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POWERS AND LOGARITHMS

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Dedicated to Professor Ivan H. Dimovski
on the occasion of his 70th birthday

Abstract

There are applied power mappings in algebras with logarithms induced by a given linear operator D in order to study particular properties of powers of logarithms. Main results of this paper will be concerned with the case when an algebra under consideration is commutative and has a unit and the operator D satisfies the Leibniz condition, i.e. $D(xy) = xDy + yDx$ for $x, y \in \text{dom } D$. Note that in the Number Theory there are well-known several formulae expressed by means of some combinations of powers of logarithmic and antilogarithmic mappings or powers of logarithms and antilogarithms (cf. for instance, the survey of Schinzel S[1].

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1. Algebras with logarithms

We recall some notions and properties which will be used in the sequel.

Let X be a linear space over a field $\mathbb{F}$ of scalars of the characteristic zero. Recall that $L(X)$ is the set of all linear operators with domains and ranges in X and $L_0(X) = \{A \in L(X) : \text{dom } A = X\}$.

If X is an algebra over $\mathbb{F}$ with a $D \in L(X)$ such that $x, y \in \text{dom } D$ implies $xy, yx \in \text{dom } D$, then we shall write $D \in \mathbf{A}(X)$. The set of all *commutative* algebras belonging to $\mathbf{A}(X)$ will be denoted by $\mathsf{A}(X)$. If $D \in \mathbf{A}(X)$, then

$$f_D(x,y) = D(xy) - c_D[xDy + (Dx)y] \quad \text{for} \quad x, y \in \text{dom } D,$$

where c_D is a scalar dependent on D only. Clearly, f_D is a bilinear (i.e. linear in each variable) form which is symmetric when X is commutative, i.e. when $D \in \mathsf{A}(X)$. This form is called a *non-Leibniz component*. Non-Leibniz components have been introduced for right invertible operators $D \in \mathsf{A}(X)$ (cf. PR[1]). If $D \in \mathbf{A}(X)$, then the product rule in X can be written as follows:

$$D(xy) = c_D[xDy + (Dx)y] + f_D(x,y) \quad \text{for} \quad x, y \in \text{dom } D.$$

If $D \in \mathsf{A}(X)$ is right invertible, then the algebra X is said to be a *D-algebra*.

We shall consider in $\mathbf{A}(X)$ the following sets:

• the set of all *multiplicative* mappings (not necessarily linear) with domains and ranges in X:

$$M(X) = \{A : \text{dom } A \subset X,\ A(xy) = A(x)A(y) \text{ for } x, y \in \text{dom } A\};$$

• the set $I(X)$ of all invertible elements belonging to X;

• the set $R(X)$ of all right invertible operators belonging to $L(X)$;

• the set $\mathcal{R}_D = \{R \in L_0(X) \ : \ DR = I\}$ of all right inverses to a $D \in R(X)$;

• the set $\mathcal{F}_D = \{F \in L_0(X) \ : \ F^2 = F, \quad FX = \ker D \text{ and } \exists_{R \in \mathcal{R}_\mathcal{D}} FR = 0\}$ of all *initial* operators for a $D \in R(X)$;

• the set $\Lambda(X)$ of all left invertible operators belonging to $L(X)$;

• the set $\mathcal{I}(X)$ of all invertible operators belonging to $L(X)$.

Clearly, if $\ker D \neq \{0\}$, then the operator D is right invertible, but not invertible. Here the invertibility of an operator $A \in L(X)$ means that the equation $Ax = y$ has a unique solution for every $y \in X$. Elements of the kernel of a $D \in R(X)$ are said to be *constants*. If $D \in \mathcal{I}(X)$ then $\mathcal{F}_D = \{0\}$ and $\mathcal{R}_D = \{D^{-1}\}$. We also have $\text{dom } D = RX \oplus \ker D$ independently of the choice of an $\mathcal{R}_D$ (cf. PR[1]).

It is well-known that F is an initial operator for a $D \in R(X)$ if and only if there is an $R \in \mathcal{R}_D$ such that $F = I - RD$ on dom D. Moreover, if F' is any projection onto $\ker D$ then F' is an initial operator for D corresponding to the right inverse $R' = R - F'R$ independently of the choice of an $R \in \mathcal{R}_D$ (cf. PR[1]).

