Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 8, NUMBER 2 (2005) ISSN 1311-0454

$L_P \to L_Q$ - ESTIMATES FOR THE FRACTIONAL ACOUSTIC POTENTIALS AND SOME RELATED OPERATORS [1]

Alexey N. Karapetyants *, Denis N. Karasev **, Vladimir A. Nogin ***

Dedicated to Acad. Bogoljub Stanković,
on the occasion of his 80-th birthday

Abstract

We obtain the $L_p \to L_q$ - estimates for the fractional acoustic potentials in $\mathbb{R}^n$, which are known to be negative powers of the Helmholtz operator, and some related operators. Some applications of these estimates are also given.

Mathematics Subject Classification: 47B38, 31B10, 42B20, 42B15

Key Words and Phrases: fractional calculus, complex powers, acoustic potentials, $L_p \to L_q$ - estimates, $\mathcal{L}$ - characteristic, symbol, range

1. Introduction

We obtain the $L_p \to L_q$ - estimates, and the estimates from L_p into $L_r + L_s$, for the fractional acoustic potentials A^γ and some related operators. Some applications of these estimates are also given.

[1]This paper has been supported by Russian Fond of Fundamental Investigations under Grant No. 40–01–008632 a.

The acoustic potentials A^γ are well-known in the multidimensional fractional calculus as complex fractional powers, of the order $-\gamma$ with $\operatorname{Re}\gamma > 0$, of the Helmholtz operator

$$I + \Delta,$$

where Δ is the Laplace operator in $\mathbb{R}^n$. Together with the operator A^γ, we consider the potentials A_b^γ which realize the complex powers

$$A_b^\gamma = (-\Delta + 2ib \cdot \mathcal{D})^{-\gamma/2}, \quad 0 < \operatorname{Re}\gamma < n+1, \tag{1}$$

where

$$b \cdot \mathcal{D} = \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j}, \quad b_j \in \mathbb{R},$$

and the potentials $H_\theta^\gamma \varphi$ defined via Fourier transform by the equality

$$\widehat{H_\theta^\gamma \varphi}(\xi) = (|\xi| - 1 - i0)^{-\gamma/2}(1 + |\xi|)^{-\theta}\widehat{\varphi}(\xi), \tag{2}$$

where $0 < \operatorname{Re}\gamma < n+1, \quad \operatorname{Re}\theta > 0$.

Our interest in potentials (1) is caused by the fact that their symbols have singularities on the sphere $|\xi + b| = |b|$. The case of operators of such a kind (whose symbols have singularities "spread" over various manifolds in $\mathbb{R}^n$) is the most difficult for studying their complex powers; it has been considered by now for special-type operators only. Operators (1) are connected with the acoustic potentials by the relation

$$A_b^\gamma = |b|^{-1} U^{-1} A^\gamma U, \quad (U\varphi)(x) = \exp\left(-i\frac{b \cdot x}{|b|}\right)\varphi\left(-i\frac{x}{|b|}\right). \tag{3}$$

Operators (2) are close to A^γ in the sense that their symbols have the same order of singularities on the unit sphere (but different orders of decrease at infinity). Here the following natural question arises, which explains our interest in these operators: *what is the difference between the $\mathcal{L}$-characteristics $\mathcal{L}(A^\gamma)$ and $\mathcal{L}(H_\theta^\gamma)$ of the operators A^γ and H_θ^γ?* (The $\mathcal{L}$ - characteristic $\mathcal{L}(A)$ of the operator A^γ is the set of all pairs $(1/p, 1/q)$ for which this operator is bounded from L_p into L_q.)

As a matter of fact, this is the question about the influence of the function $(1 + |\xi|)^{-\theta}$ on mapping properties of operator (2).

We construct some convex sets in the $(1/p, 1/q)$-plane for which the operators under consideration are bounded from L_p into L_q and point out the domains, where they are not bounded. We also obtain some $L_p \to$

$L_q + L_r$ - estimates for the operator A_b^γ, which are then applied to describe the range $A_b^\gamma(L_p)$. The importance of such a description is beyond any doubt; it is explained by the fact that these classes can be regarded as the natural domains of complex powers with positive real parts of the operator $-\Delta+2ib\cdot\mathcal{D}$. In this connection, we note that although an explicit expression for these complex powers was obtained in [4], the range $A_b^\gamma(L_p)$ has never been described as yet.

Direct analysis of the estimates, obtained for the operators A^γ and H_θ^γ, shows that their $\mathcal{L}$ – characteristics may essentially differ from each other (see Remark 6).

In connection with fractional powers of operators, we refer, for example, to the books [7, 13, 15]. Here we choose the approach developed in [13, 15] (see also the survey papers [12, 10, 11, 14]), where fractional powers with negative real parts of various differential operators in partial derivatives are treated as the corresponding fractional potentials.

We note that some $L_p \to L_q$ - estimates for the operator A^γ, $0 < \operatorname{Re}\gamma < n+1$, were established in [5]. Here we essentially complement these results proving the $L_p \to L_q$ - estimates for the operator A^γ when $\left(\frac{1}{p}, \frac{1}{q}\right)$ belongs to an interval which passes through the point $D = \left(\frac{1}{2} + \frac{\operatorname{Re}\gamma}{2(n+1)}, \frac{1}{2} - \frac{\operatorname{Re}\gamma}{2(n+1)}\right)$ orthogonally to the line of duality $1/p + 1/q = 1$ (so far only the relation $D \in \mathcal{L}(A^\gamma)$, $0 < \gamma < n+1$, has been proved; see [9]). This allows us, in particular, to describe the $\mathcal{L}$-characteristic of the operator A^γ in the case of real γ, $\gamma \geq 1$, $\gamma \neq 2, 4, \ldots$ (see Remark 2). We also note that the range $A^\gamma(L_p)$ was described in [5] for $0 < \operatorname{Re}\gamma < 2$.

We observe that the principal difficulties, which arise when studying mapping properties of operator (2), are caused by the following reason. It seems impossible to establish a suitable integral representation for this operator convenient for obtaining the $L_p \to L_q$ – estimates. To overcome these difficulties, we split H_θ^γ into a sum of the acoustic potential and some multiplier operators with the known $\mathcal{L}$ – characteristics and then apply the results obtained for the operator A^γ.

2. Main results and some comments

Throughout the paper, the symbol $(A, B, \ldots, K)$ denotes the open polygon in $\mathbb{R}^2$ with the vertices at the points $A, B, \ldots, K$; $[A, B, \ldots, K]$ stands for its closure.

