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CERTAIN PROPERTIES OF FRACTIONAL CALCULUS OPERATORS ASSOCIATED WITH GENERALIZED MITTAG-LEFFLER FUNCTION

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Dedicated to Acad. Bogoljub Stanković
on the occasion of his 80th birthday

Abstract

This paper deals with the study of an entire function of the form

$$E_{\beta,\gamma}^{\delta}(z) := \sum_{n=0}^{\infty} \frac{(\delta)_n}{\Gamma(\beta n+\gamma)\ n!}\ z^n,$$

where $\beta > 0$ and $\gamma > 0$. For $\delta = 1$, it reduces to Mittag-Leffler function $E_{\beta,\gamma}(z)$. Certain relations that exist between $E_{\beta,\gamma}^{\delta}(z)$ and the Riemann-Liouville fractional integrals and derivatives are investigated. It has been shown that the fractional integration and differentiation operators transform such functions with power multipliers into the functions of the same form. Some of the results given earlier by Kilbas and Saigo follow as special cases.

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1. Introduction and preliminaries

The function defined by the series representation

$$E_\beta(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\beta n + 1)} z^n \quad (\beta > 0, z \in \mathbb{C}) \tag{1}$$

and its generalization

$$E_{\beta,\gamma}(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\beta n + \gamma)} z^n \quad (\beta > 0, \gamma > 0, z \in \mathbb{C}) \tag{2}$$

were introduced and studied by Mittag-Leffler [21, 22], Wiman [25, 26], Agarwal [1], Humbert [10] and Humbert and Agarwal [11], where $\mathbb{C}$ is the set of complex numbers. The main properties of these functions are given in the book by Erdélyi *et al.* [4, Section 18.1] and a more extensive and detailed account on Mittag-Leffler functions is presented in Dzherbashyan [2, Chapter 2]. In particular, the functions (1) and (2) are entire functions of order $\rho = 1/\beta$ and type $\sigma = 1$; see, for example, [2, p.118].

The Mittag-Leffler function is not given in the tables of Laplace transforms, where it naturally occurs in the derivation of the inverse Laplace transform of the functions of the type $p^\varepsilon/(a + bp^\beta)$. This function also occurs in the solution of certain boundary value problems involving fractional integro-differential equations of Volterra-type [24]. During the various developments of Fractional Calculus in the last three decades this function has gained importance on account of its applications in the fields of physical, mathematical and engineering sciences. Hille and Tamarkin [9] have presented a solution of the Abel-Volterra type equation in terms of Mittag-Leffler function. For a detailed account of various properties, generalizations and applications of this function, the reader may refer to an excellent work of Dzherbashyan [2], Kilbas and Saigo [12, 13, 14, 15], Gorenflo and Mainardi [8], Gorenflo, Luchko and Rogosin [7] and Gorenflo, Kilbas and Rogosin [6].

By means of the series representation a generalization of (2) is introduced by Prabhakar [23] as:

$$E^{\delta}_{\beta,\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_n}{\Gamma(\beta n + \gamma)\; n!} z^n, \tag{3}$$

where $\beta, \gamma, \delta \in \mathbb{C}$ ($\mathrm{Re}(\beta) > 0$). It is an entire function of order $[\mathrm{Re}(\beta)]^{-1}$ [23, p. 7]. It is a special case of Wright's generalized hypergeometric function

[27, 28] as well as H-function [5] as shown in (5) and (6), below. For various properties of the function defined by (3), see [17].