Suppose that $D \in \mathbf{A}(X)$. Let $\Omega_r, \Omega_l : \text{dom } D \longrightarrow 2^{\text{dom } D}$ be multifunctions defined as follows:

$$\Omega_r u = \{x \in \text{dom } D : Du = uDx\}, \quad \Omega_l u = \{x \in \text{dom } D : Du = (Dx)u\} \tag{1.1}$$

for $u \in \text{dom } D$. The equations

$$Du = uDx \quad \text{for } (u,x) \in \text{graph } \Omega_r \ , \quad Du = (Dx)u \quad \text{for } (u,x) \in \text{graph } \Omega_l \tag{1.2}$$

are said to be the *right* and *left basic equations*, respectively. Clearly,

$$\Omega_r^{-1} x = \{u \in \text{dom } D : Du = uDx\}, \quad \Omega_l^{-1} x = \{u \in \text{dom } D : Du = (Dx)u\} \tag{1.3}$$

for $x \in$ dom D. The multifunctions Ω_r, Ω_l are well-defined and dom $\Omega_r \cap$ dom $\Omega_l \supset \ker D$.

Suppose that $(u_r, x_r) \in$ graph Ω_l, $(u_l, x_l) \in$ graph Ω_r, L_r, L_l are selectors of Ω_r, Ω_l, respectively, and E_r, E_l are selectors of Ω_r^{-1}, Ω_l^{-1}, respectively. By definitions, $L_r u_r \in$ dom Ω_r^{-1}, $E_r x_r \in$ dom Ω_r, $L_l u_l \in$ dom Ω_l^{-1}, $E_l x_l \in$ dom Ω_l and the following equations are satisfied:

$$Du_r = u_r D L_r u_r, \quad DE_r x_r = (E_r x_r) D x_r;$$

$$Du_l = (DL_l u_l) u_l, \quad DE_l x_l = (Dx_l)(E_l x_l).$$

Any invertible selector L_r of Ω_r is said to be a *right logarithmic mapping* and its inverse $E_r = L_r^{-1}$ is said to be a *right antilogarithmic mapping*. If $(u_r, x_r) \in$ graph Ω_r and L_r is an invertible selector of Ω_r, then the element $L_r u_r$ is said to be a *right logarithm* of u_r and $E_r x_r$ is said to be a *right antilogarithm* of x_r. By $G[\Omega_r]$ we denote the set of all pairs (L_r, E_r), where L_r is an invertible selector of Ω_r and $E_r = L_r^{-1}$. Respectively, any invertible selector L_l of Ω_l is said to be a *left logarithmic mapping* and its inverse $E_l = L_l^{-1}$ is said to be a *left antilogarithmic mapping*. If $(u_l, x_l) \in$ graph Ω_l and L_l is an invertible selector of Ω_l, then the element $L_l u$ is said to be a *left logarithm* of u_l and $E_l x_l$ is said to be a *left antilogarithm* of x_l. By $G[\Omega_l]$ we denote the set of all pairs (L_l, E_l), where L_l is an invertible selector of Ω_l and $E_l = L_l^{-1}$.

If $D \in \mathsf{A}(X)$ then $\Omega_r = \Omega_l$ and we write $\Omega_r = \Omega$ and $L_r = L_l = L$, $E_r = E_l = E$, $(L, E) \in G[\Omega]$. Selectors L, E of Ω are said to be *logarithmic* and *antilogarithmic* mappings, respectively. For any $(u, x) \in G[\Omega]$ elements Lu, Ex are said to be *logarithm* of u and *antilogarithm* of x, respectively. The multifunction Ω has been examined in PR[2].