For $0 < \mathrm{Re}\,\gamma < n+1$, we put

$$A = \left(1, 1 - \frac{n - \mathrm{Re}\,\gamma + 1}{2n}\right), \quad A' = \left(\frac{\mathrm{Re}\,\gamma + n - 1}{2n}, 0\right),$$

$$B = \left(1 - \frac{(n+1)^2 + \mathrm{Re}\,\gamma(n-1)}{2n(n+1)}, \frac{n - \mathrm{Re}\,\gamma + 1}{2n}\right),$$

$$B' = \left(\frac{\mathrm{Re}\,\gamma + n - 1}{2n}, \frac{(n-1)(n - \mathrm{Re}\,\gamma + 1)}{2n(n+1)}\right),$$

$$D = \left(\frac{1}{2} + \frac{\mathrm{Re}\,\gamma}{2(n+1)}, \frac{1}{2} - \frac{\mathrm{Re}\,\gamma}{2(n+1)}\right), \quad E = (1, 0),$$

$$G = \left(1 - \frac{(n - \mathrm{Re}\,\gamma + 1)(n-1)}{2n(n+3)}, \frac{n - \mathrm{Re}\,\gamma + 1}{2n}\right),$$

$$G' = \left(\frac{\mathrm{Re}\,\gamma + n - 1}{2n}, \frac{(n - \mathrm{Re}\,\gamma + 1)(n-1)}{2n(n+3)}\right),$$

$$H = \left(\frac{n - \mathrm{Re}\,\gamma + 1}{2n}, \frac{n - \mathrm{Re}\,\gamma + 1}{2n}\right), \quad H' = \left(\frac{n + \mathrm{Re}\,\gamma - 1}{2n}, \frac{n + \mathrm{Re}\,\gamma - 1}{2n}\right),$$

$$K = \left(\frac{1}{2} + \frac{\mathrm{Re}\,\gamma}{n+1}, \frac{1}{2}\right), \quad K' = \left(\frac{1}{2}, \frac{1}{2} - \frac{\mathrm{Re}\,\gamma}{n+1}\right),$$

$$L = \left(1, 1 - \frac{\mathrm{Re}\,\gamma}{n}\right), \quad L' = \left(\frac{\mathrm{Re}\,\gamma}{n}, 0\right), \quad O = (1, 1), \quad O' = (0, 0).$$

To formulate the main statements, we introduce the following sets (see also **Picture 1**):

$$L_1(\gamma, n) = \begin{cases} (A', B', B, A, E) \cup (A, E] \cup (A', E], & 1 \le \mathrm{Re}\,\gamma < n+1 \\ (A', G', K', K, G, A, E) \cup (A, E] \cup (A', E], & 0 < \mathrm{Re}\,\gamma < 1; \end{cases}$$

$$L_1^*(\gamma, n) = \begin{cases} L_1(\gamma, n) \cup [K', K], & 0 < \mathrm{Re}\,\gamma < 1 \\ L_1(\gamma, n) \cup (B', B), & 1 \le \gamma < n, \quad \gamma \ne 1, 2, \dots, n-1 \\ L_1(\gamma, n), & 1 < \mathrm{Re}\,\gamma < n+1, \mathrm{Im}\,\gamma \ne 0; \end{cases}$$

$L_2(\gamma, n) = [O, O', L, L'] \setminus (\{L'\} \cup \{L\})$ if $0 < \mathrm{Re}\,\gamma < n$, $L_2(\gamma, n) = [O', O, E]$ if either $n < \mathrm{Re}\,\gamma < n+1$ or $\mathrm{Re}\,\gamma = n, \mathrm{Im}\,\gamma \ne 0$, and $L_2(\gamma, n) = [O', O, E] \setminus \{E\}$ if $\gamma = n$.

The following theorem provides the $L_p \to L_q$ - estimates for the acoustic potential

$$(A^\gamma \varphi)(x) = \int_{\mathbb{R}^n} h_\gamma(|y|)\varphi(x-y)\,dy,$$

where

$$h_\gamma(|y|) = \zeta_{n,\gamma}|y|^{\frac{\gamma-n}{2}} H^{(1)}_{\frac{n-\gamma}{2}}(|y|),$$

$\zeta_{n,\gamma} = 2^{(\gamma-n)/2}\pi^{1-n/2}\dfrac{i}{\Gamma(\gamma/2)}$, $H^{(1)}_\nu(z)$ is the Hankel function of the first kind.

Theorem 1. *Let* $0 < \operatorname{Re}\gamma < n+1$. *Then*

$$\mathcal{L}(A^\gamma) \supset L_1^*(\gamma, n) \cap L_2(\gamma, n). \tag{4}$$

We also indicate the domains in the $(1/p, 1/q)$ plane where the operator A^γ is not bounded from L_p into L_q.

Remark 1. ([5]) The $\mathcal{L}$ - characteristic $\mathcal{L}(A^\gamma)$ does not contain the points of the sets $L_2(\gamma, n)$, $[A, H, O]$, and $[A', H', O']$ and the points above the straight line $B'B$ if $0 < \gamma < n+1$.

In the case $\gamma = n$, the point $\{E\}$ does not belong to $\mathcal{L}(A^\gamma)$ also.

Remark 2. From Theorem 1 and Remark 1 we deduce that in the case $1 \le \gamma < n+1$, $\gamma \ne 2, 4, \dots$, the $\mathcal{L}$ - characteristic $\mathcal{L}(A^\gamma)$ is exactly the set $L_1^*(\gamma, n) \cap L_2(\gamma, n)$.

The next theorem provides the $L_p \to L_q$ - estimates for operator (2).

Theorem 2. *Let* $0 < \operatorname{Re}\gamma < n+1$. *Then*

$$\mathcal{L}(H_\theta^\gamma) \supset L_1^*(\gamma, n) \cap L_2(\theta + \gamma/2, n). \tag{5}$$

Moreover, $\mathcal{L}(H_\theta^\gamma)$ *does not contain the points of the set* $L_2(\theta + \gamma/2, n)$ *if* $\operatorname{Re}\theta > \frac{\operatorname{Re}\gamma(n-1)}{2(n+1)}$.

Remark 3. The inequality $\operatorname{Re}\theta > \frac{\operatorname{Re}\gamma(n-1)}{2(n+1)}$ means that the straight line $L'L$ lyes below $B'B$ (see **Picture 1**). Thus, the intersection of the sets on the right-hand side of (5) is always not empty in this case.

