A GENERALIZED CONVOLUTION WITH A WEIGHT FUNCTION FOR THE FOURIER COSINE AND SINE TRANSFORMS

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Dedicated to Professor Ivan H. Dimovski
on the occasion of his 70th birthday

Abstract

A generalized convolution with a weight function for the Fourier cosine and sine transforms is introduced. Its properties and applications to solving a system of integral equations are considered.

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1. Introduction

Let F_c and F_s be the Fourier cosine and sine transforms, respectively:

$$F_c(f)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(y) \cos xy \, dy, \quad x > 0, \tag{1}$$

$$F_s(f)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(y) \sin xy \, dy, \quad x > 0. \tag{2}$$

A convolution of two functions f and g for the Fourier cosine transform was given in 1941 by Churchill [1]:

$$(f \underset{F_c}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int\limits_0^\infty f(y) \left(g(|x - y|) + g(x + y)\right) dy, \quad x > 0 \qquad (3)$$

with the factorization property

$$F_c(f \underset{F_c}{*} g)(y) = (F_c f)(y) (F_c g)(y), \quad \forall y > 0. \qquad (4)$$

A convolution with the weight function $\gamma(y) = \sin y$ of functions f and g for the Fourier sine transform F_s was studied in [2], [7]:

$$(f \underset{F_s}{*} g)(x) = \frac{1}{2\sqrt{2\pi}} \int\limits_0^\infty f(t) \left(g(x + t + 1) + g(|x - t + 1|)\text{sign}(x - t + 1)\right.$$

$$\left. + \ g(|x + t - 1|)\text{sign}(x + t - 1) + g(|x - t - 1|)\text{sign}(x - t - 1)\right) dt, \quad x > 0,$$

$$(5)$$

for which the factorization property holds:

$$F_s(f \underset{F_s}{*} g)(y) = \sin y\, (F_s f)(y)\, (F_s g)(y), \quad \forall y > 0. \qquad (6)$$

A generalized convolution for the Fourier sine and cosine transforms was first introduced by Churchill in [1]:

$$(f \underset{1}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int\limits_0^\infty f(y) \left(g(|x - y|) - g(x + y)\right) dy, \quad x > 0, \qquad (7)$$

that satisfies the factorization property

$$F_s(f \underset{1}{*} g)(y) = (F_s f)(y)(F_c g)(y), \qquad y > 0. \qquad (8)$$

Yakubovich et al. [11], [12], [13] studied some special cases of generalized convolutions for nonconvolution integral transforms. In 1998, Kakichev and Nguyen Xuan Thao [3] proposed a constructive method of defining a generalized convolution for three (possibly different) integral transforms K_1, K_2, K_3 with a weight function $\gamma(x)$ with a factorization property

$$K_1(f \overset{\gamma}{*} g)(x) = \gamma(x)(K_2 f)(x)(K_3 g)(x).$$

In recent years, generalized convolutions, for instance, for the transforms of Stieltjes [15], Hilbert [6], [14], Hankel [16], H-transforms [4], I-transforms

[9], the Fourier cosine and sine transforms [8] have been studied. For example, a generalized convolution for the Fourier cosine and sine transforms has been introduced in [8], as follows:

$$(f \underset{2}{*} g(x) = \frac{1}{\sqrt{2\pi}} \int\limits_0^\infty f(y) \left(\text{sign}(y-x)g(|y-x|) + g(y+x)\right) dy, \ x > 0. \quad (9)$$

It satisfies a factorization property

$$F_c(f \underset{2}{*} g)(y) = (F_s f)(y)(F_s g)(y), \ \forall y > 0. \quad (10)$$

In this paper we give a new generalized convolution with a weight function for the Fourier cosine and sine transforms. We prove some of its properties as well as we point out some of its relationships to several well-known convolutions and apply this notion to solve a system of integral equations.

2. A generalized convolution

DEFINITION 1. A generalized convolution with the weight function $\gamma(y) = \sin y$ for the Fourier cosine and sine transforms of functions f and g is defined by

$$(f \overset{\gamma}{*} g)(x) = \frac{1}{2\sqrt{2\pi}} \int\limits_0^\infty f(y) \left[(g(|x+y-1|) + g(|x-y+1|) \right.$$
$$\left. - g(x+y+1) - g(|x-y-1|)) \, dy, \quad \forall x > 0. \quad (11)$$

We denote by $L(R_+)$ the set of all functions f defined on $(0, \infty)$ such that $\int\limits_0^\infty |f(x)|dx < \infty$.