Clearly, by definition, for all $(L_r, E_r) \in G[\Omega_r]$, $(u_r, x_r) \in \text{graph } \Omega_r$, $(L_l, E_l) \in G[\Omega_l]$, $(u_l, x_l) \in \text{graph } \Omega_l$ we have

$$E_r L_r u_r = u_r,\ L_r E_r x_r = x_r; \quad E_l L_l u_l = u_l,\ L_l E_l x_l = x_l; \tag{1.4}$$

$$DE_r x_r = (E_r x_r) D x_r, \quad D u_r = u_r D L_r u_r; \tag{1.5}$$

$$DE_l x_l = (D x_l)(E_l x_l), \quad D u_l = (D L_l u_l) u_l.$$

A right (left) logarithm of zero is not defined. If $(L_r, E_r) \in G[\Omega_r]$, $(L_l, E_l) \in G[\Omega_l]$, then $L_r(\ker D \setminus \{0\}) \subset \ker D$, $E_r(\ker D) \subset \ker D$, $L_l(\ker D \setminus \{0\}) \subset \ker D$, $E_l(\ker D) \subset \ker D$. In particular, $E_r(0), E_l(0) \in \ker D$.

If $D \in R(X)$, then logarithms and antilogarithms are uniquely determined up to a constant.

If $D \in \mathbf{A}(X)$ and if D satisfies the *Leibniz condition*: $D(xy) = xDy + (Dx)y$ for $x, y \in \text{dom } D$, then X is said to be a *Leibniz algebra*.

Let $D \in \mathsf{A}(X)$. A logarithmic mapping L is said to be of the *exponential type* if $L(uv) = Lu + Lv$ for $u, v \in \text{dom } \Omega$. If L is of the exponential type, then $E(x + y) = (Ex)(Ey)$ for $x, y \in \text{dom } \Omega$. We have proved that a logarithmic mapping L is of the exponential type **if and only if** X is a commutative *Leibniz algebra* (cf. PR[2]). In commutative Leibniz algebras with a right invertible operator D $u \in \text{dom } \Omega$ **if and only if** $u \in I(X)$ (cf. PR[2]). The Leibniz condition is also a necessary and sufficient condition for the Trigonometric Identity to be satisfied.

By $\mathbf{Lg}(D)$ ($\mathbf{Lg}_r(D)$, $\mathbf{Lg}_l(D)$) we denote the class of these algebras with unit $e \in \text{dom } \Omega$ for which $D \in R(X)$ and there exist invertible selectors of Ω (Ω_r, Ω_l, respectively), i.e. there exist $(L, E) \in G[\Omega]$ ($(L_r, E_r) \in G[\Omega_r]$, $(L_l, E_l) \in G[\Omega_l]$, respectively).

By $\mathbf{Lg}_{\#}(D)$ we denote the class of these commutative algebras with a left invertible D for which there exist invertible selectors of Ω, i.e. there exists $(L, E) \in G[\Omega]$. Clearly, if D is left invertible then $\ker D = \{0\}$. Thus the multifunction Ω is single-valued and we may write: $\Omega = L$. On the other hand, if $\ker D = \{0\}$, then either X is not a Leibniz algebra or X has no unit (cf. PR[2]).

Suppose that either $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ and $D \in \mathbf{A}(X)$ with unit e is a complete linear metric space. Write $x^0 = e$ and

$$\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for } x \in X \tag{1.6}$$

whenever this series is convergent. The function e^x is said to be an *exponential function*. Observe that here we write e for the number

$$\sum_{n=0}^{\infty} \frac{1}{n!}$$

in order to distinguish between this number and the unit e of the algebra X.

If $X \in \mathbf{Lg}(D)$ with unit $e \in \operatorname{dom} \Omega^{-1}$ is a complete linear metric space then we write

$$\mathcal{E}_D(X) = \{x \in \operatorname{dom} \Omega^{-1} \ : \ \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ is convergent}\}. \tag{1.7}$$

By definition, $\mathrm{e}^{x+y} = \mathrm{e}^x \mathrm{e}^y = \mathrm{e}^y \mathrm{e}^x$ and $\mathrm{e}^0 = e$. The same definition can be used for $X \in \mathbf{Lg}_{\#}(D)$.

