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NUMERICAL RESULTS FOR THE GENERALIZED MITTAG-LEFFLER FUNCTION

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Dedicated to Acad. Bogoljub Stanković
on the occasion of his 80th birthday

Abstract

Results of extensive calculations for the generalized Mittag-Leffler function $E_{0.8,0.9}(z)$ are presented in the region $-8 \leq \operatorname{Re} z \leq 5$ and $-10 \leq \operatorname{Im} z \leq 10$ of the complex plane. This function is related to the eigenfunction of a fractional derivative of order $\alpha = 0.8$ and type $\beta = 0.5$.

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1. Introduction

The right-sided, resp. left-sided, fractional derivative $\mathrm{D}_{a\pm}^{\alpha,\beta} f(x)$ of $f(x)$ of order $0 < \alpha < 1$ and type $0 \leq \beta \leq 1$ with respect to x was introduced in [6]. It is defined by

$$\mathrm{D}_{a\pm}^{\alpha,\beta} f(x) = \left(\pm \mathrm{I}_{a\pm}^{\beta(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} (\mathrm{I}_{a\pm}^{(1-\beta)(1-\alpha)} f) \right)(x) \tag{1}$$

for functions for which the expression on the right hand side exists. Here the right-sided Riemann-Liouville fractional integral of order $\alpha > 0, \alpha \in \mathbb{R}$ of a locally integrable function f is defined as

$$(\mathrm{I}^{\alpha}_{a+}f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y)\,\mathrm{d}y \tag{2a}$$

for $x > a$, the left-sided Riemann-Liouville fractional integral is defined as

$$(\mathrm{I}^{\alpha}_{a-}f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^a (y-x)^{\alpha-1} f(y)\,\mathrm{d}y \tag{2b}$$

for $x < a$. The familiar Riemann-Liouville fractional derivative [14, 15]

$$\mathrm{D}^{\alpha}_{a\pm}f(x) = \mathrm{D}^{\alpha,0}_{a\pm}f(x) = \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{I}^{(1-\alpha)}_{a\pm}f)(x) \tag{3}$$

corresponds to $a > -\infty$ and type $\beta = 0$. Derivatives of order α and type $\beta = 1$ were already introduced by Liouville [11, p. 10]. They were later applied by Caputo and coworkers [1]. It is interesting to note that derivatives that would correspond to the degenerate cases $\beta \geq 1$ were apparently already discussed by Dzherbashyan and Nersesyan [15, p. 88]. To the best of our knowledge none of these authors discussed the operators in (1) interpolating between type $\beta = 0$ and $\beta = 1$.

For $\alpha = 1$ the fractional derivative reduces to the ordinary first order derivative operator whose eigenfunction is the exponential function. Given the great importance of the eigenfunction of the derivative operator in all applications it is of interest to investigate also the eigenfunctions of the fractional derivative operators [6].

Using the basic fractional integral

$$\mathrm{I}^{\alpha}_{a+}(x-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(x-a)^{\alpha+\beta}\,, \tag{4}$$

it is readily seen that the eigenvalue equation

$$\mathrm{D}^{\alpha,\beta}_{0+}f(x) = \lambda\, f(x) \tag{5}$$

is solved by [6]

$$f(x) = x^{(1-\beta)(\alpha-1)} E_{\alpha,\alpha+\beta(1-\alpha)}(\lambda x^{\alpha}). \tag{6}$$

Here the generalized Mittag-Leffler function is defined by

$$E_{a,b}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak+b)} \tag{7}$$

for all $a > 0, b \in \mathbb{C}, z \in \mathbb{C}$. Of course, $E_{a,b}(z)$ generalizes the exponential function $\exp(z) = E_{1,1}(z)$.

For $\beta = 0$ the well known result [15]

$$f(x) = x^{(\alpha-1)} E_{\alpha,\alpha}(\lambda x^\alpha) \tag{8}$$

is recovered. Of special interest for applications in physics is also the case $\beta = 1$ [6]. It has the solution

$$f(x) = E_\alpha(\lambda x^\alpha) \tag{9}$$

where $E_\alpha(z) = E_{\alpha,1}(z)$ denotes the ordinary Mittag-Leffler function.