Some important special cases of this function are enumerated below:

(i) $E_{\beta}(z) = E^{1}_{\beta,1}(z)$;

(ii) $E_{\beta,\gamma}(z) = E^{1}_{\beta,\gamma}(z)$;

(iii) $\beta\delta E^{\delta+1}_{\beta,\gamma}(z) = (1+\beta\delta-\gamma)E^{\delta}_{\beta,\gamma}(z) + E^{\delta}_{\beta,\gamma-1}(z)$;

(iv) $\Phi(\gamma,\delta;z) = \Gamma(\delta)E^{\gamma}_{1,\delta}(z)$, where $\Phi(\gamma,\delta;z)$ is the Kummer confluent hypergeometric function [3, p. 248, Eq. 1];

(v) $Z^{\mu}_{m}(z;k) = \Gamma(km+\mu+1)E^{-m}_{k,\mu+1}(z^k)$, where $m, k \in \mathbb{R}^{+} = [0,\infty)$ and $Z^{\mu}_{m}(\cdot)$ is a one set of the biorthogonal polynomial pair discussed by Konhauser [19];

(vi) If $\mu \in \mathbb{R}^{+}$, we have

$$E^{\gamma}_{m,\delta}(z) = \frac{1}{\Gamma(\delta)}\ {}_1F_m\left(\gamma; \mathit{\Delta}(\delta;m); \frac{z}{m^m}\right), \tag{4}$$

where ${}_1F_m(\cdot)$ is the generalized hypergeometric function and the symbol $\mathit{\Delta}(a;m)$ represents the sequence of parameters $a/m, (a+1)/m, \cdots,$ $(a+m-1)/m$;

(vii) $E^{\delta}_{\beta,\gamma}(z)$ has the forms:

$$E^{\delta}_{\beta,\gamma}(z) = \frac{1}{\Gamma(\delta)}\ {}_1\Psi_1\left[\begin{matrix}(\delta,1)\\(\gamma,\beta)\end{matrix};z\right] \tag{5}$$

$$= \frac{1}{\Gamma(\delta)}\ H^{1,1}_{1,2}\left[-z\,\middle|\,\begin{matrix}(1-\delta,1)\\(0,1),(1-\gamma,\beta)\end{matrix}\right] \tag{6}$$

$$= \frac{1}{2\pi\omega\Gamma(\delta)}\int_{\Omega}\frac{\Gamma(-s)\Gamma(\delta+s)}{\Gamma(\gamma+s\beta)}\,(-z)^s ds, \tag{7}$$

where ${}_1\Psi_1(\cdot)$ and $H^{1,1}_{1,2}(\cdot)$ are respectively Wright's generalized hypergeometric function [27] and H-function [5]. In (7), $\omega = \sqrt{-1}$ and the contour Ω is a straight line parallel to the imaginary axis separating the poles of $\Gamma(-s)$ at the points $s = \nu$ ($\nu \in \mathbb{N}_0 = \{0,1,2,\cdots\}$) from those of $\Gamma(\delta+s)$ at the points $s = -\delta-\nu$ ($\nu \in \mathbb{N}_0$). The poles of the integrand in (7) are assumed to be simple.

Formula (7) gives the Mellin-Barnes integral representation for the generalized Mittag-Leffler function $E^{\delta}_{\beta,\gamma}(z)$. A detailed account of H-function is available from the monographs of Mathai and Saxena [20] and Kilbas and Saigo [16].

Another generalization of Wiman function defined by (1) was recently introduced by Kilbas and Saigo [12] in terms of a special entire function of the form

$$E_{\alpha,m,l}(z) = \sum_{n=0}^{\infty} c_n z^n \quad \text{with} \quad c_n = \prod_{i=0}^{n-1} \frac{\Gamma(\alpha[im+l]+1)}{\Gamma(\alpha[im+l+1]+1)} \quad (n \in \mathbb{N}_0), \tag{8}$$

where an empty product is to be interpreted as unity. Certain properties of this function associated with Riemann-Liouville fractional integrals and derivatives are obtained and exact solutions of certain integral equations of Abel-Volterra type are derived by their applications [12, 14, 15]. The order and type of the above entire function, defined by (8) alongwith its recurrence relations, connection with hypergeometric functions and differential formulas are obtained by Gorenflo, Kilbas and Rogosin [6]. Also, see [6, 8] in this connection. In a recent paper, Kilbas, Saigo and Saxena [18] obtained a closed form solution of a fractional generalization of a free electron laser equation of the form:

