Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 7, NUMBER 4 (2004) ISSN 1311-0454

SUGGESTION FROM THE PAST?

J. A. Tenreiro Machado [1], **Isabel S. Jesus** [2]

Abstract

The generalization of the concept of derivative to non-integer values goes back to the beginning of the theory of differential calculus. Nevertheless, its application in physics and engineering remained unexplored up to the last two decades. Recent research motivated the establishment of strategies taking advantage of the Fractional Calculus (FC) in the modeling and control of many phenomena. In fact, many classical engineering applications deserve a closer attention and a new analysis in the viewpoint of FC. Bearing these ideas in mind, this work addresses the partial differential equations that model the electrical transmission lines. The distributed characteristics of this system may lead to design techniques, for integrated circuits, capable of implementing directly fractional-order impedances and, therefore, constitutes an alternative to exploring fractal geometries and dielectric properties.

Mathematics Subject Classification: 26A33 (main), 35A22, 78A25, 93A30
Key Words and Phrases: fractional calculus, partial differential equations, transmission lines, impedance

*

The main person responsible for a complete mathematical analysis of signal propagation on transmissions lines was Olivier Heaviside who published a book, in 1880, based on Maxwell electromagnetic theory [1].

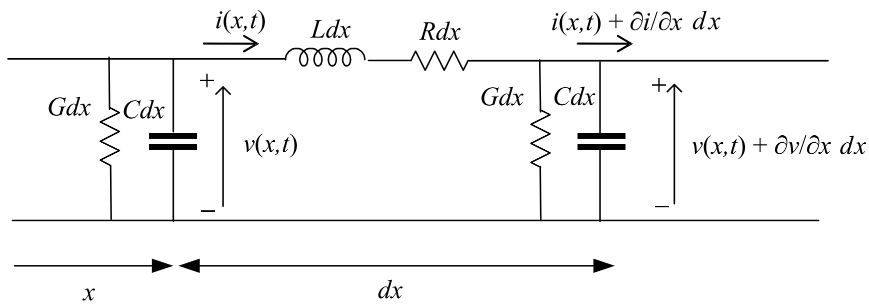

Figure 1: Electrical circuit of an infinitesimal portion of a uniform transmission line.

During the twentieth century, the electrical power transmission, telecommunication and microwave engineering, and the subsequent development of innumerable applications, made popular the introduction of transmission line theory in electrical engineering curricula [2-3].

The differential equations for a uniform transmission line are found by considering an infinitesimal length dx located at coordinate x. This line section has series inductance and resistance Ldx and Rdx and shunt conductance and capacitance Gdx and Cdx, as depicted in Figure 1. The application of the Kirchoff laws to the circuit leads to the set of partial differential equations:

$$\partial v(x,t)/\partial x = -L\partial i(x,t)/\partial t - Ri(x,t), \quad (1a)$$

$$\partial i(x,t)/\partial x = -C\partial v(x,t)/\partial t - Gv(x,t), \quad (1b)$$

where t represents the time, v - voltage and i - electrical current.

A few simple calculations allow us to eliminate one variable and to explicit the differential equation either to v or to i, yielding:

$$\partial^2 v(x,t)/\partial x^2 = LC\partial^2 v(x,t)/\partial t^2 + (LG+RC)\partial v(x,t)/\partial t + RGv(x,t), \quad (2a)$$

$$\partial^2 i(x,t)/\partial x^2 = LC\partial^2 i(x,t)/\partial t^2 + (LG+RC)\partial i(x,t)/\partial t + RGi(x,t). \quad (2b)$$

It is interesting to note that when $L = 0$ and $G = 0$, equation (2) reduces to the equivalent of the heat diffusion equation, where v and i are the analogs of the temperature and the heat flux, respectively.

To analyze the transmission lines in the frequency domain it is considered the Fourier transform operator F such that $I(x,j\omega) = F\{i(x,t)\}$ and

$V(x,i\omega) = F\{v(x,t)\}$ (with $j = (-1)^{1/2}$) and equations (1) are transformed to:

$$dV(x, j\omega)/dx = -Z(i\omega)I(x, j\omega), \tag{3a}$$

$$dI(x, j\omega)/dx = -Y(i\omega)V(x, j\omega), \tag{3b}$$

where $Z(i\omega) = R + j\omega L$ and $Y(i\omega) = G + j\omega C$. In the same line of thought equations (2) are transformed to:

$$d^2V(x, j\omega)/dx^2 = -Z(i\omega)Y(i\omega)V(x, j\omega) \tag{4}$$

which as a solution in the frequency domain of the type, is:

$$V(x, j\omega) = A_1e^{\gamma x} + A_2e{-}^{\gamma x}, \tag{5a}$$

$$I(x, j\omega) = Z_c^{-1}(A_2e^{-\gamma x} - A_1e^{\gamma x}), \tag{5b}$$

where $Z_c(j\omega) = \left[Z(j\omega)Y^{-1}(j\omega)\right]^{1/2} = [(R + j\omega L)/(G + j\omega C)]^{1/2}$ (characteristic impedance) and $\gamma(j\omega) = [Z(j\omega)Y(j\omega)]^{1/2} = \alpha(\omega) + j\beta(\omega)$.

These expressions have two terms corresponding to waves traveling in opposite directions: the term proportional to $e^{-\gamma x}$ is due to the signal applied at the line input while the term $e^{\gamma x}$ represents the reflected wave.