Note also that for $\lambda = 0$ equation (6) reduces to a power law

$$f(x) = \frac{x^{(1-\beta)(\alpha-1)}}{\Gamma((1-\beta)(\alpha-1)+1)}, \tag{10}$$

because $E_{a,b}(0) = 1/\Gamma(b)$.

An early comprehensive treatise of the Mittag-Leffler function $E_a(z)$ was given soon after its introduction in [13]. The generalized Mittag-Leffler function $E_{a,b}(z)$ appears to have been first introduced in [17]. It was later discussed in [10] (but does not appear in [9]). A summary of the properties of $E_a(z)$ and $E_{a,b}(z)$ as well as more references to the early literature can be found in [3]. The generalized Mittag-Leffler function $E_{a,b}(z)$ is an example of an entire function of order $1/a$, and it is completely monotone if and only if $0 < a \le 1$ and $b \ge a$ [16, 12].

The generalized Mittag-Leffler function appears in applications not only as the eigenfunction of generalized fractional derivatives. The eigenfunction $x^{(\alpha-1)} E_{\alpha,\alpha}(x^\alpha)$ of the usual Riemann-Liouville derivative given in eq. (8) appears in the theory of continuous time random walks [8, 7] as the waiting time density corresponding to a master equation with fractional time derivative of order α and type $\beta = 1$. On the other hand the function $E_\alpha(x^\alpha)$ from eq. (9) is known to appear in the theory of renewal processes [4].

2. Problem and objective

In view of the growing importance and applications of the Mittag-Leffler function it becomes necessary to investigate its behaviour not only on the real axis, but in the whole complex plane. While some numerical calculations have been reported in the literature [8, 5] mostly for small real arguments, there seem to be no results available in the complex plane.

The objective of this paper is to study the generalized Mittag-Leffler function numerically in the complex plane. As an example we present extensive calculations for the generalized Mittag-Leffler function $E_{0.8,0.9}(z)$. This function is related to the eigenfunction of a fractional derivative of order $\alpha = 0.8$ and type $\beta = 0.5$.

3. Methodical remarks

The Mittag-Leffler function can be calculated from its series expansion in eq. (7) which converges for all $z \in \mathbb{C}$. This method of evaluation is numerically useful for small z and we use it inside a small disc around the origin. This is indicated as the crosshatched region in **Figure 1**. For other values of z, however, we insert Hankel's integral representation [2]

$$\frac{1}{\Gamma(ak+b)} = \frac{1}{2\pi i} \int\limits_{\mathcal{L}_H} e^s s^{-(ak+b)} \mathrm{d}s \tag{11}$$

for the reciprocal Γ-function into eq. (7). Hankel's contour $\mathcal{L}_H$ starts at $-\infty$ below the real axis, encircles the origin counterclockwise, and returns to $-\infty$ above the real axis. Summing the geometric series in the resulting expression gives Mittag-Leffler's integral representation [3]

$$E_{a,b}(z) = \frac{1}{2\pi i} \int\limits_{\mathcal{L}_H} \frac{s^{a-b} e^s}{s^a - z} \mathrm{d}s \tag{12}$$

valid for all $z \in \mathbb{C}$. Here the contour is chosen to encircle the disc $|s| \leq |z|^{1/a}$.

Being interested mainly in the case $0 < a < 2$, we follow [18] to arrive at the integral representations

$$E_{a,b}(z) = \frac{1}{2\pi i a} \int\limits_{\mathcal{L}_{2/3}} \frac{s^{(1-b)/a} e^{s^{1/a}}}{s - z} \mathrm{d}s \tag{13a}$$

if z lies to the left of the contour $\mathcal{L}_{2/3}$, and

$$E_{a,b}(z) = \frac{z^{(1-b)/a}}{a} e^{z^{1/a}} + \frac{1}{2\pi i a} \int\limits_{\mathcal{L}_1} \frac{s^{(1-b)/a} e^{s^{1/a}}}{s - z} \mathrm{d}s \tag{13b}$$

if z lies to the right of the contour $\mathcal{L}_1$. In the first formula the contour $\mathcal{L}_{2/3}$ emanates from infinity in the lower half plane along the line $\arg(z) =$

$-2a\pi/3$, avoids the origin using a circular arc, and proceeds towards infinity in the upper half plane along the line $\arg(z) = 2a\pi/3$. The contour $\mathcal{L}_1$ emanates from infinity in the lower half plane along the line $\arg(z) = -a\pi$, avoids the origin, and proceeds towards infinity in the upper half plane along the line $\arg(z) = a\pi$. The regions to the left of $\mathcal{L}_{2/3}$ and to the right of $\mathcal{L}_1$ have nonzero intersection and this allows us to avoid the case where z falls directly onto the integration contour. In **Figure 1** we show the contours and the regions.