Remark 4. Imbedding (5) means that the operator H_θ^γ, initially defined on functions in the Schwartz class $\mathcal{S}$, is extended to the whole space L_p to a bounded operator from L_p into L_q if

$$(1/p, 1/q) \in L_1^*(\gamma, n) \cap L_2(\theta + \gamma/2, n).$$

In the rest of this section we state some results on the operator A_b^γ. In view of (3), the operator A_b^γ has the same boundedness properties as A^γ has, that is,

$$\mathcal{L}(A_b^\gamma) = \mathcal{L}(A^\gamma)$$

and the $L_p \to L_r + L_s$ - estimates for A_b^γ and A^γ are also the same. We apply these estimates to describe the range $A_b^\gamma(L_p)$ in terms of the operator N_b^γ (left) inverse to A_b^γ.

We observe that the inversion of potentials $f = A_b^\gamma \varphi$, $\varphi \in L_p$, $0 < \operatorname{Re}\gamma < n+1, 1 \le p < \frac{2n}{n+\operatorname{Re}\gamma - 1}$, was constructed in [4] in the form

$$\begin{aligned} N_b^\gamma f(x) &= \lim_{\varepsilon\to 0}^{(L_{p,\mu})} N_{b,\varepsilon}^\gamma f(x) \\ &= \lim_{\varepsilon\to 0}^{(L_{p,\mu})} \int_{\mathbb{R}^n} N_{\gamma,b,\varepsilon}(y) f(x-y)dy, \quad \mu > \frac{n+\operatorname{Re}\gamma - 1}{2}p, \end{aligned} \tag{6}$$

where $N_{\gamma,b,\varepsilon}(x) = |b|^{n+\gamma} e^{ib\cdot x} S_{\varepsilon,\gamma}(|b|x)$,

$$S_{\varepsilon,\gamma}(x) = \frac{(2\pi)^{-\frac{n}{2}}}{|x|^{\frac{n-2}{2}}} \int_0^\infty t^{n/2} e^{-\varepsilon t^2} \frac{(t^2-1)^{n+\gamma/2}}{(t^2+(\varepsilon+i)^2)^n} J_{\frac{n-2}{2}}(t|x|)dt;$$

the limit in $L_{p,\mu}$-norm in (6) can be replaced with the almost everywhere limit.

To formulate the corresponding result, we denote

$$\frac{1}{p_0(\gamma)} = \begin{cases} \frac12 + \frac{(\operatorname{Re}\gamma - 1)(n+3)(n+1-2n\operatorname{Re}\gamma)}{2n((n+1)(3n+1)-\operatorname{Re}\gamma(n^2+6n+1))} & \text{if} \quad 0 < \operatorname{Re}\gamma < \frac{n+1}{2n}, \\ \frac12 + \frac{\operatorname{Re}\gamma}{n+1} - \frac{1}{2n} \quad \text{if} \quad \frac{n+1}{2n} \le \operatorname{Re}\gamma < 2. \end{cases}$$

We note that the number $1/p_0(\gamma)$ is the abscissa of the point, where the straight line $1/q = (n-1)/(2n)$ meets the straight lines $B'B$ and $K'G'$ in the cases $(n+1)/(2n) \le \operatorname{Re}\gamma < 2$ and $0 < \operatorname{Re}\gamma < (n+1)/(2n)$, respectively.

Theorem 3. *Let $0 < \operatorname{Re}\gamma < 2$. Suppose that $1/p \in [1/p_0(\gamma), 1]$ if either $1 \le \gamma < 2$ or $(n+1)/(2n) \le \operatorname{Re}\gamma < 1$ and $1/p \in (1/p_0(\gamma), 1]$ otherwise. Then*

$$A_b^\gamma(L_p) = \{f \in L_{q_1} + L_{q_2} : \ N_b^\gamma f \in L_p\}, \tag{7}$$

where q_1, q_2 are such numbers that $2n/(n+1) < q_1'$ and $q_2' \le \infty$ and the operator A_b^γ is bounded from L_p into $L_{q_1} + L_{q_2}$ (see Corollary 1 below).

3. Preliminaries

3.1. Notation

In the sequel, besides the above notation, we shall also use the following: S^{n-1} is the unit sphere in $\mathbb{R}^n$; $\mathcal{R}_0 = \{\varphi : \ \varphi = Ff, \ f \in L_1\}$ is the Wiener ring; $L_{p,\mu} = \{f(x) : \ (1+|x|^2)^{-\mu/2} \ f(x) \in L_p\}$. Following [3], we denote by L_p^q the space of all distributions $k \in S'$ such that $\|k * f\|_q \leq C\|f\|_p$, where $f \in S$, the constant $C > 0$ not depending on f. The Fourier dual space $F(L_p^q)$ is denoted by M_p^q.

3.2. Analyticity of an integral depending on a parameter

The following lemma is taken from [13] (Lemma 1.31).

LEMMA 1. *Let $f(x,z)$ be an analytic function in the domain $D \subset \mathbb{C}$ for almost all $x \in \Omega \subseteq \mathbb{R}^n$. If $f(x,z)$ admits an integrable dominant:*

$$m(x) := \sup_{z \in D} |f(x,z)| \in L_1(\Omega),$$

then

$$F(z) = \int_\Omega f(x,z)dx$$

is an analytic function in D.

3.3. Some estimates for the acoustic potential and the operator A_b^γ

We need the following two theorems proved in [5].

THEOREM 4. *Let $0 < \operatorname{Re}\gamma < n+1$. Then*

$$\mathcal{L}(A^\gamma) \supset L_1(\gamma,n) \cap L_2(\gamma,n)$$

if $0 < \operatorname{Re}\gamma < n$,

$$\mathcal{L}(A^\gamma) \supset L_1(\gamma,n)$$

if either $n < \operatorname{Re}\gamma < n+1$ or $\operatorname{Re}\gamma = n$ and $\operatorname{Im}\gamma \neq 0$,

$$\mathcal{L}(A^\gamma) \supset L_1(\gamma,n) \setminus \{E\}$$

if $\gamma = n$.

In the case of real γ, the relation $\{D\} \in \mathcal{L}(A^\gamma)$ holds.