THEOREM 1. Let $f, g \in L(R_+)$. Then the generalized convolution with the weight-functions $\gamma(y) = \sin y$ for the Fourier cosine and sine transforms belongs to $L(R_+)$, and the following factorization property holds:

$$F_c(f \overset{\gamma}{*} g)(y) = \sin y \, (F_s f)(y)(F_c g)(y), \quad \forall y > 0. \quad (12)$$

P r o o f. Based on (11) and the fact that f and $g \in L(R_+)$ we have

$$\int\limits_0^\infty |(f \overset{\gamma}{*} g)|dx \leq \frac{1}{2\sqrt{2\pi}} \int\limits_0^\infty \int\limits_0^\infty |f(y)| \cdot |[g(|x+y-1|) + g(|x-y+1|)$$

$$-g(x+y+1) - g(|x-y-1|)]| \, dy dx \leq \frac{1}{2\sqrt{2\pi}} \int\limits_0^\infty |f(y)| \left[\int\limits_0^\infty |g(|x+y-1|)| dx \right.$$

$$\left. + \int\limits_0^\infty |g(|x-y-1|)| dx + \int\limits_0^\infty |g(x+y+1)| \, dx + \int\limits_0^{+\infty} |g(x-y-1)| \, dx \right] dy.$$

$$\tag{13}$$

On the other hand,

$$\int\limits_0^\infty |(g(x+y+1)| \, dx + \int\limits_0^\infty |g(|x-y-1|)| \, dx$$

$$= \int\limits_{y+1}^\infty |g(t)| \, dt + \int\limits_{-y-1}^\infty |g(|t|)| \, dt = 2 \int\limits_0^\infty |g(t)| \, dt. \tag{14}$$

Similarly,

$$\int\limits_0^\infty |g(|x-y+1|)| \, dx + \int\limits_0^\infty |g(|x+y-1|)| \, dx$$

$$= \int\limits_{1-y}^\infty |g(|t|)| \, dt + \int\limits_{y-1}^\infty |g(|t|)| \, dt = 2 \int\limits_0^\infty |g(t)| \, dt. \tag{15}$$

¿From (13), (14) and (15) it follows that

$$\int\limits_0^\infty \left| (f \overset{\gamma}{*} g)(x) \right| dx \leq \sqrt{\frac{2}{\pi}} \int\limits_0^\infty |f(t)| dt \int\limits_0^\infty |g(t)| dt < \infty.$$

So $\left(f \overset{\gamma}{*} g \right)(x) \in L(R_+)$.

Now we prove the factorization property (12). Since

$$\sin x \, (F_s f) \, (x) \, (F_c g) \, (x) = \frac{2}{\pi} \int\limits_0^\infty \int\limits_0^\infty \sin x \, \sin xu \, \cos xv \, f(u) \, g(v) \, dudv$$

and

$$\sin x \, \sin xu \, \cos xv = \frac{1}{4} \left[\cos x(u-v-1) + \cos x(u+v-1) \right.$$

$$- \cos x(u + v + 1) - \cos x(u - v + 1)],$$

we obtain

$$\sin x \, (F_s f) \, (x) \, (F_c g) \, (x) = \frac{1}{2\pi} \int\limits_0^\infty \int\limits_0^\infty [\cos x(u - v - 1) + \cos x(u + v - 1)$$

$$- \cos x(u + v + 1) - \cos x(u - v + 1)] \, f(u)g(v)dudv. \qquad (16)$$

Taking a substitution $y = u$, $t = u + v + 1$, we get

$$\frac{1}{2\pi} \int\limits_0^\infty \int\limits_0^\infty \cos x(u+v+1) f(u)g(v)dudv = \frac{1}{2\pi} \int\limits_0^\infty \int\limits_{y+1}^\infty \cos xt \, f(y)g(t-y-1)dtdy$$

$$= \frac{1}{2\pi} \int\limits_0^\infty \int\limits_0^\infty \cos xt \, f(y)g(|t-y-1|)dtdy - \frac{1}{2\pi} \int\limits_0^\infty \int\limits_0^{y+1} \cos xt \, f(y)g(y-t+1)dtdy.$$

$$(17)$$

Similarly, with a substitution $y = u, t = v - u - 1$, we have

$$\frac{1}{2\pi} \int\limits_0^\infty \int\limits_0^\infty \cos x(u-v+1) \, f(u)g(v)dudv = \frac{1}{2\pi} \int\limits_0^\infty \int\limits_{-y-1}^\infty \cos xt \, f(y)g(t+y+1)dtdy$$