The integrals are then evaluated numerically using a Gauss-Lobatto scheme to obtain the subsequent results.

4. Results

In this section we present the results of extensive numerical calculations of the Mittag-Leffler function $E_{0.8,0.9}(z)$ in the complex plane. The function was evaluated on a grid of 801×481 points, i.e. for 385281 numbers z with $-8 \le \operatorname{Re} z \le 5$ and $-10 \le \operatorname{Im} z \le 10$.

First we show a three-dimensional plot of the real and imaginary parts. On these surfaces the contour line with $\operatorname{Re} E_{0.8,0.9}(z) = 0$, resp. $\operatorname{Im} E_{0.8,0.9}(z) = 0$ is shown as a thick solid line. The gray scale reflects function values. White corresponds to high function values and black to low values. Note the different scales on the axes.

In **Figures 4 and 5** we show the real part, $\operatorname{Re} E_{0.8,0.9}(z)$, and imaginary part, $\operatorname{Im} E_{0.8,0.9}(z)$, but now truncated at ± 3. The axes are scaled such that the lengths on each axis are equal. A different viewpoint has been chosen to make the thick contour line for level 0 better visible.

Next, **Figure 6** shows a combined contour plot of the real and imaginary part of $E_{0.8,0.9}(z)$. The contour lines of the real part are drawn as solid lines whereas those for the imaginary part are shown dashdotted. The thick lines represent the contour lines for level 0 which have already been indicated in **Figures 2 and 3**. The intersection of a thick solid line and a thick dash-dotted line represents a zero.

Finally we plot the logarithm of the absolute value as $\log_{10}(|E_{0.8,0.9}(z)|)$ in **Figure 7**.

5. Discussion

The behaviour of $\operatorname{Re} E_{a,b}(z)$ is characterized by the wedge $|\arg z| \le a\pi/2$ having its apex at $z = 0$ and opening angle $a\pi$. For $a \to 1$ the wedge opens and the function approaches the exponential function. For $a \to 0$ the wedge closes and approaches the positive real axis.

Inside the wedge the real and imaginary parts oscillate and increase to infinity. Outside of the wedge they decrease to zero. Along the delimiting lines of the wedge the function approaches $1/a$ in an oscillatory fashion. Inside the wedge the function $E_{a,b}(z)$ grows as $E_{a,b}(z) \sim e^{z^{1/a}}$ with $z \to \infty$. This can be verified for $a = 0.8$ from **Figure 7**, where the logarithm of the absolute value is seen to increase linearly for $z \to \infty$.

Contrary to the exponential function (corresponding to $a = 1$) the function shows zeros in the complex plane. These are best seen in **Figure 6** as the intersection of thick solid and thick dash-dotted lines, and in **Figure 7** as singularities in the logarithm of the absolute value. The value of b influences the position of the zeros. For $E_{0.8,0.9}(z)$ we find the first pair of conjugate zeros at $1.09 \pm 4.20i$ with an accuracy of roughly 0.001 in real and imaginary part.

The function is purely real, resp. purely imaginary, on the lines where the imaginary resp. real part vanish. These are shown in **Figures 2 through 5**. To the best of our knowledge these lines have not been studied before. Their behaviour changes dramatically as the parameters a and b are changed.