Theorem 5. *Let $0 < \operatorname{Re}\gamma < n+1$, $1 \le p < 2n/(n+\operatorname{Re}\gamma - 1)$. Then the operator A^γ is bounded from L_p into $L_{q_1} + L_{q_2}$, where q_1 and q_2 are such that*

1) $(1/p, 1/q_1) \in L_1(\gamma, n)$ *and* $(1/p, 1/q_2) \in L_2(\gamma, n)$
   if $0 < \operatorname{Re}\gamma < n$ *and* $\operatorname{Im}\gamma \ne 0$,

2) $(1/p, 1/q_1) \in L_1(\gamma, n) \cup \{D\}$ *and* $(1/p, 1/q_2) \in L_2(\gamma, n)$
   if $0 < \gamma < n$,

3) $(1/p, 1/q_1) \in L_1(\gamma, n)$*and* $p \le q_2 \le \infty$
   if $n \le \gamma < n+1$,

4) $(1/p, 1/q_1) \in L_1(\gamma, n) \cup \{D\}$ *and* $p \le q_2 \le \infty$
   if either $n < \operatorname{Re}\gamma < n+1$ *or* $\operatorname{Re}\gamma = n$ *and* $\operatorname{Im}\gamma \ne 0$,

5) *the conditions of item 3) are fulfilled, except for the case* $p = 1$, *in which we assume that* $1 \le q_2 < \infty$ *if* $\gamma = n$.

In view of (3), we have

Theorem 6. *Let $0 < \operatorname{Re}\gamma < n+1$, $1 \le p < 2n/(n+\operatorname{Re}\gamma - 1)$. The statement of Theorem 5 is also valid for the operator A_b^γ.*

Corollary 1. *Let the conditions of Theorem 3 be fulfilled. Then there exist such numbers $q_1, q_2 \in \left[1, \frac{2n}{n-1}\right)$ that the operator A_b^γ is bounded from L_p into $L_{q_1} + L_{q_2}$.*

3.4. Estimates for the Bochner-Riesz operators

We consider the Bochner-Riesz operator

$$(B^{\gamma/2}\varphi)(x) = \int\limits_{\mathbb{R}^n} |y|^{(\gamma-n)/2} J_{(n-\gamma)/2}(|y|)\varphi(x-y)\,dy, \; 0 < \operatorname{Re}\gamma < n+1, \quad (8)$$

where $J_\nu(z)$ is the Bessel function of the order ν.

The following statement was proved in [6] (see also [2] and [4] for some estimates in the case of real γ).

THEOREM 7. *Let $0 < \operatorname{Re}\gamma < n+1$. The imbedding*

$$\mathcal{L}(B^{\gamma/2}) \supset \begin{cases} L_1(\gamma, n) & \text{if } \operatorname{Im}\gamma \neq 0 \\ L_1(\gamma, n) \cup \{D\} & \text{if } \operatorname{Im}\gamma = 0 \end{cases} \tag{9}$$

is valid.

3.5. On some conditions of absolute integrability of Fourier integrals

The following theorems are taken from [16].

THEOREM 8. *Let $f(x) \in C^N(\mathbb{R}^n)$, $N = [n/2]+1$, and let there exist constants $C > 0$ and $\delta > 0$ such that*

$$|D^j f(x)| \leq C|x|^{-\delta-|j|}, \quad x \in \mathbb{R}^n,$$

for all $1 \leq j \leq N$. Then $f(x) \in \mathcal{R}_0$.

THEOREM 9. *Let $f(x) \in C^N(\mathbb{R}^n \setminus \{0\})$, $N = [n/2]+1$, have a compact support and*

$$|D^j f(x)| \leq C|x|^{\delta-|j|}, \quad x \in \mathbb{R}^n \setminus \{0\},$$

for all $1 \leq j \leq N$. Then $f(x) \in \mathcal{R}_0$.

4. Some auxiliary statements

4.1. Boundedness results for some multiplier operators

We need the following lemmas.

LEMMA 2. *Let $m(\xi) \in M_{p_0}^{q_0}$ for some (p_0, q_0), $1 \leq p_0 \leq q_0 \leq \infty$, and let $\omega(\xi) \in C_0^\infty(\mathbb{R}^n)$. Then $m(\xi)\omega(\xi) \in M_p^q$ for all (p, q) such that $1 \leq p \leq p_0$ and $q_0 \leq q \leq \infty$.*

This statement follows from the relation $m(\xi)\omega(\xi) \in M_{p_0}^\infty$ and the convexity and duality arguments.

LEMMA 3. *Let the function $\psi(r) \in C^\infty(0,\infty)$, be such that $0 \leq \psi(r) \leq 1$, $\psi(r) = 0$ if $r \leq 1$ and $\psi(r) = 1$ if $r \geq 2$. Set $k_\lambda = F^{-1}\left(\psi(|\xi|)|\xi|^{-\lambda}\right)$, $0 < \operatorname{Re}\lambda < n$, where the inverse Fourier transform is treated in the $\mathcal{S}'$ - sense. Then $k_\lambda \in L_p^q$ if and only if $(1/p, 1/q) \in L_2(\lambda, n)$.*

P r o o f. The "if" part is derived from the fact that the distribution k_λ agrees with the function $k_\lambda(x)$ which possess the following properties:

$k_\lambda(|x|)$ is smooth in $\mathbb{R}^n \setminus \{0\}$ and for every $M > 0$,

$|k_\lambda(|x|)| \le C|x|^{-M}$ as $|x| \to \infty$,

$k_\lambda(|x|) = 2^\lambda \pi^{n/2}\Gamma(\frac{\lambda}{2})/\Gamma(\frac{n-\lambda}{2})|x|^{\lambda-n} + u(|x|)$,

where $u(r)$ is a smooth function (see [8] for the case of real γ; the case of complex γ is considered in the same way).

The "only if" part follows from the fact that the kernel $\chi_1(y)|y|^{\lambda-n}$ does not belong to L_p^q if $(1/p, 1/q) \in [O', O, E] \setminus L_2(\lambda, n)$, where $\chi_1(y)$ is the characteristic function of the unit ball in $\mathbb{R}^n$. ∎

Proceeding, we establish some $L_p \to L_q$ - estimates for the Bochner-Riesz operator (8) and the operator

$$(G^{\gamma/2}\varphi)(x) = (A^\gamma\varphi)(x) - e^{-i\pi\gamma/2}\frac{\Gamma(1-\gamma/2)}{2^{\gamma/2}(2\pi)^{n/2}}(B^{\gamma/2}\varphi)(x). \tag{10}$$

The corresponding results are contained in Theorems 10 – 12, which provide the boundedness of the mentioned operators from L_p into L_q when $\left(\frac{1}{p}, \frac{1}{q}\right)$ belongs to the interval which passes through the point $\{D\}$ orthogonally to the line of duality.