X is said to be a complete *m-pseudoconvex algebra*, if it is an algebra and a complete locally pseudoconvex space with the topology induced by a sequence $\{\| \cdot \|_n\}$ of *submultiplicative* p_n-homogeneous F-norms, i.e. such pseudonorms that

$$\|xy\|_n \leq \|x\|_n \|y\|_n \quad \text{for all } x, y \in X, \ n \in \mathbb{N}.$$

2. Powers

We begin with the following

DEFINITION 2.1. Let $X \in \mathbf{Lg_r}(D) \cap \mathbf{Lg_l}(D)$. Write for $\lambda \in \mathbb{F}$:

$$E_{r,\lambda} u = E_r(\lambda L_r u) \quad \text{if } (L_r, E_r) \in G[\Omega_r], \ u \in \operatorname{dom} \Omega_{\mathrm{r}}, \tag{2.1}$$

$$E_{l,\lambda} u = E_l(\lambda L_l u) \quad \text{if } (L_l, E_l) \in G[\Omega_l], \ u \in \operatorname{dom} \Omega_{\mathrm{l}}.$$

If $X \in \mathbf{Lg}(D)$, then we write

$$E_\lambda u = E(\lambda L u) \quad \text{if } (L, E) \in G[\Omega], \ u \in \operatorname{dom} \Omega. \tag{2.1'}$$

The mappings $E_{r,\lambda}$, $E_{l,\lambda}$ and E_λ are said to be of *the power type with exponent* λ or, if it does not lead to any misunderstanding, shortly, *power mappings*.

Note 2.1. Without any additional assumptions, just by definitions, left and right logarithms and antilogarithms of elements qe, where e is the unit of X and $q \in \mathbb{Q}$, are well-defined (provided that $c_D \neq 0$). In a standard way we obtain extensions of left and right logarithms and antilogarithms to $\mathbb{R}$ and $\mathbb{C}$ in Leibniz algebras (cf. for details PR[2], also PR[3]).

We recall without proofs (which can be found either in PR[2] or in PR[3]) the following properties of powers. For the sake of brevity, we shall consider only the commutative case. We get

PROPOSITION 2.1. *Suppose that* $X \in \mathbf{Lg}(D)$, $(L, E) \in G[\Omega]$ *and the mappings* E_λ *are defined by Formulae (2.1′). Then for all* λ, $\mu \in \mathbb{F}$ *we have* $E_\lambda(\text{dom } \Omega) \subset \text{dom } \Omega$, $LE_\lambda = \lambda L$ *and* $E_\lambda E_\mu = E_{\lambda\mu}$, *i.e. these mappings are multiplicative functions of the parameter* λ.

THEOREM 2.1. *Suppose that* $X \in \mathbf{Lg}(\mathbf{D})$ *is a Leibniz algebra and* $(L, E) \in G_[\Omega]$. *Then for all* $\lambda \in \mathbb{F}$ *and* $u \in I(X) \cap \text{dom } D$ $E_\lambda u^{-1} = (E_\lambda u)^{-1}$. *If* $D \in R(X)$ *then* $E_\lambda \in M(X)$.

PROPOSITION 2.2. *Suppose that* $X \in \mathbf{Lg}(D)$ *is a Leibniz algebra and* $(L, E) \in G[\Omega]$. *Then for all* λ, $\mu \in \mathbb{F}$ *and* $u \in \text{dom } \Omega$

$$(E_\lambda u)(E_\mu u) = E_{\lambda+\mu} u; \quad E_\lambda u,\ E_{-\lambda} u \in I(X) \quad \text{and} \quad (E_\lambda u)^{-1} = E_{-\lambda} u.$$

Proposition 2.2 does not hold in the noncommutative case (cf. PR[2]).

PROPOSITION 2.3. *Suppose that* $X \in \mathbf{Lg}(D)$ *and* $(L, E) \in G_{R,1}[\Omega]$ *for an* $R \in \mathcal{R}_D$[1]. *If* $\lambda \in \mathbb{F}$ *and* $u, v \in \text{dom } \Omega$, $E_\lambda u$, $E_\lambda v \in I(X)$, *then there is a* $z \in \ker D$ *such that*

$$(E_\lambda u)(E_\lambda v) = E\{c_D \lambda L v$$
$$+ R[c_D \lambda (E_\lambda v)^{-1} u^{-1} (Du)(E_\lambda v) + (E_\lambda v)^{-1} (E_\lambda u) f_D(E_\lambda u, E_\lambda v)] + z.$$

COROLLARY 2.1. *Suppose that all assumptions of Proposition 2.3 are satisfied and* $c_D = 0$. *Then the mappings* E_λ *are not defined for* $\lambda \neq 1$. *If* $\lambda = 1$ *then* $E_1 = I|_{\text{dom } \Omega}$.