$$= \frac{1}{2\pi} \int\limits_0^\infty \int\limits_0^\infty \cos xt \, f(y)g(t+y+1)dtdy + \frac{1}{2\pi} \int\limits_0^\infty \int\limits_{-y-1}^0 \cos xt \, f(y)g(t+y+1)dtdy.$$

$$(18)$$

Furthermore,

$$\int\limits_0^\infty \int\limits_{-y-1}^0 \cos xt \, f(y)g(t + y + 1)dtdy = - \int\limits_0^\infty \int_{y+1}^0 \cos xt \, f(y)g(y - t + 1)dtdy$$

$$= \int_0^\infty \int_0^{y+1} \cos xt \, f(y)g(y - t + 1)dtdy. \qquad (19)$$

¿From (17), (18) and (19) we obtain

$$-\frac{1}{2\pi} \int_0^\infty \int_0^\infty [\cos x(u + v + 1) + \cos x(u - v + 1)] \, f(u)g(v)dudv$$

$$= -\frac{1}{2\pi} \int_0^\infty \int_0^\infty \cos xt \, [g(|t - y - 1|) + g(t + y + 1)] \, f(y) dt dy. \qquad (20)$$

Similarly,

$$\frac{1}{2\pi} \int_0^\infty \int_0^\infty \cos x(u + v - 1) f(u) g(v) du dv = \frac{1}{2\pi} \int_0^\infty \int_{y-1}^\infty \cos xt f(y)$$

$$\times \, g(t - y + 1) dt dy = \frac{1}{2\pi} \int_0^\infty \int_0^\infty \cos xt f(y)$$

$$\times \, g(|t - y + 1|) dt dy - \frac{1}{2\pi} \int_0^\infty \int_0^{y-1} \cos xt \, f(y) \, g(|t - y + 1|) dt dy. \qquad (21)$$

With a substitution $y = u, t = v - u + 1$ we get

$$\frac{1}{2\pi} \int_0^\infty \int_0^\infty \cos x(u - v - 1) f(u) g(v) du dv = \frac{1}{2\pi} \int_0^\infty \int_{1-y}^\infty \cos xt \, f(y)$$

$$\times \, g(t + y - 1) dt dy = \frac{1}{2\pi} \int_0^\infty \int_0^\infty \cos xt \, f(y)$$

$$\times \, g(|t + y - 1|) dt dy - \frac{1}{2\pi} \int_0^\infty \int_0^{1-y} \cos xt \, f(y) \, g(|t + y - 1|) dt dy. \qquad (22)$$

On the other hand,

$$\int_0^\infty \int_0^{y-1} \cos xt f(y) g(|t-y+1|) dt dy = -\int_0^\infty \int_{1-y}^0 \cos xt f(y) g(|t+y-1|) dt dy. \qquad (23)$$

¿From (21), (22), and (23) we have

$$\frac{1}{2\pi} \int_0^\infty \int_0^\infty [\cos x(u + v - 1) + \cos x(u - v - 1)] \, f(u) g(v) du dv$$

$$= \frac{1}{2\pi} \int_0^\infty \int_0^\infty \cos xt \, [g(|t - y + 1|) + g(|t + y - 1|)] \, f(y) dt dy. \qquad (24)$$

Finally, from (16), (20) and (24),

$$\sin x (F_s f)(x)(F_c f)(x) = \frac{1}{2\pi} \int\limits_0^\infty \cos xy \Big\{ \int_0^\infty f(y) \qquad (25)$$

$$\times \, [g(|x + y - 1|) + g(|x - y + 1|) - g(x + y + 1) - g(|x - y - 1|)] \, dy \Big\} dx.$$

The last equality and (11) yield

$$\sin x \, (F_s f)(x)(F_c g)(x) = F_c(f \overset{\gamma}{*} g)(x).$$

The proof is complete.   ∎

THEOREM 2. *In the space $L(R_+)$ the generalized convolution (11) is noncommutative:*

$$(f \overset{\gamma}{*} g)(x) = -(g \overset{\gamma}{*} f)(x) + \frac{1}{2\pi}[\text{sign}(1-x)(f \underset{L}{*} g)(|x-1|) + (f \underset{L}{*} g)(x+1)]. \quad (26)$$