$$D^{\alpha}_{\tau} a(\tau) = \lambda \int_0^{\tau} t^{\delta} a(\tau - t) E^{b}_{\rho,\delta+1}(i\nu t^{\rho}) dt + \beta \tau^{\sigma} E^{\gamma}_{\rho,\sigma+1}(i\nu\tau^{\rho}) \tag{9}$$

$$(0 \leqq \tau \leqq 1)$$

where $\beta, \lambda \in \mathbb{C}$, $\nu, b, \beta \in \mathbb{R}$, $\alpha > 0$, $\rho > 0$, $\sigma > -1$, $\delta > -1$ and $E^{b}_{\rho,\delta+1}(z)$ is the generalized Mittag-Leffler function defined by (3). The object of this paper is to derive the relations that exist between the generalized Mittag-Leffler function defined by (3) and the left- and right-sided operators of Riemann-Liouville fractional calculus [24]. The results derived in this paper are believed to be new.

The operators are defined by (see Samko, Kilbas and Marichev [24, Sect. 2]) for $\alpha > 0$:

$$\left(I^{\alpha}_{0+} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\, dt; \tag{10}$$

$$\left(I^{\alpha}_{-} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} \frac{f(t)}{(t-x)^{1-\alpha}}\, dt; \tag{11}$$

$$\begin{aligned}\left(D_{0+}^{\alpha} f\right)(x) &= \left(\frac{d}{dx}\right)^{[\alpha]+1} \left(I_{0+}^{1-\{\alpha\}} f\right)(x);\\ &= \frac{1}{\Gamma(1-\{\alpha\})} \left(\frac{d}{dx}\right)^{[\alpha]+1} \int_0^x \frac{f(t)}{(x-t)^{\{\alpha\}}}\, dt; \end{aligned} \tag{12}$$

$$\begin{aligned}\left(D_{-}^{\alpha} f\right)(x) &= \left(\frac{d}{dx}\right)^{[\alpha]+1} \left(I_{-}^{1-\{\alpha\}} f\right)(x);\\ &= \frac{1}{\Gamma(1-\{\alpha\})} \left(-\frac{d}{dx}\right)^{[\alpha]+1} \int_x^\infty \frac{f(t)}{(t-x)^{\{\alpha\}}}\, dt, \end{aligned} \tag{13}$$

where $[\alpha]$ means the maximal integer not exceeding α and $\{\alpha\}$ is the fractional part of α.

2. Properties of generalized Mittag-Leffler function

In this section we derive several interesting properties of the generalized Mittag-Leffler function $E_{\beta,\gamma}^{\delta}(z)$ defined by (3) associated with Riemann-Liouville fractional integrals and derivatives.

THEOREM 1. *Let $\alpha > 0, \beta > 0, \gamma > 0$ and $a \in \mathbb{R}$. Let I_{0+}^{α} be the left-sided operator of Riemann-Liouville fractional integral* (10). *Then there holds the formula*

$$\left(I_{0+}^{\alpha}[t^{\gamma-1} E_{\beta,\gamma}^{\delta}(at^{\beta})]\right)(x) = x^{\alpha+\gamma-1} E_{\beta,\alpha+\gamma}^{\delta}(ax^{\beta}). \tag{14}$$

P r o o f. By virtue of (3) and (10) we have

$$\begin{aligned} K &\equiv \left(I_{0+}^{\alpha}[t^{\gamma-1} E_{\beta,\gamma}^{\delta}(at^{\beta})]\right)(x)\\ &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \sum_{n=0}^{\infty} \frac{(\delta)_n}{\Gamma(\beta n+\gamma)\; n!}\; a^n t^{n\beta+\gamma-1} dt. \end{aligned}$$