4.2. Estimates for the Bochner-Riesz operator and the operator $G^{\gamma/2}$

We start with the case of the Bochner-Riesz operator.

THEOREM 10. *The following imbeddings are valid:*

$$[K', K] \subset \mathcal{L}(B^{\gamma/2}), \quad 0 < \operatorname{Re}\gamma < 1, \tag{11}$$

$$(B', B) \subset \mathcal{L}(B^{\gamma/2}), \quad 1 \le \gamma < n+1. \tag{12}$$

P r o o f. Imbeddings (11), (12) were proved in [1] in the cases $0 < \gamma < 1$ and $1 \le \gamma < n+1$, respectively. Thus, it remains to prove (11) in the case of complex γ, $0 < \operatorname{Re}\gamma < 1$. To this end, it suffices to verify that $\{K\} \in \mathcal{L}(B^{\gamma/2})$, $0 < \operatorname{Re}\gamma < 1$. Equivalently, we need to prove the estimate

$$\|B^{\gamma/2}\varphi\|_2 \le C\|\varphi\|_p, \tag{13}$$

where $\varphi \in L_p$, $1/p = 1/2 + \operatorname{Re}\gamma/(n+1)$, the constant C not depending on φ.

Together with operator (8), we consider the corresponding multiplier operator $\widetilde{B}^{\gamma/2}$ with the symbol $\frac{2^{\gamma/2}(2\pi)^{n/2}}{\Gamma(1-\gamma/2)}(1-|\xi|^2)_+^{-\gamma/2}$.

We first establish (13) for $\varphi \in \mathcal{S}$. In view of the relation

$$\frac{\Gamma(1-\frac{\gamma}{2})}{2^{\frac{\gamma}{2}}(2\pi)^{\frac{n}{2}}}(B^{\frac{\gamma}{2}}\varphi)(x) = F^{-1}((1-|\xi|^2)_+^{-\frac{\gamma}{2}}\widehat{\varphi}(\xi))(x), \quad \varphi \in \mathcal{S}, \tag{14}$$

it suffices to prove the estimate

$$\|\widetilde{B}^{\gamma/2}\varphi\|_2 \le C\|\varphi\|_p, \quad \varphi \in \mathcal{S}. \tag{15}$$

Estimate (15) will follow from the equality

$$\langle \widetilde{B}^{\gamma/2}\varphi, \widetilde{B}^{\gamma/2}\varphi\rangle = \langle \widetilde{B}^{\mathrm{Re}\,\gamma}\varphi, \varphi\rangle, \ \varphi \in \mathcal{S}, 0 < \mathrm{Re}\,\gamma < 1. \tag{16}$$

Let us prove (16). Applying the Parseval equality, we obtain

$$\langle \widetilde{B}^{\gamma/2}\varphi, \widetilde{B}^{\gamma/2}\varphi\rangle = (2\pi)^{-n}\int_{\mathbb{R}^n} |\widehat{\varphi}(\xi)|^2(1-|\xi|^2)_+^{-\mathrm{Re}\,\gamma}d\xi. \tag{17}$$

On the other hand, for $0 < \mathrm{Re}\,\beta < 1/2$ we have

$$\langle \widetilde{B}^{\beta/2}\varphi, \varphi\rangle = (2\pi)^{-n}\int_{\mathbb{R}^n} |\widehat{\varphi}(\xi)|^2(1-|\xi|^2)_+^{-\beta/2}d\xi. \tag{18}$$

As both sides of (18) are analytic with respect to β in the strip $0 < \mathrm{Re}\,\beta < 1$, equality (18) also holds for such β. Setting $\beta = 2\,\mathrm{Re}\,\gamma$ in (18), we see that the right hand sides of (17) and (18) coincide, which yields (16).

By virtue of (16) and (14) we have

$$\|\widetilde{B}^{\mathrm{Re}\,\gamma}\varphi\|_2^2 \le \|\widetilde{B}^{\mathrm{Re}\,\gamma}\varphi\|_{p'}\|\varphi\|_p \le C\|\varphi\|_p^2. \tag{19}$$

The last inequality in (19) follows from the relation $\{D\} \in \mathcal{L}(B^{\mathrm{Re}\,\gamma})$ (see [2]).

Thus, we have proved (13) for $\varphi \in \mathcal{S}$. As is seen from Theorem 7, the operator $B^{\gamma/2}$ is bounded from L_p into L_q, $0 \le 1/q < 1/2$, which implies the validity of (13) for $\varphi \in L_p$ as well. ■

From Theorems 7 and 10 we deduce the following statement, which is of special interest itself.

THEOREM 11. *Let* $0 < \mathrm{Re}\,\gamma < n+1$. *Then*

$$\mathcal{L}(B^{\gamma/2}) \supset L_1^*(\gamma, n).$$

THEOREM 12. *Under the additional condition* $\gamma \neq 2, 4, \ldots$, *in the case* $1 \leq \gamma < n+1$, *the statement of Theorem 10 is also valid for the operator* $G^{\gamma/2}$.

P r o o f. The proof of imbedding

$$[K', K] \subset \mathcal{L}(G^{\gamma/2}),\ \ 0 < \mathrm{Re}\,\gamma < 1,$$

is much in lines with that of (11).

Let us prove the imbedding

$$(B', B) \subset \mathcal{L}(G^{\gamma/2}),\ \ 1 \leq \gamma < n+1, \quad \gamma \neq 2, 4, \ldots, \tag{20}$$

which is non-trivial in view of the fact that the right-hand side of (10) tends to ∞ as $\gamma \to 2k$.

For every fixed $\gamma \in [1, n+1)$ and the point $(1/p, 1/q) \in (B', B)$, $1/q + 1/p > 1$, we consider the family of the operators

$$\begin{aligned} (T^z\varphi)(x) &= \pi^{-1}\Gamma\left(\gamma_0(1-z) + \frac{n+1}{2}z\right)\sin\left(\gamma_0(1-z) + \frac{n+1}{2}z\right) \\ &\times \left(A^{2\gamma_0(1-z)+(n+1)z}\varphi\right)(x) - \frac{e^{i\pi(\gamma_0(1-z)+z(n+1)/2)}(2\pi)^{n/2}}{2^{\gamma_0(1-z)+z(n+1)/2}} \\ &\times \left(B^{\gamma_0(1-z)+z(n+1)/2}\varphi\right)(x), \quad 0 \leq \mathrm{Re}\,z \leq 1, \quad \varphi \in C_0^\infty \end{aligned} \tag{21}$$

$\gamma_0 = \frac{(q(1-p)+p)(n+1)}{2p}$, $0 < \gamma_0 < 1$, and apply to it the Stein-Weiss interpolation theorem (see [17]). It is not difficult to verify that this is the family of admissible growth in the sense of definition given in [17] (the proof is direct and we omit it).