[1]Let F be an initial operator for a $D \in R(X)$ corresponding to an $R \in \mathcal{R}_D$. We denote by $G_{R,1}[\Omega]$ the set of these selectors of Ω for which $FLu = 0$ for all $u \in \text{dom } D$ (cf. PR[2])

Corollary 2.1 implies that for multiplicative D the mappings E_λ are not defined (cf. Note 2.1).

Clearly, we can extend Definition 2.1 to left invertible operators. We get

Proposition 2.4. *Suppose that* $X \in \mathbf{Lg}_\#(D)$, $(L, E) \in G[L]$ *and the mapping* E_λ *is defined by (2.1′). Let* $\lambda \in \mathbb{F}$. *Then* $E_\lambda(\mathrm{dom}\ L) \subset \mathrm{dom}\ L$ *and* $LE_\lambda = \lambda L$ *(cf. Proposition 2.1).*

Proposition 2.5. *Suppose that* $X \in \mathbf{Lg}_\#(D)$ *and* $(L, E) \in G[L]$. *Let* $\lambda \in \mathbb{F}$. *Then* $E_\lambda \in M(X)$ *and* $DE_\lambda u = \lambda(E_{\lambda-1}u)Du$ *for* $u \in \mathrm{dom}\ L$ *(cf. Proposition 2.2).*

In general, we have the following

Proposition 2.6. *Suppose that* $X \in \mathbf{Lg}_\#(D)$ *and* $(L, E) \in G[L]$. *If* $\lambda \in \mathbb{F}$ *and* $u, v \in \mathrm{dom}\ L$, $E_\lambda u, E_\lambda v \in I(X)$, *then*

$$(E_\lambda u)(E_\lambda v) = E\{c_D\lambda(Lu+Lv)+S[(E_\lambda u)^{-1}(E_\lambda v)^{-1}f_D(E_\lambda u, E_\lambda v)]\}\ (S \in \mathcal{L}_D).$$

(cf. Proposition 2.3).

Corollary 2.2. *Suppose that all assumptions of Proposition 2.6 are satisfied and* $c_D = 0$. *Then the mapping* E_λ *is not defined for* $\lambda \neq 1$. *If* $\lambda = 1$, *then* $E_1 = I|_{\mathrm{dom}\ \Omega}$ *(cf. Corollary 2.1).*

Definition 2.2. Suppose that either $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, X is a complete m-pseudoconvex Leibniz algebra with unit e, either $X \in \mathbf{Lg}(D)$ or $X \in \mathbf{Lg}_\#(D)$, $e \in \mathrm{dom}\ \Omega^{-1}$ and $(L, E) \in G[\Omega]$ (Recall that for $D \in \Lambda(X)$ we have $\Omega = L$). Write

$$\mathcal{E}'_D(X) = \{u \in \mathrm{dom}\ L : \lambda Lu \in \mathcal{E}_D(X) \quad \text{for } (L, E) \in G[\Omega],\ \lambda \in \mathbb{F}\} \tag{2.3}$$

(cf. Formula (1.6)) and

$$u^\lambda = \mathrm{e}^{\lambda Lu} \quad \text{for } u \in \mathcal{E}'_D(X),\ \lambda \in \mathbb{F}. \tag{2.4}$$

The function u^λ is said to be a *power function.*

Proposition 2.7. *Suppose that either* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{F} = \mathbb{C}$, $X \in \mathbf{Lg}(D)$ *is a Leibniz complete* m*-pseudoconvex algebra with unit* $e \in \mathrm{dom}\ \Omega^{-1}$, $(L, E) \in G[\Omega]$ *and* D *is closed. Then for* $\lambda \in \mathbb{F}$:

