕ_ϵ is jointly analytic in (x, ϵ). The analyticity of $\phi_0(x)$ in x is a classical result. Then we deduce the result by using the analyticity of the matrix B_ϵ in ϵ (which is obvious from (17)) and the fact that the function $R(\epsilon)\phi_0(x)$ is jointly analytic in (x, ϵ). From relation (13) we then deduce the analyticity of the functions $q^{(j)}(\epsilon)$ for all $j = 1, \cdots, m$.

The analyticity of the matrix-operator B_ϵ, with respect to ϵ, allows to write in a neighborhood of 0 the expansion

$$B_\epsilon = I_m + \epsilon B_1 + \epsilon^2 B_2 + \cdots$$

Thus for ϵ sufficiently small, say for $|\epsilon| < \epsilon_4$ where $0 < \epsilon_4 \leq \epsilon_3$, we write the uniform expansion:

$$\phi_\epsilon = \sum_{k \geq 0} \epsilon^k q_k.$$

If we replace B_ϵ and ϕ_ϵ by their asymptotic expansions as $\epsilon \to 0$, then the relation (18) becomes:

$$(\sum_{n=0}^{+\infty} \epsilon^n B_n)(\sum_{r=0}^{+\infty} \epsilon^r q_r) = q_0 + \sum_{n=1}^{+\infty} \epsilon^n R_n q_0,$$

where we have considered that $q_0 = \phi_{\epsilon=0}$ and R_n is the $n-th$ Taylor coefficient of $R(\epsilon)$ $(R(0) = 0)$. Then,

$$\sum_{k=0}^{n} B_k q_{n-k} = q_0 + R_n q_0. \tag{19}$$

The $j-th$ composite of the vector $B_k q_{n-k}$ is given by

$$(B_k q_{n-k})_j = \sum_{i=1}^{m} < q_0^{(j)}, q_k^{(i)} > q_{n-k}^{(i)}.$$

For $j = 1, \cdots, m$, $q_n^{(j)}$ means the $n-th$ Taylor coefficient in the expansion of the function $q_\epsilon^{(j)}$ for ϵ in a neighborhood of zero. Now, the relation (19) implies,

$$(B_0 q_n)_j + \sum_{k=1}^{n} \sum_{i=1}^{m} < q_0^{(j)}, q_k^{(i)} > q_{n-k}^{(i)} = q_0^{(j)} + R_n q_0^{(j)}.$$

Moreover, the fact that $B_0 = I_m$, we deduce the Taylor coefficient of $q_\epsilon^{(j)}$:

$$q_n^{(j)} = -\sum_{k=1}^{n} \sum_{i=1}^{m} < q_0^{(j)}, q_k^{(i)} > q_{n-k}^{(i)} + q_0^{(j)} + R_n q_0^{(j)}.$$

On the other hand,

$$q_n^{(j)} = -\sum_{r=1}^{m} < q_0^{(j)}, q_n^{(r)} > q_0^{(r)} - \sum_{k=1}^{n-1} \sum_{i=1}^{m} < q_0^{(j)}, q_k^{(i)} > q_{n-k}^{(i)} + q_0^{(j)} + R_n q_0^{(j)}. \tag{20}$$

But, if we take the inner product of $q_n^{(j)}$ with the element of the orthonormal basis $q_0^{(r)}; r = 1, \cdots, m$, we obtain:

$$q_n^{(j)} = \sum_{i=1}^{m} < q_0^{(i)}, q_n^{(j)} > q_0^{(i)} + \sum_{r=1}^{m} \big(\sum_{k=1}^{n-1} \sum_{i=1}^{m} < q_0^{(j)}, q_k^{(i)} >< q_0^{(r)}, q_{n-k}^{(i)} >$$

$$-\delta_r^j - < q_0^{(r)}, R_n q_0^{(j)} > \big) q_0^{(r)} - \sum_{k=1}^{n-1} \sum_{i=1}^{m} < q_0^{(j)}, q_k^{(i)} > q_{n-k}^{(i)} + q_0^{(j)} + R_n q_0^{(j)}. \tag{21}$$