Here $f \underset{L}{*} g$ denotes the convolution for the Laplace integral transform

$$(f \underset{L}{*} g)(x) = \int_0^x f(t)\, g(x-t)\, dt.$$

P r o o f. Indeed, with the substitutions $t = x+y-1$, $t = y-x-1$, $t = x+y+1$, $t = y-x+1$, respectively, we have

$$(f \overset{\gamma}{*} g)(x) = \frac{1}{2\sqrt{2\pi}}\Big[\int_{x-1}^{\infty} g(|t|)f(t-x+1)dt + \int_{-(x+1)}^{\infty} g(|t|)f(t+x+1)dt$$

$$- \int_{x+1}^{\infty} g(|t|)f(t-x-1)dt - \int_{1-x}^{\infty} g(|t|)f(t+x-1)dt\Big] \quad (27)$$

$$= \frac{1}{2\sqrt{2\pi}}\Big\{\int_0^{\infty}[f(|t-x+1|)+f(t+x+1)-f(|t-x-1|)-f(|t+x-1|)]$$

$$\times g(t)dt + \int_{x-1}^0 g(|t|)f(t-x+1)dt + \int_{-(x+1)}^0 g(|t|)f(t+x+1)dt$$

$$- \int_{x+1}^0 g(|t|)f(t-x-1)dt - \int_{1-x}^0 g(|t|)f(t+x-1)dt\Big\}$$

$$= \frac{1}{2\sqrt{2\pi}}\Big\{\int_0^{\infty}[f(|x-t-1|)+f(x+t+1)-f(|x-t+1|)-f(|x+t-1|)]$$

$$\times g(t)dt + \int_{x-1}^0 g(|t|)f(t-x+1)dt + \int_{-(x+1)}^0 g(|t|)f(t+x+1)dt$$

$$- \int_{x+1}^0 g(|t|)f(t-x-1)dt - \int_{1-x}^0 g(|t|)f(t+x-1)dt\Big\}$$

$$= -(g \overset{\gamma}{*} f)(x) + \frac{1}{2\sqrt{2\pi}}\Big\{\int_{x-1}^0 g(|t|)f(t-x+1)dt$$

$$- \int_{1-x}^0 g(|t|)f(t+x-1)dt - \int_{x+1}^0 g(|t|)f(t-x-1)dt + \int_{-(x+1)}^0 g(|t|)f(t+x+1)dt\Big\}.$$

With the substitution $u = -t$ we get

$$\int_0^{x-1} g(|t|)f(|t-x+1|)du \quad = \quad -\int_0^{1-x} g(|u|)f(|u+x-1|)dv$$

$$\int_0^{-(x+1)} g(|t|)f(t+x+1)du \quad = \quad -\int_0^{1+x} g(u)f(|u-x-1|)du,$$

therefore,

$$\int_0^{1-x} g(|u|)f(|u+x-1|)du - \int_0^{x-1} g(|u|)f(|u-x+1|)dv$$

$$= \begin{cases} 2\int_0^{1-x} g(u)f(1-x-u)du, & x < 1 \\ -2\int\limits_0^{x-1} g(u)f(x-1-u)du, & x \geq 1 \end{cases}$$

$$= 2\operatorname{sign}(1-x)\int\limits_0^{|x-1|} g(u)f(|x-1|-u)du = 2\operatorname{sign}(1-x)(f\underset{L}{*}g)(|x-1|).$$

Hence and with the substitutions $u = t - x + 1$, $u = t + x - 1$, $u = t - x - 1$, $u = t + x + 1$ we obtain

$$\int_{x-1}^0 g(|t|)f(t-x+1)dt - \int_{1-x}^0 g(|t|)f(t+x-1)dt - \int_{x+1}^0 g(t)f(t-x-1)dt$$

$$+ \int_{-(x+1)}^0 g(|t|)f(t+x+1)dt = \int_0^{1-x} f(|u|)g(|u+x-1|)du$$

$$- \int_0^{x-1} f(|u|)g(|u-x+1|)du - \int_0^{-(x+1)} f(|u|)g(|u+x+1|)du$$

$$+ \int_0^{x+1} f(u)g(|u-x-1|)du = 2\operatorname{sign}(1-x)(f\underset{L}{*}g)(|x-1|)$$

$$+ 2\int\limits_0^{(x+1)} f(u)g(x+1-u)du = 2\operatorname{sign}(1-x)(f\underset{L}{*}g)(|x-1|) + 2(f\underset{L}{*}g)(x+1).$$