For every $\sigma \in \mathbb{R}$ we have

$$\begin{aligned} \|T^{1+i\sigma}\varphi\|_\infty &\leq \left(\frac{\|h_{n+1+i\sigma(n+1-2\gamma_0)}\|_\infty}{\Gamma\left((n+1)/2 + i\sigma((n+1)/2-\gamma_0)\right)}\right. \\ &+ \left. e^{\pi|\sigma|((n+1)/2-\gamma_0)}\|b_{(n+1+i\sigma(n+1-2\gamma_0))/2}\|_\infty\right)\|\varphi\|_1 \\ &\leq Ce^{\pi|\sigma|((n+1)/2-\gamma_0)}\|\varphi\|_1, \end{aligned} \tag{22}$$

where $h_{n+1+i\sigma(n+1-2\gamma_0)}(|y|)$ and $b_{(n+1+i\sigma(n+1-2\gamma_0))/2}(|y|)$ are the kernels of the acoustic potential and the Bochner-Riesz operator, respectively.

Applying the same arguments as in the proof of imbedding (11) and the estimate

$$|\Gamma(\mu+i\nu)|^{-1} \le e^{\pi|\nu|},$$

we have

$$\|T^{i\sigma}\varphi\|_2 = \frac{\langle G^{\gamma_0(1-i\sigma)+i\sigma(n+1)/2}\varphi, G^{\gamma_0(1-i\sigma)+i\sigma(n+1)/2}\varphi\rangle}{|\Gamma(1-\gamma_0+i\sigma(\gamma_0-(n+1)/2))|^2}$$
$$= \frac{\langle G^{2\gamma_0}\varphi,\varphi\rangle}{|\Gamma(1-\gamma_0+i\sigma(\gamma_0-(n+1)/2))|^2} \le Ce^{\pi|\sigma|(\frac{n+1}{2}-\gamma_0)}\|\varphi\|_{p_0}, \quad (23)$$

where $1/p_0 = (4\gamma_0+n+1)/(2n+2)$. The last inequality follows from the relations $\{D\} \in \mathcal{L}(A^\gamma)$ and $\{D\} \in \mathcal{L}(B^{\gamma/2})$, $0<\gamma<n+1$; see Theorems 4 and 7.

Setting $1/p_t = \frac{1-t}{p_0}+t$, $1/q_t = \frac{1-t}{2}$, $t = \frac{\gamma-\gamma_0}{(n+1)/2-\gamma_0}$, and interpolating between (22) and (23), we arrive at the inequality

$$\|G^{\gamma/2}\varphi\|_q \le C\|\varphi\|_p,$$

where $\varphi \in C_0^\infty$, the constant C not depending on φ. As the operator $G^{\gamma/2}$ is bounded from L_p into L_r, $1/p-\gamma/n \le 1/r < 1/q$, by virtue of Theorems 7 and 4, this estimates also holds for $\varphi \in L_p$.

Thus, we have proved the boundedness of operator (10) from L_p into L_q for the points $(1/p,1/q) \in (B',B)$ such that $1/p+1/q>1$. We obtain (20) due to duality. ∎

REMARK 5. In the case $\gamma = 2,4,\dots$, the question about the validity of imbedding (12) remains open.

5. Proof of the main results

5.1. Proof of Theorem 1

The statement of Theorem 1 follows from Theorems 4, 10, 12 and equality (10).

5.2. Proof of Theorem 2

P r o o f. The operator

$$\left(H_\theta^\gamma\varphi\right)(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}(|\xi|-1-i0)^{-\gamma/2}(1+|\xi|)^{-\theta}\widehat{\varphi}(\xi)e^{-ix\cdot\xi}d\xi \qquad (24)$$

is well-defined on functions in $\mathcal{S}$ if $0 < \operatorname{Re}\gamma < 2$. In the case $\operatorname{Re}\gamma \geq 2$, the integral on the right-hand side of (24) is treated in the sense of analytic continuation into the domain $\Omega_\gamma = \{\gamma \in \mathbb{C} : 0 < \operatorname{Re}\gamma < n+1, \gamma \neq 2, 4, \dots\}$, which can be constructed as follows:

$$\begin{aligned}\left(H_\theta^\gamma \varphi\right)(x) = \quad & \frac{(2\pi)^{-n}}{(1-\frac{\gamma}{2})\dots(\ell-\frac{\gamma}{2})} \int_0^\infty (\rho - 1 - i0)^{\ell-\gamma/2} \\ & \times \left(\frac{d}{d\rho}\right)^\ell \left(\frac{\rho^{n-1}}{(1+\rho)^\theta} \int_{S^{n-1}} \widehat{\varphi}(\rho\sigma) e^{-i\rho(x\cdot\sigma)} d\sigma\right) d\rho,\end{aligned}$$

where $\ell = \left[\frac{n-1}{2}\right]+1$. We observe that both sides of this equality are analytic in Ω_γ in view of Lemma 1.

Let the functions $\chi(r)$ and $\omega(r)$ be such that $\chi(r), \omega(r) \in C^\infty(0, +\infty)$, $0 \leq \chi(r), \omega(r) \leq 1$, $\chi(r) = 1$ if $r \geq 2N$ and $\chi(r) = 0$ if $r \leq N$; $\omega(r) = 1$ if $r \in [1-\delta/2, 1+\delta/2]$ and $\omega(r) = 0$ otherwise (here $\delta, N > 0$ and $1+\delta < N$).