Interchanging the order of integration and summation and evaluating the inner integral by beta-function formula, it gives

$$K = x^{\alpha+\gamma-1} \sum_{n=0}^{\infty} \frac{(\delta)_n}{\Gamma(\alpha+\beta n+\gamma)\; n!}\; (ax^{\beta})^n = x^{\alpha+\gamma-1} E_{\beta,\alpha+\gamma}^{\delta}(ax^{\beta}).$$

The interchange of the order of integration and summation is permissible under the conditions stated with the theorem dues to convergence of the integrals involved in the process. This completes the proof of Theorem 1. ■

COROLLARY 1.1. *For* $\alpha > 0, \beta > 0, \gamma > 0$ *and* $a \in \mathbb{R}$, *there holds the formula*

$$\left(I_{0+}^{\alpha}[t^{\gamma-1}E_{\beta,\gamma}(at^{\beta})]\right)(x) = x^{\alpha+\gamma-1}E_{\beta,\alpha+\gamma}(ax^{\beta}). \tag{15}$$

REMARK 1. The formula (15) is a known relation [24, Table 9.1, Formula 23]. If we set $\beta = \alpha$ in (15), then in view of the relation [4, p. 210, Eq. 23]

$$E_{\alpha,\gamma}(x) = \frac{1}{\Gamma(\gamma)} + xE_{\alpha,\alpha+\gamma}(x) \tag{16}$$

it follows that

$$\left(I_{0+}^{\alpha}[t^{\gamma-1}E_{\alpha,\gamma}(at^{\alpha})]\right)(x) = \frac{x^{\gamma-1}}{a}\left[E_{\alpha,\gamma}(ax^{\alpha}) - \frac{1}{\Gamma(\gamma)}\right] \quad (a \neq 0). \tag{17}$$

We now give a lemma.

LEMMA 1. *For* $a \in \mathbb{R}$ *there holds the formula*

$$ax^{\beta}E_{\beta,\gamma}^{\delta}(ax^{\beta}) = E_{\beta,\gamma-\beta}^{\delta}(ax^{\beta}) - E_{\beta,\gamma-\beta}^{\delta-1}(ax^{\beta}). \tag{18}$$

P r o o f. The formula (18) is easily verified by virtue of the relation $(n+1)(\delta)_n = (\delta)_{n+1} - (\delta-1)_{n+1}$. ■

Theorem 1 and Lemma 1 imply

THEOREM 2. *Let* $\alpha > 0, \beta > 0, \gamma > 0, a \in \mathbb{R}$ $(a \neq 0)$ *and let* I_{0+}^{α} *be the left-sided operator of Riemann-Liouville fractional integral* (10). *Then there holds the formula*

$$\left(I_{0+}^{\alpha}[t^{\gamma-1}E_{\beta,\gamma}^{\delta}(at^{\beta})]\right)(x)$$

$$= \frac{1}{a}x^{\alpha+\gamma-\beta-1}\left[E_{\beta,\alpha+\gamma-\beta}^{\delta}(ax^{\beta}) - E_{\beta,\alpha+\gamma-\beta}^{\delta-1}(ax^{\beta})\right]. \tag{19}$$

COROLLARY 2.1. *For $\alpha > 0, \beta > 0, \gamma > 0$ with $\alpha + \gamma > \beta$ and for $a \in \mathbb{R}$ $(a \neq 0)$, there holds the formula*

$$\Big(I_{0+}^{\alpha}[t^{\gamma-1}E_{\beta,\gamma}(at^{\beta})]\Big)(x)$$

$$= \frac{1}{a}x^{\alpha+\gamma-\beta-1}\left[E_{\beta,\alpha+\gamma-\beta}(ax^{\beta}) - \frac{1}{\Gamma(\alpha+\gamma-\beta)}\right]. \qquad (20)$$