Therefore,

$$(f\overset{\gamma}{*}g)(x) = -(g\overset{\gamma}{*}f)(x) + \frac{1}{\sqrt{2\pi}}\left[\operatorname{sign}(1-x)(f\underset{L}{*}g)(|x-1|) + (f\underset{L}{*}g)(x+1)\right].$$

The theorem is proved. ∎

THEOREM 3. *In the space $L(R_+)$ the generalized convolution (11) is not associative and satisfies the following equalities:*

a) $f \overset{\gamma}{*} (g \overset{\gamma}{*} h) = g \overset{\gamma}{*} (f \overset{\gamma}{*} h)$;

b) $f \overset{\gamma}{*} (g \overset{\gamma}{*} h) = (f \underset{F_s}{*} g) \overset{\gamma}{*} h$, *where $(f \underset{F_s}{*} g)$ is the Fourier-sine convolution with a weight-function (5);*

c) $f \overset{\gamma}{*} (g \underset{F_s}{*} h) = (f \overset{\gamma}{*} g) \underset{F_c}{*} h$, *where $(g \underset{F_c}{*} h)$ is the convolution for the Fourier cosine transform (3);*

d) $f \overset{\gamma}{*} (g \underset{2}{*} h) = (f \underset{F_s}{*} g) \underset{2}{*} h$, *where $(g \underset{2}{*} h)$ is the generalized convolution with a weight-function for the Fourier cosine and sine transforms (9);*

e) $f \underset{1}{*} (g \overset{\gamma}{*} h) = (f \underset{1}{*} h) \overset{\gamma}{*} g$, *where $(f \underset{1}{*} h)$ is the generalized convolution for the Fourier sine and cosine transforms (7).*

P r o o f. **a)** From the factorization property, we have

$$F_c(f \overset{\gamma}{*} (g \overset{\gamma}{*} h))(x) = \sin x (F_s f)(x) \, F_c(g \overset{\gamma}{*} h)(x)$$

$$= \sin x \, (F_s f)(x) \, \sin x (F_s g)(x) \, (F_c h)(x) = \sin x \, (F_s g)(x) \, F_c(f \overset{\gamma}{*} h)(x)$$

$$= F_c \left\{ g \overset{\gamma}{*} (f \overset{\gamma}{*} h) \right\} (x), \quad \forall x > 0,$$

which implies that

$$(f \overset{\gamma}{*} g) \overset{\gamma}{*} h = g \overset{\gamma}{*} (f \overset{\gamma}{*} h).$$

b)
$$F_c(f \overset{\gamma}{*} (g \overset{\gamma}{*} h))(x) = \sin x (F_s f)(x) \, F_c(g \overset{\gamma}{*} h)(x)$$

$$= \sin x (F_s f)(x) \, \sin x (F_s g)(x) \, (F_c h)(x) = F_s (f \underset{F_s}{*} g)(x) \, \sin x \, (F_c h)(x)$$

$$= F_c \left\{ (f \underset{F_s}{*} g)(x) \overset{\gamma}{*} h \right\} (x), \quad \forall x > 0.$$

Hence $f \overset{\gamma}{*} (g \overset{\gamma}{*} h) = (f \underset{F_s}{*} g) \overset{\gamma}{*} h$.

The proofs for **c), d), e)** are similar to those of **a)** and **b)**. The theorem is proved. ∎

THEOREM 4. *The generalized convolution (11) is related to the convolution (7) as follows*

$$(f \overset{\gamma}{*} g)(x) = \frac{1}{2}[(f \underset{1}{*} g)(x+1) - (f \underset{1}{*} g)(x-1)\,\text{sign}(x-1)]. \qquad (28)$$

P r o o f. Indeed,

i) For $x \geq 1$, from Definition 1 we obtain

$$
\begin{aligned}
(f \overset{\gamma}{*} g)(x) &= \frac{1}{2}\Big\{ \frac{1}{\sqrt{2\pi}} \int_0^\infty [g(|x-y+1|) - g(x+y+1)]\, f(y)dy \\
&\quad -\frac{1}{2}\Big\{ \frac{1}{\sqrt{2\pi}} \int_0^\infty [(|x-y-1|) - g(x+y-1)]\, f(y)dy \Big\} \\
&= \frac{1}{2}\Big[(f \underset{1}{*} g)(x+1) - (f \underset{1}{*} g)(x-1) \Big].
\end{aligned}
$$