(i) if $u \in \mathcal{E}'_D(X)$, $\lambda \in \mathbb{F}$, *then* $\mathrm{e}^{\lambda Lu} \in \mathrm{dom}\ \Omega$, $\mathrm{e}^{\lambda Lu} = E_\lambda u = u^\lambda$ *and* $Lu^\lambda = \lambda Lu$*;*

(ii) if $u \in I(X) \cap \mathcal{E}'_D(X)$ *then*

$$Du^\lambda = \lambda u^{\lambda-1} Du; \tag{2.5}$$

(iii) in particular, if $\lambda = n \in \mathbb{N}$ *then*

$$u^\lambda = u^n = \underbrace{u \cdot ... \cdot u}_{n-times}.$$

If we restrict ourselves to commutative algebras with right invertible operators, then Definition 2.2 can be generalized in the following manner.

Definition 2.3. Suppose that $X \in \mathbf{Lg}(D)$ and $(L, E) \in G[\Omega]$. Write

$$\Upsilon(\Omega) = \{(x, y) : x \in \mathrm{dom}\ \Omega,\ yLx \in \mathrm{dom}\ \Omega^{-1}\}$$

and

$$x^y = E(yLx) \quad \text{whenever} \quad (x, y) \in \Upsilon(\Omega).$$

By definition, $Lx^y = yLx$. Indeed, $Lx^y = LE(yLx) = yLx$. Let $u = x^y$ for $(x, y) \in \Upsilon$ and let $y \in I(X)$. Then

$$x = ELx = E(y^{-1}Lx^y) = E(y^{-1}Lu) = u^{y^{-1}}.$$

Clearly, x^y is a generalization of power functions introduced by Definition 2.2 for scalar exponents, so that we call x^y also a *power function*.

Observe that by definition, $x^e = x$ and $x^{-e} = x^{-1}$, since $x^e = E(eLx) = ELx = x$ and $x^{-e} = E(-eLx) = E(-Lx) = x^{-1}$. Moreover, if $u = x^y$ for $(x, y) \in \Upsilon(\Omega)$ and $y \in I(X)$, then

$$x = ELx = E(y^{-1}Lx^y) = E(y^{-1}Lu) = u^{y^{-1}}.$$

Definition 2.3 will be very useful in order to establish the relationship between the number e and the unit e of an algebra under consideration.

Theorem 2.2. *Suppose that* $X \in \mathbf{Lg}(D)$ *is a Leibniz algebra with unit* e *and* $(L, E) \in G[\Omega]$. *Then the power function* x^y *has the following properties:*

(i) if (x, a),$(x, b) \in \Upsilon(\Omega)$, *then* $a + b \in \Upsilon(\Omega)$ *and* $x^a x^b = x^{a+b}$*;*

(ii) if $(x,a),(y,a) \in \Upsilon(\Omega)$, *then* $(xy,a) \in \Upsilon(\Omega)$ *and* $x^a y^a = (xy)^a$*;*

(iii) if $(x,y) \in \Upsilon(\Omega)$ *then* $x^y \in \operatorname{dom} D$ *and* $Dx^y = x^y[(Dy)Lx + yx^{-1}Dx]$, *in particular, if* $x \in \ker D$ *then* $Dx^y = x^y(Dy)Lx$, *if* $y \in \ker D$ *then* $Dx^y = yx^{y-e}Dx$*;*

(iv) if $(x,y) \in \Upsilon(\Omega)$ *and* $x,y \in \ker D$, *then* $x^y \in \ker D$, *in other words: a constant to a constant power is again a constant;*

(v) if $(x,y) \in \Upsilon(\Omega)$ *and* $a = Ey$, *then* $x^{La} = a^{Lx}$*;*

(vi) if $(x,y) \in \Upsilon(\Omega)$ *and* $a = Lx$, *then* $(Ea)^y = y^{Ea}$*;*

(vii) $e^{\lambda e} = e$ *whenever* $\lambda \in \mathbb{F}$*;*

(viii) if $x \in \operatorname{dom} \Omega$, *then* $x^0 = e$ (cf. (1.7))*;*

(ix) if $(x,u),(x^u,v) \in \Upsilon(\Omega)$, *then* $(x,uv) \in \Upsilon(\Omega)$ *and* $(x^u)^v = x^{uv}$*;*