REMARK 2. When $\beta = \alpha$ in (20), it yields the following result given by Kilbas and Saigo [14, p. 359, Eq. 20]

$$\Big(I_{0+}^{\alpha}[t^{\gamma-1}E_{\alpha,\gamma}(at^{\alpha})]\Big)(x) = \frac{x^{\gamma-1}}{a}\left[E_{\alpha,\gamma}(ax^{\alpha}) - \frac{1}{\Gamma(\gamma)}\right] \quad (a \neq 0); \qquad (21)$$

$$\Big(I_{0+}^{\alpha}[E_{\alpha}(at^{\alpha})]\Big)(x) = \frac{1}{a}\left[E_{\alpha}(ax^{\alpha}) - 1\right] \quad (a \neq 0). \qquad (22)$$

THEOREM 3. *Let $\alpha > 0, \beta > 0, \gamma > 0, a \in \mathbb{R}$ and let I_{-}^{α} be the right-sided operator of Riemann-Liouville fractional integral* (11). *Then there holds the formula*

$$\Big(I_{-}^{\alpha}[t^{-\alpha-\gamma}E_{\beta,\gamma}^{\delta}(at^{-\beta})]\Big)(x) = x^{-\gamma}E_{\beta,\alpha+\gamma}^{\delta}(ax^{-\beta}). \qquad (23)$$

P r o o f. By virtue of (3) and (11) we find

$$K \equiv \Big(I_{-}^{\alpha}[t^{-\alpha-\gamma}E_{\beta,\gamma}^{\delta}(at^{-\beta})]\Big)(x)$$

$$= \frac{1}{\Gamma(\alpha)}\int_{x}^{\infty} t^{-\alpha-\gamma}(t-x)^{\alpha-1}\sum_{n=0}^{\infty}\frac{(\delta)_n}{\Gamma(\beta n+\gamma)\; n!}\; a^n t^{-\beta n}dt.$$

Interchanging the order of integration and summation and evaluating the inner integral, we obtain

$$K = \sum_{n=0}^{\infty}\frac{(\delta)_n}{\Gamma(\beta n+\alpha+\gamma)\; n!}\; a^n x^{-\beta n-\gamma} = x^{-\gamma}E_{\beta,\alpha+\gamma}^{\delta}(ax^{-\beta}).$$

■

Corollary 3.1. *For $\alpha > 0, \beta > 0, \gamma > 0, a \in \mathbb{R}$ there holds the formula*

$$\left(I_{-}^{\alpha}[t^{-\alpha-\gamma}E_{\beta,\gamma}(at^{-\beta})]\right)(x) = x^{-\gamma}E_{\beta,\alpha+\gamma}(ax^{-\beta}). \tag{24}$$

Theorem 4. *Let $\alpha > 0, \beta > 0, \gamma > 0, a \in \mathbb{R}$ $(a \neq 0)$ and let I_{-}^{α} be the right-sided operator of Riemann-Liouville fractional integral* (11). *Then there holds the formula*

$$\left(I_{-}^{\alpha}[t^{-\alpha-\gamma}E_{\beta,\gamma}^{\delta}(at^{-\beta})]\right)(x)$$

$$= \frac{1}{a}x^{\beta-\gamma}\left[E_{\beta,\alpha+\gamma-\beta}^{\delta}(ax^{-\beta}) - E_{\beta,\alpha+\gamma-\beta}^{\delta-1}(ax^{-\beta})\right]. \tag{25}$$

The proof can be developed on similar lines to that of Theorem 3.