ii) Let $0 < x < 1$. We have

$$
\begin{aligned}
(f \overset{\gamma}{*} g)(x) &= \frac{1}{2}\Big\{ \frac{1}{\sqrt{2\pi}} \int_0^\infty [g(|x-y+1|) - g(x+y+1)]\, f(y)dy \\
&\quad -\frac{1}{\sqrt{2\pi}} \int_0^\infty [g(|x-y-1|) - g(|x+y-1|)]\, f(y)dy \Big\} \\
&= \frac{1}{2}\Big\{ \frac{1}{\sqrt{2\pi}} \int_0^\infty [g(|x-y+1|) - g(x+y+1)]\, f(y)dy \\
&\quad -\frac{1}{\sqrt{2\pi}} \int_0^\infty [g(|x-y-1|) - g(|x+y-1|)]\, f(y)dy \Big\} \\
&= \frac{1}{2}\Big[(f \underset{1}{*} g)(x+1) + (f \underset{1}{*} g)(1-x) \Big] \\
&= \frac{1}{2}\Big[(f \underset{1}{*} g)(x+1) - (f \underset{1}{*} g)(|x-1|)\text{sign}(x-1) \Big].
\end{aligned}
$$

Hence

$$(f \overset{\gamma}{*} g)(x) = \frac{1}{2}\Big[(f \underset{1}{*} g)(x+1) - (f \underset{1}{*} g)(|x-1|)\text{sign}(x-1) \Big].$$

The theorem is proved.   ■

THEOREM 5. *In the space $L(R_+)$ the generalized convolution (11) does not have a unit element.*

P r o o f. Suppose that there exists a unit element of the generalized convolution (11) in $L(R_+) : e \overset{\gamma}{*} g = g \overset{\gamma}{*} e = g$, for any function $g \in L(R_+)$. It follows that $F_c(e \overset{\gamma}{*} g)(y) = (F_c g)(y), \ \forall y > 0$. Hence, $\sin y \, (F_s e)(y) \, (F_c g)(y) = (F_c g)(y), \ \forall y > 0$. The last equation is equivalent to the equality

$$(F_c g)(y)[\sin y (F_s e)(y) - 1] = 0, \ \ \forall y > 0$$

for any function $g(y) \in L(R_+)$. Choosing g such that $(F_c g)(y) \neq 0, \ \forall y > 0$, we see that $\sin y (F_s e)(y) - 1 = 0$ or $(F_s e)(y) = \frac{1}{\sin y}, \ \forall y > 0$.

Thus, $(F_s e)(y)$ does not approach 0 as y tends to infinity, as it must be if $e \in L(R_+)$, and so $e \notin L(R_+)$. This is a contradiction. The theorem is proved. ∎

Set $L(R_+, e^x) = \{h(x), \text{ for all } e^x h(x) \in L(R_+)\}$

THEOREM 6. (Titchmarch-type theorem) *Let* $f, g \in L(R_+, e^x)$. *If*
$$(f \overset{\gamma}{*} g)(x) = 0, \ \ \forall x > 0,$$
then either $f(x) = 0$ *or* $g(x) = 0, \ \forall x > 0$.

P r o o f. Under the hypothesis $(f \overset{\gamma}{*} g)(x) = 0, \ \forall x > 0$, it follows that $F_c(f \overset{\gamma}{*} g)(y) = 0, \ \forall y > 0$. By virtue of Theorem 1,

$$\sin y (F_s f)(y)(F_c g)(y) = 0, \ \ \forall y > 0. \tag{29}$$

Consider the Fourier cosine integral transform

$$(F_c g)(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(x) \cos(yx) dx, \quad y \in R_+.$$

Since

$$\left| \frac{d^n}{dy^n} \left(\cos(yx) g(x) \right) \right| = \left| g(x) x^n \cos \left(yx + n\frac{\pi}{2} \right) \right| \leq |g(x) x^n|$$

$$= |e^{-x} x^n g_1(x)| = |e^{-x} x^n| \, |g_1(x)| \leq C |g_1(x)|, \quad g_1(x) = e^x \, g(x) \in L(R_+),$$

for x large enough, due to Weierstrass' criterion, the integral

$$\int_0^\infty \frac{d^n}{dy^n} \left[\cos(yx) g(x) \right] dx$$

uniformly converges on R_+. Therefore, based on the differentiability of integrals depending on parameter, we conclude that $(F_c g)(y)$ is analytic. Similarly, $(F_s f)(y)$ is analytic. So from (29) we have $(F_s f)(y) = 0, \ \forall y > 0$ or $(F_c g)(y) = 0, \ \forall y > 0$. It follows that either $f(x) = 0$ or $g(x) = 0, \ \forall x > 0$. The theorem is proved. ∎