For a transmission line of length l it is usual to adopt as variable the distance up to the end given by:

$$y = l - x. \tag{6}$$

If V_2 and I_2 represent the voltage and current at the end of the transmission line, then the Fourier transforms of equation (1) at coordinate y are given by:

$$V(x, j\omega) = V_2ch(\gamma y) + I_2Z_csh(\gamma y), \tag{7a}$$

$$I(x, j\omega) = V_2Z_c^{-1}sh(\gamma y) + I_2ch(\gamma y). \tag{7b}$$

Therefore, for a loading impedance $Z_2(j\omega)$ we have

$$V_2(j\omega) = Z_2(j\omega)I_2(j\omega)$$

and the input impedance $Z_i(j\omega)$ of the transmission line results:

$$Z_i(j\omega) = [Z_2ch(\gamma y) + Z_csh(\gamma y)]\left[Z_2Z_c^{-1}sh(\gamma y) + ch(\gamma y)\right]^{-1}. \tag{8}$$

Typically are considered three cases at the end of the line, namely a short circuit, an open circuit and an adapted line, that simplify equation (8) yielding:

$$V_2 = 0, Z_2(j\omega) = 0, Z_i(j\omega) = Z_c(j\omega)th(\gamma l), \tag{9a}$$

$$I_2 = 0, Z_2(j\omega) = \infty, Z_i(j\omega) = Z_c(j\omega)cth(\gamma l), \tag{9b}$$

$$Z_2(j\omega) = Z_c(j\omega), Z_i(j\omega) = Z_c(j\omega). \tag{9c}$$

The classical perspective is to study lossless lines (i.e., $R = 0$ and $G = 0$), reasonable in power systems, and approximations in the frequency domain leading to two-port networks with integer order elements. This is surprising because the transcendental equations (8) and (9) may lead both to integer and fractional-order expressions. For example, in the case of an adapted line (with R, C, L, $G \in \Re^+$), we can have half-order fractional capacitances and half-order fractional inductances according with the expressions:

$$L = 0,\ G = 0 \Rightarrow Z_c(j\omega) = \left[(j\omega)^{-1}RC^{-1}\right]^{1/2}, \tag{10a}$$

$$R = kL,\ G = kC\ (k \in \Re) \quad \Rightarrow Z_c(j\omega) = (RL^{-1})^{1/2}, \tag{10b}$$

$$R = 0,\ C = 0 \Rightarrow Z_c(j\omega) = (j\omega LG^{-1})^{1/2}. \tag{10c}$$

Since conditions (9a) and (9b) are easier to implement in practice than condition (9c), we can take advantage of the asymptotic expansions of $th(\gamma l)$ and $cth(\gamma l)$. In fact, knowing that for low frequencies we have $\omega \to 0$, $th(\gamma l) \to \gamma l$, $cth(\gamma l) \to (\gamma l)^{-1}$ and that for high frequencies $\omega \to \infty$, $th(\gamma l) \to 1$, $cth(\gamma l) \to 1$, than we obtain the following approximations for the short circuit an open circuit cases, respectively:

$$Z_i(j\omega) = \begin{cases} Z(j\omega)l, \omega \to 0 \\ Z_c(j\omega), \omega \to \infty \end{cases}, \tag{11a}$$

$$Z_i(j\omega) = \begin{cases} [Y(j\omega)l]^{-1}, \omega \to 0 \\ Z_c(j\omega), \omega \to \infty \end{cases}. \tag{11b}$$

We conclude that both cases approximate condition (9c) at high frequencies.

These results, overlooked in the (integer-order point of view) classical textbooks, suggest possible strategies for implementing fractional-order impedances, somehow as standard microstrips and striplines work in microwave circuits. In fact, this hardware strategy of implementing fractional-order derivatives has been recently pointed out in order to avoid computational approximation schemes [4-9]. Therefore, an alternative to exploring fractal geometries and dielectric properties [10-11] to achieve fractional capacitors we can also turn our attention to the distributed characteristics of this type of system in order to design integrated circuits capable of implementing directly fractional derivatives.

References

[1] Olivier Heaviside, *Electromagnetic Theory*. London (1893).

[2] Richard B. Adler, Lan Jen Chu, Robert M. Fano, *Electromagnetic Energy Transmission and Radiation*. The M.I.T. Press (1969).

[3] Robert A. Chipman, *Transmission Lines*. McGraw-Hill (1968).

[4] Alain Le Méhauté, Raoul R. Nigmatullin, Laurent Nivanen, *Flèches du Temps et Géométrie Fractale*. Edition Hermes, 2^e edition (1998).

[5] Svante Westerlund, *Dead Matter has Memory!*. Causal Consulting, Kalmar, Sweden (2002).

[6] A. Oustaloup, *La Commande CRONE: Commande Robuste d'Ordre Non Entier*. Hermes (1991).

[7] J. A. Tenreiro Machado, Analysis and design of fractional-order digital control systems. *Systems Analysis-Modelling-Simulation*, Gordon & Breach Science Publishers, **27** (1997), 107-122.

[8] Igor Podlubny, *Fractional Differential Equations*. Academic Press, San Diego (1999).

[9] YangQuan Chen, Blas M. Vinagre, A new IIR-type digital fractional order diferentiator. *Signal Processing*, **83**, No. 11 (Nov. 2003), 2359-2365.

[10] Hirad Samavati, Ali Hajimiri, Arvin Shahani, Gitty N. Nasserbakht, Thomas H. Lee, Fractal capacitors. *IEEE Journal of Solid-State Circuits*, **33**, No. 12 (Dec. 1998), 2035-2041.

[11] Gary W. Bohannan, *Analog Realization of a Fractional Control Element – Revisited*. Wavelength Electronics, Inc. (Oct. 27, 2002).

Institute of Engineering of Porto
Dept. of Electrotechnical Engineering
Rua Dr. António Bernardino de Almeida
4200-072 Porto, PORTUGAL

Received: June 22, 2004
Revised: September 27, 2004

e-mails: [1] *jtm@dee.isep.ipp.pt* , [2] *isj@dee.isep.ipp.pt*