We remember that

$$P(0) q_n^{(j)} = \sum_{i=1}^{m} < q_0^{(i)}, q_n^{(j)} > q_0^{(i)}.$$

Then the relation (21) becomes,

$$(I - P(0)) \big(q_n^{(j)} + \sum_{k=1}^{n-1} \sum_{i=1}^{m} < q_0^{(j)}, q_k^{(i)}) > q_{n-k}^{(i)} - R_n q_0^{(j)} \big) = 0. \tag{22}$$

In other words, it is obvious that $R_n q_0^{(j)} \notin Ker(L_0(\lambda_0))$ for all $j = 1, \cdots, m$ and therefore,

$$(q_n^{(j)} + \sum_{k=1}^{n-1} \sum_{i=1}^{m} < q_0^{(j)}, q_k^{(i)} > q_{n-k}^{(i)} - R_n q_0^{(j)}) \notin Ker(L_0(\lambda_0)).$$

Thus relation (22) implies that,

$$q_n^{(j)} + \sum_{k=1}^{n-1} \sum_{i=1}^{m} < q_0^{(j)}, q_k^{(i)} > q_{n-k}^{(i)} - R_n q_0^{(j)} = 0.$$

(ii) In order to find out the coefficients in (16), our method is based on expanding the expression $L_\epsilon(\lambda_j(\epsilon))$ for ϵ near zero. To handle this we have to expand, first, the operator-valued function $L_\epsilon(\lambda)$ around $\epsilon = 0$ and so the resulting expression around $\lambda = \lambda_0$. The kernel G given by (4) satisfies the following uniform expansion

$$G(\gamma_\epsilon(s,t), \gamma_\epsilon(s',t'); \lambda) = \sum_{n \geq 0} G_n(\gamma(s,t), \gamma(s',t'); \lambda) \epsilon^n,$$

where the Taylor coefficients $G_n(\gamma(s,t),\gamma(s',t');\lambda)$ can be computed easily. As an immediate consequence, for $|\epsilon| < \epsilon_4$, we can write

$$L_\epsilon(\lambda) = L_0(\lambda) + \sum_{n=1}^{+\infty} L_n(\lambda)\epsilon^n, \quad \lambda \in D_{r_0}(\lambda_0),$$

where the Taylor coefficient $L_n(\lambda)$ $(n \geq 1)$ is an operator-valued function with kernel $G_n(\gamma(s,t),\gamma(s',t');\lambda)$. Obviously, $G_n(\gamma(s,t),\gamma(s',t');\lambda)$ is analytic in $\lambda \in D_{r_0}(\lambda_0)$ and then we obtain the following uniform expansion:

$$G_n(\gamma(s,t),\gamma(s',t');\lambda) = \sum_{k=0}^{+\infty}(\lambda-\lambda_0)^k G_k^{(n)}(\gamma(s,t),\gamma(s',t')).$$

Now, the following expansion holds

$$L_\epsilon(\lambda) = \sum_{n=0}^{+\infty}\sum_{k=0}^{+\infty} \epsilon^n(\lambda-\lambda_0)^k l_k^{(n)} \quad \text{for } (\epsilon,\lambda) \in]-\epsilon_4,\epsilon_4[\times D_{r_0}(\lambda_0), \tag{23}$$

where the Taylor coefficient $l_k^{(n)}$ $(n \geq 0, \quad k \geq 0)$ is operator-valued function with kernel $G_k^{(n)}(\gamma(s,t),\gamma(s',t'))$.
Next, we can write $\lambda_j(\epsilon) = \lambda_0 + \lambda_1^{(j)}\epsilon + \cdots + \lambda_n^{(j)}\epsilon^n + \cdots$, for $|\epsilon| < \epsilon_4$ and to find out the Taylor coefficients $\lambda_n^{(j)}$ we have to insert this expansion of $\lambda_j(\epsilon)$ into relation (23). So we obtain

$$L_\epsilon(\lambda_j(\epsilon)) = \sum_{n=0}^{+\infty}(\sum_{k=0}^{n} F_{k,n-k})\epsilon^n, \quad \text{for } |\epsilon| < \epsilon_4, \tag{24}$$

where

$$F_{n,0} = l_0^{(n)}$$

and

$$F_{n,k} = \sum_{i=1}^{k}(\sum_{r_1+\cdots+r_i=k} \lambda_{r_1}^{(j)}\lambda_{r_2}^{(j)}\cdots\lambda_{r_i}^{(j)})l_i^{(n)}. \tag{25}$$