3. A system of integral equations

Consider a system of integral equations

$$
\begin{cases}
\lambda_{11} f(y) + \lambda_{12} \int\limits_{0}^{\infty} \varphi(t) g_1(y,t) dt &= k(y), \\
\lambda_{21} \int\limits_{0}^{\infty} f(t) \psi_1(y,t) dt + \lambda_{22} g(y) &= h(y), \qquad y > 0.
\end{cases}
\tag{30}
$$

Here, $\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}$ are complex numbers and $\varphi, \psi, k, h \in L(R_+)$, f and g are unknown functions of $L(R_+)$ and

$$
g_1(y,t) = \frac{1}{\sqrt{2\pi}} \left[g(|t-y|) + g(t+y) \right],
$$

$$
\psi_1(y,t) = \frac{1}{\sqrt{2\pi}} \left[\psi(|y+t-1|) + \psi(|y-t+1|) - \psi(y+t+1) - \psi(|y-t-1|) \right].
$$

THEOREM 7. *Let* $1 + C \operatorname{sign} y \, (F_s\varphi)(y)(F_c\psi)(y) \neq 0$. *Then there exists a solution in* $L(R_+)$ *of* (30) *which is defined by*

$$
f(y) = \frac{1}{\lambda} \left\{ \lambda_{22} k(y) - \lambda_{12}(\varphi \underset{1}{*} h)(y) - (k \underset{1}{*} l)(y) + \lambda_{12} \left((\varphi \underset{1}{*} h) \underset{1}{*} l \right)(y) \right\} \in L(R_+),
$$

$$
g(y) = \frac{1}{\lambda} \left\{ \lambda_{11} h(y) - \lambda_{21}(k \overset{\gamma}{*} \psi)(y) - \lambda_{11}(h \underset{F_c}{*} l)(y) + \lambda_{21}((k \overset{\gamma}{*} \psi) \underset{F_c}{*} l)(y) \right\} \in L(R_+).
$$

Here, $l \in L(R_+)$ *is defined by*

$$
(F_c l)(y) = \frac{C \, F_c(\varphi \overset{\gamma}{*} \psi)(y)}{1 + C F_c(\varphi \overset{\gamma}{*} \psi)(y)}, \quad \lambda = \lambda_{11}\lambda_{22}, \;\; C = -\frac{\lambda_{12}\lambda_{21}}{\lambda_{11}\lambda_{22}}.
$$

P r o o f. System (30) can be written in the form

$$
\begin{cases}
\lambda_{11} f(y) + \lambda_{12}(\varphi \underset{1}{*} g)(y) &= k(y), \\
\lambda_{21}(f \overset{\gamma}{*} \psi)(y) + \lambda_{22} g(y) &= h(y), \qquad y > 0.
\end{cases}
$$

Using the factorization properties of the convolutions (7) and (11) we have

$$
\begin{cases}
\lambda_{11}(F_s f)(y) + \lambda_{12}(F_s\varphi)(y)(F_c g)(y) &= (F_s k)(y), \\
\lambda_{21} \sin y (F_s f)(y)(F_c\psi)(y) + \lambda_{22}(F_c g)(y) &= (F_c h)(y), \qquad y > 0.
\end{cases}
$$

Since,

$$\Delta = \begin{vmatrix} \lambda_{11} & \lambda_{12}(F_s\varphi)(y) \\ \lambda_{21}(F_c\psi)(y)\sin y & \lambda_{22} \end{vmatrix}$$

$$= \lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21}\sin y(F_s\varphi)(y)(F_c\psi)(y)$$

$$= \lambda_{11}\lambda_{12}\left[1 - \frac{\lambda_{12}\lambda_{21}}{\lambda_{11}\lambda_{22}}\sin y(F_s\varphi)(y)(F_c\psi)(y)\right]$$

$$= \lambda[1 + C\operatorname{sign} y(F_s\varphi)(y)(F_c\psi)(y)(y)] \neq 0,$$

$$\Delta_1 = \begin{vmatrix} (F_sk)(y) & \lambda_{12}(F_s\varphi)(y) \\ (F_ch)(y) & \lambda_{22} \end{vmatrix} = (F_sk)(y)\lambda_{22} - \lambda_{12}(F_s\varphi)(y)(F_ch)(y).$$