We base ourselves on the following representation for the symbol $h_\theta^\gamma(|\xi|)$ of the operator H_θ^γ:

$$\begin{aligned} h_\theta^\gamma(|\xi|) \quad = \quad & \chi(|\xi|)(|\xi| - 1 - i0)^{-\gamma/2}(1+|\xi|)^{-\theta} \\ & +\omega(|\xi|)(|\xi| - 1 - i0)^{-\gamma/2}(1+|\xi|)^{-\theta} \\ & +(1-\chi(|\xi|))(1-\omega(|\xi|))(|\xi| - 1 - i0)^{-\gamma/2}(1+|\xi|)^{-\theta} \\ & \equiv m_{\gamma,\theta}(|\xi|) + n_{\gamma,\theta}(|\xi|) + l_{\gamma,\theta}(|\xi|). \end{aligned} \tag{25}$$

In accordance with (25), in the Fourier pre-images we have

$$\left(H_\theta^\gamma \varphi\right)(x) = \left(M_\theta^\gamma \varphi\right)(x) + \left(N_\theta^\gamma \varphi\right)(x) + \left(L_\theta^\gamma \varphi\right)(x), \;\; \varphi \in \mathcal{S}, \gamma \in \Omega_\gamma, \tag{26}$$

where M_θ^γ, N_θ^γ, and L_θ^γ are the multiplier operators with the symbols $m_{\gamma,\theta}(|\xi|)$, $n_{\gamma,\theta}(|\xi|)$, and $l_{\gamma,\theta}(|\xi|)$, respectively.

In the case $\operatorname{Re}\gamma \geq 2$, the integral $\left(N_\theta^\gamma \varphi\right)(x)$ is treated in the sense of analytic continuation into the domain Ω_γ which is constructed in just the same way as that of the integral $\left(H_\theta^\gamma \varphi\right)(x)$:

$$\begin{aligned}\left(N_\theta^\gamma \varphi\right)(x) \quad = \quad & \frac{(2\pi)^{-n}}{(1-\frac{\gamma}{2})\dots(\ell-\frac{\gamma}{2})} \int_{1-\delta}^{1+\delta} (\rho - 1 - i0)^{\ell-\gamma/2} \\ & \times \left(\frac{d}{d\rho}\right)^\ell \left(\omega(\rho)\frac{\rho^{n-1}}{(1+\rho)^\theta} \int_{S^{n-1}} \widehat{\varphi}(\rho\sigma) e^{-i\rho(x\cdot\sigma)} d\sigma\right) d\rho.\end{aligned}$$

Note that both sides of this equality are analytic in Ω_γ in view of Lemma 1.

In order to obtain the estimates for the operator N_θ^γ (on functions in $\mathcal{S}$), we represent it as

$$\left(N_\theta^\gamma \varphi\right)(x) = \left(A^\gamma A_{\gamma,\theta}\varphi\right)(x), \ \gamma \in \Omega_\gamma, \tag{27}$$

where $A_{\gamma,\theta}$ is the convolution operator with the kernel $a_{\gamma,\theta} \in \mathcal{S}$ for which $\widehat{a_{\gamma,\theta}}(|\xi|) = \omega(|\xi|)(1+|\xi|)^{-\theta+\gamma/2}$.

Equality (27) is derived from the following one:

$$\begin{aligned} &F^{-1}\left((|\xi|^2 - 1 - i0)^{-\beta/2}\omega(|\xi|)(1+|\xi|)^{-\theta+\gamma/2}\widehat{\varphi}(\xi)\right)(x) \\ &= \left(A^\beta A_{\gamma,\theta}\varphi\right)(x), \quad \varphi \in \mathcal{S}, \end{aligned} \tag{28}$$

$\beta \in \Omega_\beta, x \in \mathbb{R}^n$. (In the case $\beta \geq 2$ the Fourier integral on the left-hand side of (28) is treated in the sense of analytic continuation.)

Let us prove (28). In the case $0 < \operatorname{Re}\beta < 1$, this equality is verified via Fourier transform in the L_2 - sense. Since both sides of (28) are continuous throughout $\mathbb{R}^n$, it is valid for every $x \in \mathbb{R}^n$. We fix $x \in \mathbb{R}^n$ and note that both sides of (28) are analytic (with respect to β) in Ω_β in view of Lemma 1, hence this equality holds for such β.

Setting $\beta = \gamma$ in (28), we arrive at (27).

From equality (27), Theorem 4, and Lemma 2 we obtain

$$\mathcal{L}(N_\theta^\gamma) \supset L_1(\gamma, n). \tag{29}$$

Now we consider the remaining multipliers on the right-hand side of (25). As $l_{\gamma,\theta}(|\xi|) \in \mathcal{R}_0 \cap L_1$ (the relation $l_{\gamma,\theta}(|\xi|) \in \mathcal{R}_0$ is easily verified with the aid of Theorem 8, we have

$$\mathcal{L}(L_\theta^\gamma) = [O', O, E]. \tag{30}$$

Applying the Taylor formula, we represent the symbol of the operator M_θ^γ in the form

$$m_{\gamma,\theta}(|\xi|) = \frac{\chi(|\xi|)}{|\xi|^{\gamma/2+\theta}} e^{-i\pi\gamma/2}\left(\sum_{j=0}^m \frac{C_j}{|\xi|^j} + r_m(|\xi|)\right),$$

where $m = n + 1 + [(n-3)/2 - \operatorname{Re}\gamma - \operatorname{Re}\theta]$. We note that the multiplier $\chi(|\xi|)|\xi|^{-(\gamma/2+\theta)}r_m(|\xi|)$ belongs to L_1; it is also in $\mathcal{R}_0$ by virtue of Theorem 9. From here and Lemma 3, we have

$$m_{\gamma,\theta}(|\xi|) \in M_p^q \Longleftrightarrow (1/p, 1/q) \in L_2\left(\gamma/2 + \theta\right). \tag{31}$$

From (29) - (31) we derive (5). The second statement of Theorem 2 is also evident. ∎

Remark 6. In order to illustrate the essential difference between the sets $\mathcal{L}(A^\gamma)$ and $\mathcal{L}(H_\theta^\gamma)$, we consider two opposite situations: when $\theta > \gamma/2$ and $\theta < \gamma/2$ (let γ and θ be positive real numbers).

In the former case, the straight line $M'M$ lies above $L'L$; the points between these straight lines belong to $\mathcal{L}(H_\theta^\gamma)$ but they do not belong to $\mathcal{L}(A^\gamma)$. (Moreover, we can state that $\operatorname{mes}\mathcal{L}(A^\gamma) < \operatorname{mes}\mathcal{L}(H_\theta^\gamma)$). In the latter case, the situation is directly opposite: $(M', L', L, M) \subset \mathcal{L}(A^\gamma)$ but $(M', L', L, M) \cap \mathcal{L}(H_\theta^\gamma) = \varnothing$ (see Picture 2).