Remember that

$$L_\epsilon(\lambda_j(\epsilon))q^{(j)}(\epsilon) = 0, \quad \text{for all } j = 1,\cdots,m.$$

Therefore, if we take the relations (15) and (24) at order $n \geq 1$, we can easily write

$$\sum_{l=0}^{n} \sum_{k=0}^{l} F_{k,l-k} q_{n-l}^{(j)} = 0.$$

Thus, from (25) and by simple calculus, we find for $n \geq 2$:

$$\lambda_n^{(j)} = -\frac{1}{\|l_1^{(0)} q_0^{(j)}\|^2} \Big[\sum_{i=2}^{n} (\sum_{r_1+\cdots+r_i=n} \lambda_{r_1}^{(j)} \lambda_{r_2}^{(j)} \cdots \lambda_{r_i}^{(j)}) < l_i^{(0)} q_0^{(j)}, l_1^{(0)} q_0^{(j)} >$$

$$+ \sum_{l=0}^{n-1} \sum_{k=0}^{l} < F_{k,l-k} q_{n-l}^{(j)}, l_1^{(0)} q_0^{(j)} > + \sum_{k=1}^{n} < F_{k,n-k} q_0^{(j)}, l_1^{(0)} q_0^{(j)} > \Big].$$

■

Next, we have the following lemma which seems useful to prove the fundamental result in this section.

LEMMA 4.1. *The functions given by* $\hat{u}_{i,j}(\epsilon)(x) = S(\lambda_j(\epsilon)) q^{(i)}(\epsilon)(\gamma^{-1})$ *are jointly analytic in the variables* $(x, \epsilon) \in \mathcal{K}_0 \times] - \epsilon_4, \epsilon_4[$, *where* $\mathcal{K}_0$ *is a bounded neighborhood of* $\overline{\Omega}_0$ *in* $\mathbf{R}^3$.

P r o o f. The function $\hat{u}_{i,j}(\epsilon)(x) = S(\lambda_j(\epsilon)) q^{(i)}(\epsilon)(\gamma^{-1})$ satisfies the Helmholtz equation in Ω_ϵ with the boundary conditions: $\hat{u}_{i,j}(\epsilon)|\partial\Omega_\epsilon = 0$ and $\partial_{\nu_\epsilon} \hat{u}_{i,j}(\epsilon)(\gamma_\epsilon(s,t)) = q^{(j)}(\epsilon)(s,t)$; which are jointly analytic with respect to the variables $(s, t; \epsilon) \in \mathbf{R}^2 \times] - \epsilon_4, \epsilon_4[$. The outward unit normal ν_ϵ to $\partial\Omega_\epsilon$ is given by $\frac{\nabla\gamma_\epsilon(s,t)}{|\nabla\gamma_\epsilon(s,t)|}$, as a function of $(s,t) = \gamma^{-1}(x)$. The symbol of the operator $\Delta_x = \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2$ is $\mathcal{P}(\xi_1, \xi_2, \xi_3, \epsilon) = \xi_1^2 + \xi_2^2 + \xi_3^2$. Thus $\mathcal{P}(\nu_\epsilon) = 1 > 0$. Since the surface $\partial\Omega_\epsilon$ is non characteristic for Δ_x the Cauchy-Kowaleska theorem implies that $\hat{u}_{i,j}(\epsilon)(x)$ is jointly analytic with respect to $(x; \epsilon)$ in $\{|\|x - \gamma(s,t)\|| \leq \alpha_0\} \times] - \epsilon_4, \epsilon_4[$, where α_0 is a positive constant. ■

We summarize the main result as follows.