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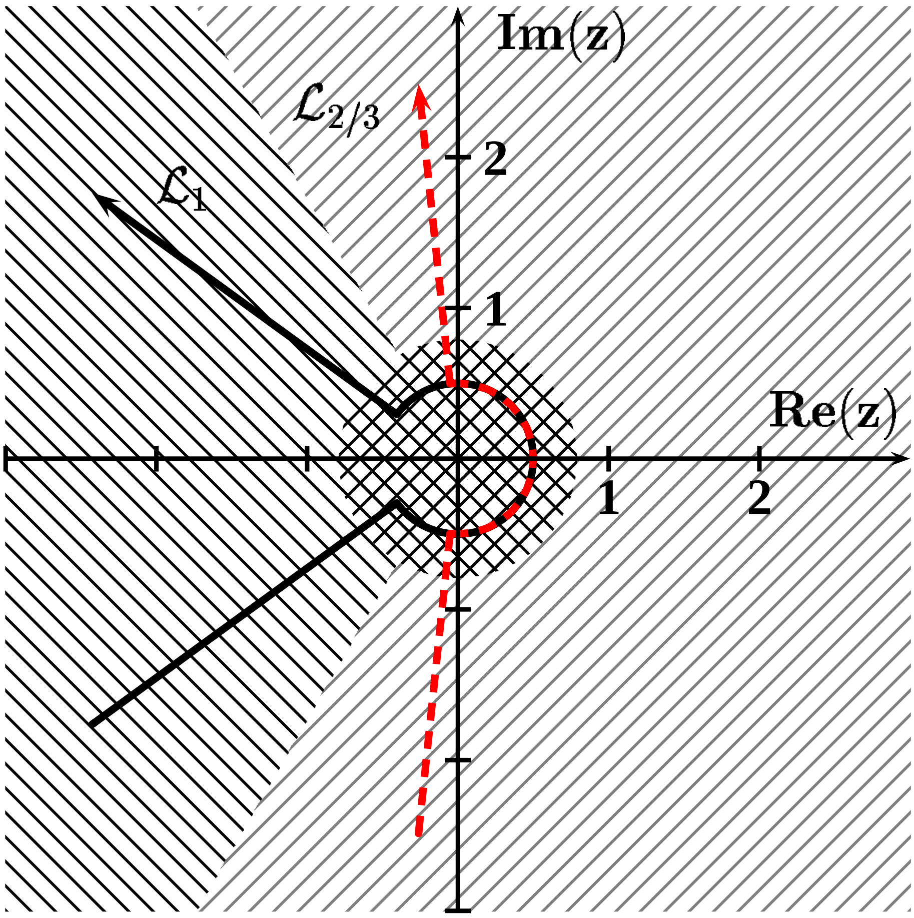

Figure 1: Integration contours and regions. The power series in eq. (7) is used in the crosshatched region. Equation (13a) is used in the region hatched with lines running at -45 degrees. Equation (13b) is used in the region hatched with lines running at +45 degrees.

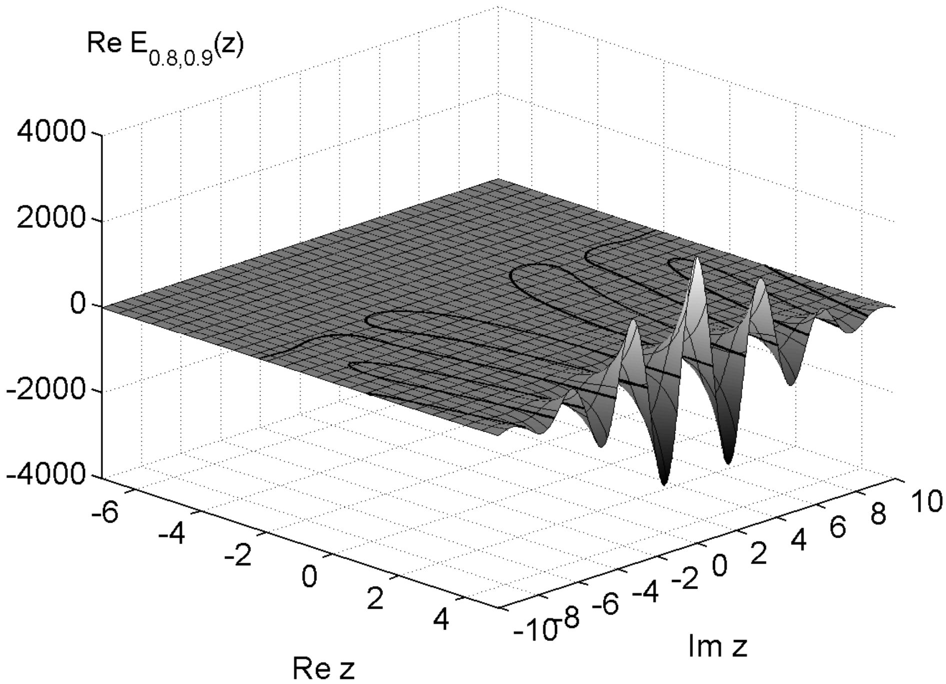

Figure 2: Real part of $E_{0.8,0.9}(z)$. The thick solid line marks the contour line $\operatorname{Re} E_{0.8,0.9}(z) = 0$.