(x) if $(x,y) \in \Upsilon(\Omega)$, *then* $(x,-y) \in \Upsilon(\Omega)$, $x^y \in I(X)$ *and* $(x^y)^{-1} = x^{-y}$*;*

(xi) if the logarithm L *is natural (i.e. if* $L(p_n e) = e \ln p_n$, *where* p_n *is the* n*-th prime* $(n \in \mathbb{N})$*), then* $(\mathrm{e}e)^x = Ex$ *whenever* $x \in \operatorname{dom} \Omega^{-1}$*;*

(xii) if X *is an* m*-pseudoconvex* D*-algebra and* $\lambda e \in \mathcal{E}_D(X)$ *for all* $\lambda \in \mathbb{F}$ *(*$\mathbb{F} = \mathbb{R}$ *or* $\mathbb{F} = \mathbb{C}$*), then* $\mathrm{e}^{\lambda e} = \mathrm{e}^{\lambda} e$, *in particular,* $\mathrm{e}^{e} = \mathrm{e}e$.

Clearly, when $X = C[0,T]$ and $D = \frac{\mathrm{d}}{\mathrm{d}t}$, the introduced power mappings coincide with the classical power functions.

Definition 2.4. Suppose that all assumptions of Definition 2.3 are satisfied. Write

$$I_n(Y) = \{x \in Y : \exists_{y \in I(Y)}\ y^n = x\} \quad \text{for } n \in \mathbb{N},\ Y \subset X\}. \tag{2.6}$$

Elements $y \in Y$ will be denoted by $y = x^{1/n}$ and called *nth roots* of x.

By definition, if $y = x^{1/n}$, then

$$x = \mathrm{e}^L x, \quad y = \mathrm{e}^{1/n} Lx = \mathrm{e}^{Lx^{1/n}} \quad \text{whenever} \quad x \in \mathcal{E}_D(X).$$

3. Powers of logarithmic mappings

In the sequel we shall admit for the sake of brevity the following condition:

[L] $X \in \mathbf{Lg}(D)$ is a Leibniz D-algebra with unit e,

(i.e. a commutative Leibniz algebra with unit and with $D \in R(X)$).

Condition [**L**] implies

$$(Lu)^m = E(mL^2u) \quad \text{for } (L,E)\in G[\Omega],\ (u,x)\in \text{graph}\ \Omega\ (m\in\mathbb{N}_0). \quad (3.1)$$

Indeed, $(Lu)^m = EL(Lu)^m = E[mL(Lu)] = E(mL^2u)$.

Definition 3.1. Suppose that Condition [**L**] holds, $(L,E)\in G[\Omega]$, $(u,x)\in$ graph Ω, $x = Lu$, $u = Lx$. Let $n\in\mathbb{N}$ be arbitrarily fixed. Write:

$$\Lambda_n u = \prod_{j=0}^{n} L^j u \quad \text{for} \quad L^j u \in \text{dom}\ \Omega \quad (j=1,...,n). \qquad (3.2)$$

Proposition 3.1. *Suppose that all assumptions of Definition 3.1 are satisfied. Then:*

$$DL^n u = (L^n u)DL^{n+1}u \qquad (n\in\mathbb{N}_0)\,. \qquad (3.3)$$

P r o o f. By definition, $Du = uDLu = uDx$. The same definition implies that for $w = Lu$ we have $DLu = Dw = wDLw = (Lu)DL^2u$. Hence $Du = uDLu = u(Lu)DL^2u$. Suppose Formula (3.3) is true for an arbitrarily fixed ($n\in\mathbb{N}$). Then, by the same reasons, $DL^{n+1}u = (L^{n+1}u)DL^{n+2}u$, i.e. (3.2) holds for $n+1$. ■

Proposition 3.2. *Suppose that all assumptions of Definition 3.1 are satisfied. Then:*

$$Du = \Big(\prod_{j=0}^{n-1} L^j u\Big)DL^n u \qquad (n\in\mathbb{N}_0)\,. \qquad (3.4)$$