Corollary 4.1. *For $\alpha > 0, \beta > 0, \gamma > 0$ with $\alpha + \gamma > \beta$ and for $a \in \mathbb{R}$ $(a \neq 0)$ there holds the formula*

$$\left(I_{-}^{\alpha}[t^{-\alpha-\gamma}E_{\beta,\gamma}(at^{-\beta})]\right)(x) = \frac{1}{a}x^{\beta-\gamma}\left[E_{\beta,\alpha+\gamma-\beta}(ax^{-\beta}) - \frac{1}{\Gamma(\alpha+\gamma-\beta)}\right]. \tag{26}$$

Remark 3. If we set $\beta = \alpha$ in (26) it reduces to the following result given earlier by Kilbas and Saigo [14, p. 360, Eq. 25], $a \neq 0$:

$$\left(I_{-}^{\alpha}[t^{-\alpha-\gamma}E_{\alpha,\gamma}(at^{-\alpha})]\right)(x) = \frac{x^{\alpha-\gamma}}{a}\left[E_{\alpha,\gamma}(ax^{-\alpha}) - \frac{1}{\Gamma(\gamma)}\right] \quad (a \neq 0); \tag{27}$$

$$\left(I_{-}^{\alpha}[t^{-\alpha-1}E_{\alpha}(at^{-\alpha})]\right)(x) = \frac{x^{\alpha-1}}{a}\left[E_{\alpha}(ax^{-\alpha}) - 1\right] \quad (a \neq 0). \tag{28}$$

We now proceed to derive certain other properties of $E_{\beta,\gamma}^{\delta}(z)$ associated with the fractional derivative operators D_{+0}^{α} and D_{-}^{α} defined by (12) and (13) respectively.

THEOREM 5. *Let $\alpha > 0, \beta > 0, \gamma > 0, a \in \mathbb{R}$ and let D_{0+}^{α} be the left-sided operator of Riemann-Liouville fractional derivative* (12). *Then there holds the formula*

$$\left(D_{0+}^{\alpha}[t^{\gamma-1}E_{\beta,\gamma}^{\delta}(at^{\beta})]\right)(x) = x^{\gamma-\alpha-1}E_{\beta,\gamma-\alpha}^{\delta}(ax^{\beta}). \tag{29}$$

P r o o f. By virtue of (3) and (12) we have

$$K \equiv \left(D_{0+}^{\alpha}[t^{\gamma-1}E_{\beta,\gamma}^{\delta}(at^{\beta})]\right)(x) = \left(\frac{d}{dx}\right)^{[\alpha]+1}\left(I_{0+}^{1-\{\alpha\}}[t^{\gamma-1}E_{\beta,\gamma}^{\delta}(at^{\beta})]\right)(x)$$

$$= \sum_{n=0}^{\infty}\frac{a^n(\delta)_n}{\Gamma(\gamma+n\beta)\Gamma(1-\{\alpha\})n!}\left(\frac{d}{dx}\right)^{[\alpha]+1}\int_0^x t^{n\beta+\gamma-1}(x-t)^{-\{\alpha\}}dt$$

$$= \sum_{n=0}^{\infty}\frac{a^n(\delta)_n}{\Gamma(\gamma+n\beta+1-\{\alpha\})n!}\left(\frac{d}{dx}\right)^{[\alpha]+1}x^{n\beta+\gamma-\{\alpha\}}$$

$$= \sum_{n=0}^{\infty}\frac{a^n(\delta)_n}{\Gamma(n\beta+\gamma-\alpha)n!}x^{\gamma+n\beta-\alpha-1} = x^{\gamma-\alpha-1}E_{\beta,\gamma-\alpha}^{\delta}(ax^{\beta}),$$

which proves the theorem. ∎

COROLLARY 5.1. *For $\alpha > 0, \beta > 0, \gamma > 0, a \in \mathbb{R}$ there holds the formula*

$$\left(D_{0+}^{\alpha}[t^{\gamma-1}E_{\beta,\gamma}(at^{\beta})]\right)(x) = x^{\gamma-\alpha-1}E_{\beta,\gamma-\alpha}(ax^{\beta}). \tag{30}$$