Therefore,

$$(F_sf)(y) = \frac{\Delta_1}{\Delta} = \frac{\Delta_1}{\lambda} - \frac{\Delta_1}{\lambda}\frac{C\sin y(F_s\varphi)(y)(F_c\psi)(y)}{1 + C\sin y(F_s\varphi)(y)(F_c\psi)(y)},$$

$$= \frac{\Delta_1}{\lambda} - \frac{\Delta_1}{\lambda}\frac{CF_c(\varphi\overset{\gamma}{*}\psi)(y)}{1 + CF_c(\varphi\overset{\gamma}{*}\psi)(y)}, \quad y > 0.$$

Due to Wiener-Levi's theorem [7] there exists a function $l \in L(R_+)$ such that

$$(F_cl)(y) = \frac{CF_c(\varphi\overset{\gamma}{*}\psi)(y)}{1 + CF_c(\varphi\overset{\gamma}{*}\psi)(y)}.$$

It follows that

$$(F_sf)(y) = \frac{1}{\lambda}[\Delta_1 - \Delta_1(F_cl)(y)] = \frac{1}{\lambda}\{(F_sk)(y)\lambda_{22}$$

$$- \lambda_{12}(F_s\varphi)(y)(F_ch)(y) - [(F_sk)(y)\lambda_{22} - \lambda_{12}(F_s\varphi)(y)(F_ch)(y)](F_cl)(y)\}$$

$$= \frac{1}{\lambda}\left\{(F_sk)(y)\lambda_{22} - \lambda_{12}F_s(\varphi\underset{1}{*}h)(y) - F_s(k\underset{1}{*}l)(y) + \lambda_{12}F_s(\varphi\underset{1}{*}h)(y)(F_cl)(y)\right\}$$

$$= \frac{1}{\lambda}\left\{(F_sk)(y)\lambda_{22} - \lambda_{12}F_s(\varphi\underset{1}{*}h)(y) - F_s(k\underset{1}{*}l)(y) + \lambda_{12}F_s\left((\varphi\underset{1}{*}h)\underset{1}{*}l\right)(y)\right\}.$$

Hence,

$$f(y) = \frac{1}{\lambda}\left\{k(y)\lambda_{22} - \lambda_{12}(\varphi\underset{1}{*}h)(y) - (k\underset{1}{*}l)(y) + \lambda_{12}\left((\varphi\underset{1}{*}h)\underset{1}{*}l\right)(y)\right\}.$$

Similarly,

$$\Delta_2 = \begin{vmatrix} \lambda_{11} & (F_sk)(y) \\ \lambda_{21}\sin y(F_c\psi)(y) & (F_ch)(y) \end{vmatrix}$$

$$= \lambda_{11}(F_c h)(y) - \lambda_{21} \sin y \, (F_s k)(y)(F_c \psi)(y) = \lambda_{11}(F_c h)(y) - \lambda_{21} F_c(k \overset{\gamma}{*} \psi)(y).$$

It follows that

$$(F_c g)(y) = \frac{\Delta_2}{\Delta} = \frac{\Delta_2}{\lambda} - \frac{\Delta_2}{\lambda}(F_c l)(y) = \frac{1}{\lambda} \left\{ \lambda_{11}(F_c h)(y) \right.$$

$$- \lambda_{21} F_c(k \overset{\gamma}{*} \psi)(y) - \lambda_{11}(F_c h)(y)(F_c l)(y) + \lambda_{21} F_c(k \overset{\gamma}{*} \psi)(y)(F_c l)(y) \right\}$$

$$= \frac{1}{\lambda} \left\{ \lambda_{11}(F_c h)(y) - \lambda_{21} F_c(k \overset{\gamma}{*} \psi)(y) \right.$$

$$- \lambda_{11} \left(F_c(h \underset{F_c}{*} l) \right)(y) + \lambda_{21} F_c \left((k \overset{\gamma}{*} \psi) \underset{F_c}{*} l \right)(y) \right\}.$$

Hence

$$g(y) = \frac{1}{\lambda} \left\{ \lambda_{11} h(y) - \lambda_{21}(k \overset{\gamma}{*} \psi)(y) - \lambda_{11}(h \underset{F_c}{*} l)(y) + \lambda_{21} \left((k \overset{\gamma}{*} \psi) \underset{F_c}{*} l \right)(y) \right\}.$$

The proof is complete. ∎

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