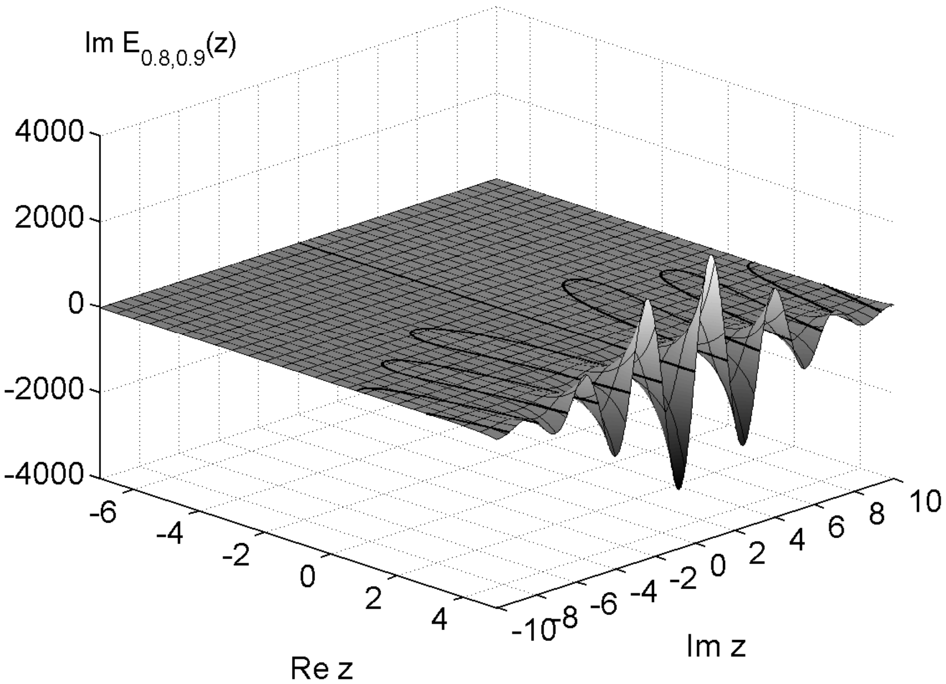

Figure 3: Imaginary part of $E_{0.8,0.9}(z)$. The thick solid line marks the contour line $\operatorname{Im} E_{0.8,0.9}(z) = 0$.

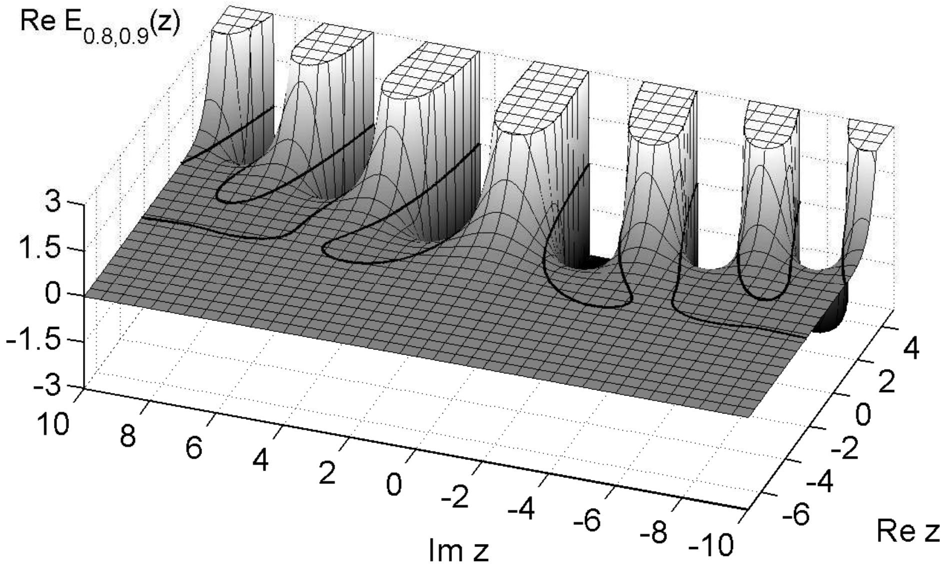

Figure 4: Real part of $E_{0.8,0.9}(z)$ truncated at ± 3. The thick solid line marks the contour line $\mathrm{Re}\, E_{0.8,0.9}(z) = 0$.