If, however, we set $\beta = \alpha$ and $\delta = 1$, then (29) also reduces to the relation ([14, p. 362, Eq. 35])

$$\left(D_{0+}^{\alpha}[t^{\gamma-1}E_{\alpha,\gamma}(at^{\alpha})]\right)(x) = \frac{x^{\gamma-\alpha-1}}{\Gamma(\gamma-\alpha)} + ax^{\gamma-1}E_{\alpha,\gamma}(ax^{\alpha}). \tag{31}$$

When $\gamma = 1$ in (31) there holds the formula

$$\left(D_{0+}^{\alpha}[E_{\alpha}(at^{\alpha})]\right)(x) = \frac{x^{-\alpha}}{\Gamma(1-\alpha)} + aE_{\alpha}(ax^{\alpha}). \tag{32}$$

Following a similar procedure, we arrive at the following theorem:

Theorem 6. *Let $\alpha > 0, \gamma > \beta > 0, a \in \mathbb{R}$ $(a \neq 0)$ and let D^{α}_{0+} be the left-sided operator of Riemann-Liouville fractional derivative* (12). *Then there holds the formula*

$$\left(D^{\alpha}_{0+}[t^{\gamma-1}E^{\delta}_{\beta,\gamma}(at^{\beta})]\right)(x)$$

$$= \frac{1}{a}x^{\gamma-\alpha-\beta-1}\left[E^{\delta}_{\beta,\gamma-\alpha-\beta}(ax^{\beta}) - E^{\delta-1}_{\beta,\gamma-\alpha-\beta}(ax^{\beta})\right]. \tag{33}$$

Corollary 6.1. *Let $\alpha > 0, \gamma > \beta > 0, a \in \mathbb{R}$ $(a \neq 0)$, then there holds the formula*

$$\left(D^{\alpha}_{0+}[t^{\gamma-1}E_{\beta,\gamma}(at^{\beta})]\right)(x)$$

$$= \frac{1}{a}x^{\gamma-\alpha-\beta-1}\left[E_{\beta,\gamma-\alpha-\beta}(ax^{\beta}) - \frac{1}{\Gamma(\gamma-\alpha-\beta)}\right]. \tag{34}$$

Theorem 7. *Let $\alpha > 0, \gamma > 0$ with $\gamma-\alpha+\{\alpha\} > 1$, and $a \in \mathbb{R}$, and let D^{α}_{-} be the right-sided operator of Riemann-Liouville fractional derivative* (13). *Then there holds the formula*

$$\left(D^{\alpha}_{-}[t^{\alpha-\gamma}E^{\delta}_{\beta,\gamma}(at^{-\beta})]\right)(x) = x^{-\gamma}E^{\delta}_{\beta,\gamma-\alpha}(ax^{-\beta}). \tag{35}$$

P r o o f. From (3) and (13) it follows that

$$K \equiv \left(D^{\alpha}_{-}[t^{\alpha-\gamma}E^{\delta}_{\beta,\gamma}(at^{-\beta})]\right)(x)$$

$$= \left(-\frac{d}{dx}\right)^{[\alpha]+1}\left(I^{1-\{\alpha\}}_{-}[t^{\alpha-\gamma}E^{\delta}_{\beta,\gamma}(at^{-\beta})]\right)(x)$$

$$= \sum_{n=0}^{\infty}\frac{a^n(\delta)_n}{\Gamma(n\beta+\gamma)\Gamma(1-\{\alpha\})n!}\left(-\frac{d}{dx}\right)^{[\alpha]+1}\int_x^{\infty}t^{-n\beta+\alpha-\gamma}(t-x)^{-\{\alpha\}}dt.$$