THEOREM 4.2. *Let* $\mathcal{K}_0$ *be a bounded neighborhood of* $\overline{\Omega}_0$ *in* $\mathbf{R}^3$. *Then there exists a constant* $\epsilon_5 = \epsilon_5(\epsilon_4) > 0$ *smaller than* ϵ_4 *such that an orthonormal basis of eigenfunctions* $(u_j(\epsilon))_j$ *corresponding to the* $\lambda_0 - group$, $(\lambda_j^2(\epsilon))_j$, *in* $H_0^1(\Omega_\epsilon)$ *can be chosen to depend holomorphically in* $(x, \epsilon) \in \mathcal{K}_0 \times] - \epsilon_5, \epsilon_5[$. *Moreover these eigenfunctions satisfy the following asymptotic formulae*

$$u_j(\epsilon) = u_0^{(j)} + \sum_{n \geq 1} u_n^{(j)} \epsilon^n,$$

where the family $u_0^{(j)}$ builds a basis of eigenfunctions of (2) associated to λ_0^2 and normalized in $L^2(\Omega_0)$. The terms $u_n^{(j)}$ are computed from the Taylor coefficients $q_n^{(j)}$ of the function $q^{(j)}(\epsilon)$.

P r o o f. Proposition 4.1 and Theorem 4.1 imply that there exists an orthonormal basis

$$(q^{(i)}(\epsilon))_{1\leq i\leq mj}(s,t)\in H_{\sharp}^{-1/2}(]0,\pi[\times]0,2\pi[) \quad \text{of} \quad Ker(L_\epsilon(\lambda_j(\epsilon))),$$

which is analytic in $\mathbf{R}^2\times]-\epsilon_4,\epsilon_4[$. We know that $S(\lambda_j(\epsilon))q^{(j)}(\epsilon)(\gamma^{-1})$ form a basis of eigenfunctions of the eigenvalue problem (2) associated to $\lambda_j^2(\epsilon)$. Using the Schmidt orthogonalization process, we construct the desired orthonormal basis. Clearly, the functions $(\hat{u}_{i,j}(\epsilon))_{ij}$, introduced in Lemma 4.1, build a basis of the eigen-spaces corresponding to the λ_0-group, $(\lambda_j(\epsilon))_j$ in $H_0^1(\Omega_\epsilon)$. We will now give the asymptotic expansion of these functions when ϵ tends to 0. To simplify notations we drop the subscripts i and j. Integral equations give

$$\hat{u}(\epsilon)(x)=\int_0^{2\pi}\int_0^{\pi}G(x,\gamma_\epsilon(s,t))q(\epsilon)(s,t)|\nabla\gamma_\epsilon(s,t)|dsdt. \tag{26}$$

The perturbed eigenvalue $\lambda(\epsilon)$ lies in a small neighborhood of λ_0 for small values of ϵ. Then, there exists $\epsilon_5>0$ ($\epsilon_5\leq\epsilon_4$), such that we have the following Taylor expansion

$$G(x,\gamma_\epsilon(s,t))|\nabla\gamma_\epsilon(s,t)|=G(x,\gamma(s,t))|\nabla\gamma(s,t)|+\sum_{k\geq 1}\epsilon^k G_k(x,\gamma(s,t);\lambda), \tag{27}$$

which holds uniformly in $x\in\bar{\mathcal{K}}_0$ and $(s,t)\in[0,\pi]\times[0,2\pi]$. Using Theorem 4.1, we have

$$q(\epsilon)(s,t)=q_0(s,t)+\sum_{k\geq 1}\epsilon^k q_k(s,t), \tag{28}$$

uniformly in $(s,t)\in[0,\pi]\times[0,2\pi]$. Substituting the last two asymptotics into (26), we find

$$\hat{u}(\epsilon)=\hat{u}(0)+\sum_{k\geq 1}\epsilon^k[\sum_{n=1}^{k}\int_0^{2\pi}\int_0^{\pi}q_{k-n}(s,t)G_n(x,\gamma(s,t);\lambda)dsdt]. \tag{29}$$

Now we use the Schmidt orthogonalization process to construct from the eigenfunctions $(\hat{u}_j(\epsilon))_j$ an orthonormal basis $(u_j(\epsilon))_j$ of the direct sum of eigenspaces associated to the λ_0-group. This method allows us to compute the asymptotic expansion of these functions. ∎

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