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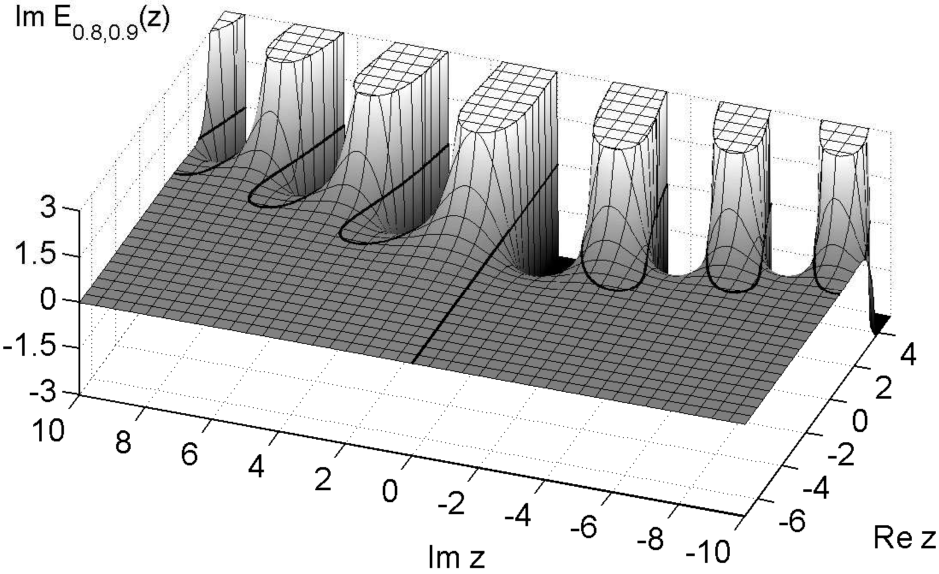

Figure 5: Imaginary part of $E_{0.8,0.9}(z)$ truncated at ± 3. The thick solid line marks the contour line $\operatorname{Im} E_{0.8,0.9}(z) = 0$.