If we set $t = x/u$, then the above expression transforms into the form

$$K = \sum_{n=0}^{\infty}\frac{a^n(\delta)_n}{\Gamma(n\beta+\gamma)\Gamma(1-\{\alpha\})n!}$$

$$\times \int_0^1 u^{n\beta-\alpha+\gamma+\{\alpha\}-2}(1-u)^{-\{\alpha\}}du\left(-\frac{d}{dx}\right)^{[\alpha]+1}x^{\alpha-n\beta-\gamma-\{\alpha\}+1}$$

$$= \sum_{n=0}^{\infty}\frac{a^n(\delta)_n}{\Gamma(n\beta+\gamma-\alpha)n!}\,x^{-n\beta-\gamma} = x^{-\gamma}E^{\delta}_{\beta,\gamma-\alpha}(ax^{-\beta}).$$

■

COROLLARY 7.1. *Let $\alpha > 0, \gamma > \beta > 0$ and $a \in \mathbb{R}$, then there holds the formula*

$$\Big(D_{-}^{\alpha}[t^{\alpha-\gamma}E_{\beta,\gamma}(at^{-\beta})]\Big)(x) = x^{-\gamma}E_{\beta,\gamma-\alpha}(ax^{-\beta}). \tag{36}$$

In a similar manner we can prove the following theorems and corollaries.

THEOREM 8. *Let $\alpha > 0, \beta > 0$ with $\gamma - [\alpha] > 1$, $a \in \mathbb{R}$ $(a \neq 0)$ and let D_{-}^{α} be the right-sided operator of Riemann-Liouville fractional derivative* (13). *Then there holds the formula*

$$\Big(D_{-}^{\alpha}[t^{\alpha-\gamma}E_{\beta,\gamma}^{\delta}(at^{-\beta})]\Big)(x)$$

$$= \frac{1}{a}x^{\beta-\gamma}\left[E_{\beta,\gamma-\alpha-\beta}^{\delta}(ax^{-\beta}) - E_{\beta,\gamma-\alpha-\beta}^{\delta-1}(ax^{-\beta})\right]. \tag{37}$$

COROLLARY 8.1. *For $\alpha > 0, \beta > 0$ with $\gamma - [\alpha] > 1$, $a \in \mathbb{R}$ $(a \neq 0)$ let $\alpha + \gamma > \beta$, there holds the formula*

$$\Big(D_{-}^{\alpha}[t^{\alpha-\gamma}E_{\beta,\gamma}(at^{-\beta})]\Big)(x)$$

$$= \frac{1}{a}x^{\beta-\gamma}\left[E_{\beta,\gamma-\alpha-\beta}(ax^{-\beta}) - \frac{1}{\Gamma(\gamma-\alpha-\beta)}\right]. \tag{38}$$

THEOREM 9. *Let $\alpha > 0, \beta > 0$ with $\gamma - [\alpha] > 0$, $a \in \mathbb{R}$ $(a \neq 0)$ and let D_{-}^{α} be the right-sided operator of Riemann-Liouville fractional derivative* (13). *Then there holds the formula*

$$\Big(D_{-}^{\alpha}[t^{\alpha-\gamma-1}E_{\beta,\gamma}^{\delta}(at^{-\beta})]\Big)(x)$$

$$= \frac{1}{a}x^{-\gamma-1}\left[(\gamma-\beta\delta)E_{\beta,\gamma-\alpha+1}^{\delta}(ax^{-\beta}) + \beta\delta E_{\beta,\gamma-\alpha-1}^{\delta+1}(ax^{-\beta})\right]. \tag{39}$$

COROLLARY 9.1. *Let $\alpha > 0, \beta > 0$ with $\gamma - [\alpha] > 0$ and $a \in \mathbb{R}$ $(a \neq 0)$, then there holds the formula*

$$\left(D_{-}^{\alpha}[t^{\alpha-\gamma-1}E_{\beta,\gamma}(at^{-\beta})]\right)(x)$$

$$= \frac{1}{a}x^{-\gamma-1}\left[(\gamma-\beta)E_{\beta,\gamma-\alpha+1}(ax^{-\beta}) + \beta E^{2}_{\beta,\gamma-\alpha+1}(ax^{-\beta})\right]. \quad (40)$$

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