5.3. Proof of Theorem 3

Making use of Theorem 6 and Corollary 1, we have to repeat the same arguments as in the proof of Theorem 2.2 from [5].

Acknowledgements. The authors are grateful to Professors S. G. Samko and N. K. Karapetyants† for useful discussions of the results and valuable comments on the manuscript of the paper.

References

[1] J.-G. Bak, D. McMichael, D. Oberlin, $L^p - L^q$ estimates off the line of duality. *J. Austral. Math. Soc., (Series A)* **58** (1995), 154-166.

[2] L. Börjeson, Estimates for the Bochner–Riesz operator with negative index. *Indiana University Mathematics Journal* **35**, No 2 (1986), 225-233.

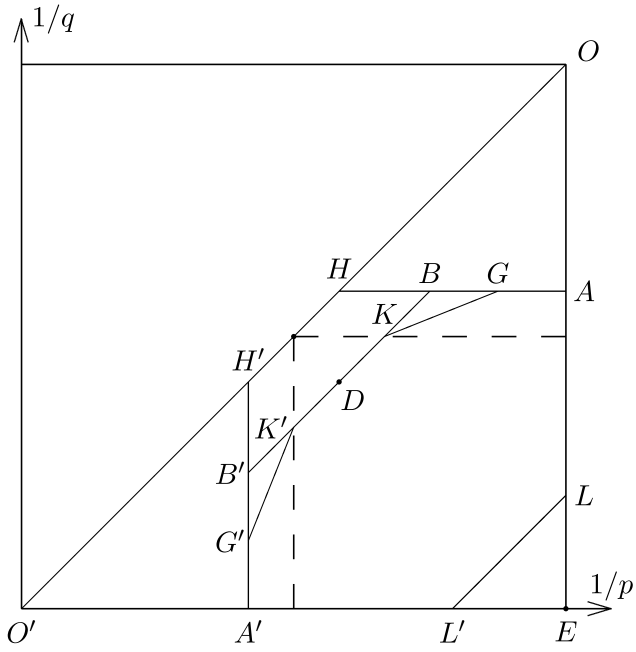
[3] L. Hörmander, Estimates for translation invariant operators in L^p spaces. *Acta Mathematica* **104** (1960), 93-140.

[4] A. N. Karapetyants, V. A. Nogin, Complex powers of the second order non-homogeneous elliptic differential operators with degenerating symbols in the spaces $L_p(\mathbb{R}^n)$. *Bol. Soc. Mat. Mexicana* **7**, No 3 (2001), 193-209.

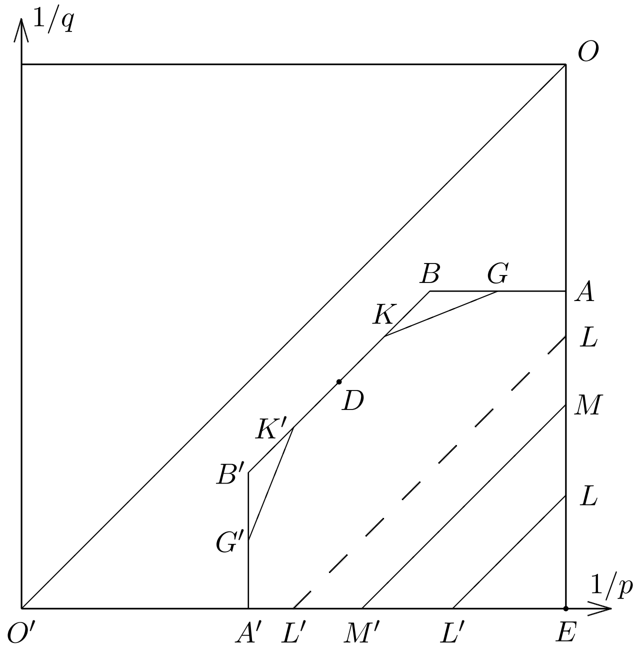
[5] D. N. Karasev, V. A., Nogin, Estimates for the acoustic potentials and their application. *Proceedings of A. Razmadze Math. Inst.* **129** (2002), 29-51.

[6] D. N. Karasev, V. A. Nogin, $(L_p \to L_q)$-estimates for the Bochner-Riesz operator of complex order. *Zeitschrift für Analysis und ihre Anwendungen.* **21**, No 4 (2002), 915-929.

[7] C. C. Martínez and A. M. Sanz, *The Theory of Fractional Powers of Operators.* North-Holland Mathematics Studies, Vol. **187**, North-Holland Publishing Co., Amsterdam (2001).



Picture 1



Picture 2

[8] A. Miyachi, On some singular Fourier multipliers. *J. Fac. Sci. Univ. Tokyo. Sec. IA.* **28** (1981), 267-315.

[9] V.A. Nogin, B.S. Rubin, Estimates for potentials with oscillating kernels that are connected with the Helmholtz equation. *Differential Equations.* **26**, No 9 (1990), 1195-1200.

[10] V. A. Nogin, S. G. Samko, Method of approximating inverse operators and its applications to inversion of potential type integral transforms. *Integral Transforms and Special Functions* **6** (1999), 1-14.

[11] V. A. Nogin, S. G. Samko, Some applications of potentials and approximative inverse operators in multi-dimensional fraction calculus. *Fractional Calculus & Applied Anaysis* **2**, No 2 (1999), 205-228.

[12] S. G. Samko, Inversion theorems for potential-type integral transforms in R^n and on S^{n-1}. *Integral Transforms and Special Functions* **1**, No 2 (1993), 145-163.

[13] S. G. Samko, *Hypersingular Integrals and Their Applications* Internat. Series "Analytical Methods and Special Functions", Vol. **5**. Taylor & Frances, London (2002).

[14] S. G. Samko, Fractional powers of operators via hypersingular integrals. *Progress in Nonlinear Differential Equations and Their Applications* **42** (2002), 259-272.

[15] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications.* Gordon & Breach. Sci. Publ., London – New York (1993).

[16] S.G. Samko, G.S. Kostetskaya, Absolute integrability of Fourier integrals. *Vestnik RUDN (Math.)* **1** (1994), 138-168.

[17] E. M. Stein, Interpolation of linear operators. *Trans. Amer. Math. Soc.* **83** (1956), 482-492.

Mathematics Department
Rostov State University
Rostov-on-Don, RUSSIA

Received: June 15, 2003

e-mails:
* *alexeyk@aaanet.ru* , ** *denis_karasev@mail.ru* , *** *vnogin@math.rsu.ru*