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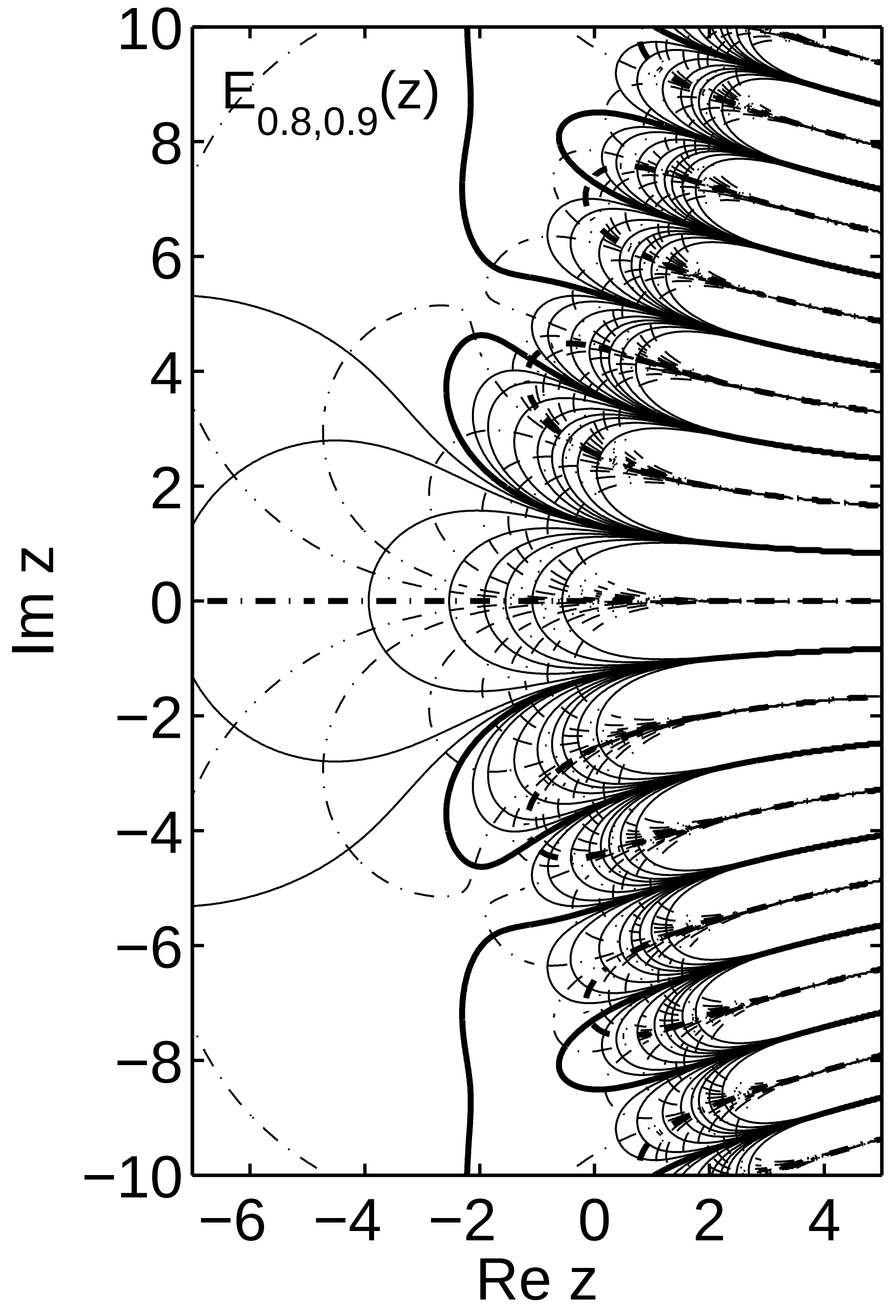

Figure 6: Contour lines $\mathrm{Re}\, E_{0.8,0.9}(z) = 0, \pm 0.01, \pm 0.02$, ± 0.05, ± 0.1, ± 0.15, ± 0.2, ± 0.3, ± 0.5 (solid) and $\mathrm{Im}\, E_{0.8,0.9}(z) = 0, \pm 0.01, \pm 0.02$, ± 0.05, ± 0.1, ± 0.15, ± 0.2, ± 0.3, ± 0.5 (dash-dotted)

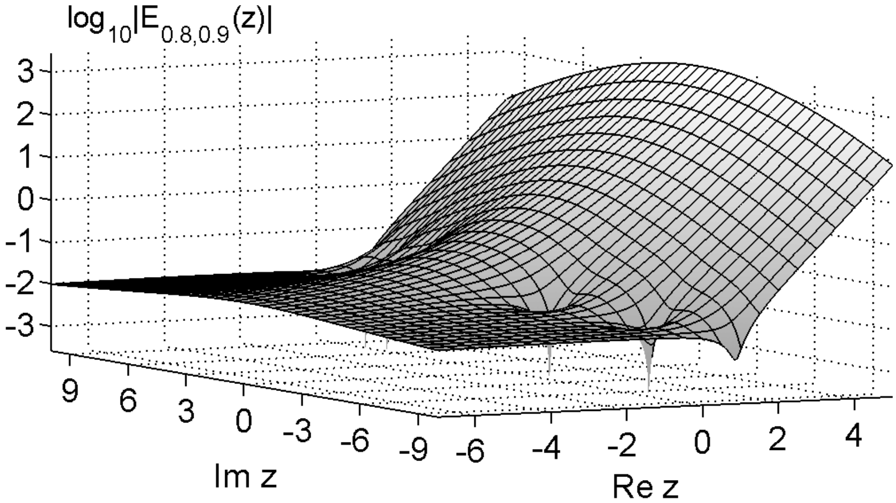

Figure 7: Decadic logarithm of absolute value $|E_{0.8,0.9}(z)|$

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