P r o o f. By induction. ■

Definition 3.1 and Formula (3.3) immediately imply

Corollary 3.1. *Suppose that all assumptions of Definition 3.1 are satisfied. Then:*

$$Du = (\Lambda_{n-1}u)DL^n u \qquad (n\in\mathbb{N})\,. \qquad (3.5)$$

DEFINITION 3.2. Suppose that all assumptions of Definition 3.1 are satisfied. Let $k_j \in \mathbb{N}$ and $a_j \in \mathrm{dom}\ \Omega$ for $j = 0, ..., n$ $(n \in \mathbb{N})$. Write:

$$\Lambda_n^{k_0,...,k_n}(a_0, ..., a_n)u = \prod_{j=0}^{n} a_j(L^j u)^{k_j} \tag{3.5}$$

and for $a_0 = ... = a_n = e$

$$\Lambda_n^{k_0,...,k_n}u = \prod_{j=0}^{n} (L^j u)^{k_j}. \tag{3.6}$$

Clearly,

$$\Lambda_n^{k_0,...,k_n}u = \Lambda_n u \quad \text{for } k_0 = k_1 = ... = k_{n+1} = 1, \tag{3.7}$$

where $\Lambda_n u$ is defined by Formula (3.2).

THEOREM 3.1. *Suppose that all assumptions of Definition 3.2 are satisfied. Then:*

$$[\Lambda_n^{k_0,...,k_n}(a_0, ..., a_n)u]^m = E(\sum_{j=01}^{n-1} La_j)E(\sum_{j=0}^{n} k_j L^{j+1}u) \quad (m \in \mathbb{N}_0). \tag{3.8}$$

P r o o f. By our assumption, X is a Leibniz algebra. Thus the logarithmic mapping L under consideration is of exponential type, i.e. $L(uv) = Lu + Lv$ for $u, v \in \mathrm{dom}\ D$. Let $n \in \mathbb{N}$ be fixed and let $m = 1$. We have

$$L\Lambda_n^{k_1,...,k_n}(a_1, ..., a_n)u = L\prod_{j=0}^{n} a_j(L^j u)^{k_j}$$

$$= \sum_{j=0}^{n} L[a_j(L^j u)^{k_j}] = \sum_{j=0}^{n} La_j + \sum_{j=0}^{n} k_j L^{j+1},$$

which implies the required Formula (3.8) for $E = L^{-1}$. Since X is a Leibniz algebra, L is of the exponential type. Thus $E = L^{-1}$ has the properties: $E(x + y) = (Ex)(Ey)$ and $E(mx) = (Ex)^m$ for $x, y \in \mathrm{dom}\ \Omega^{-1}$, $m \in \mathbb{N}_0$. Hence Theorem 3.1 and Formula 3.1 together imply the required formula (3.8). ∎

In particular, we have

$$(\Lambda_n u)^m = E(m \sum_{j=1}^{n+1} L^j u) \qquad (m, n \in \mathbb{N}_0)\,. \tag{3.9}$$

It should be mentioned that the already obtained results have some connections with the Number Theory, then also with applications in the cryptography (cf. Schinzel S[1]). There are also some other connections.

4. Functional equations for logarithms, antilogarithms and powers

Recall the classical results.

Example 4.1. (cf. Kuczma K[1]). Suppose that $X = \mathbb{R}$, $\mathbb{F} = \mathbb{R}$. Let $f \in C^\infty(\mathbb{R})$. Then all solutions of the functional equations

- $f(x+y) = f(x) + f(y)$ are $x = ct,\ (c \in \mathbb{R})$,
- $f(xy) = f(x) + f(y)$ are $x = c \log_a t,\ (a \in \mathbb{R} \setminus 0,\ c \in \mathbb{R})$,
- $f(x+y) = f(x)f(y)$ are $x = ce^{at},\ (a, c \in \mathbb{R})$,
- $f(xy) = f(x)f(y)$ are $x = ct^a,\ (a, c \in \mathbb{R})$.

Theorem 4.1. *Suppose that Condition* **[L]** *holds,* $(L, E) \in G[\Omega]$, (u, x), $(v, y) \in \text{graph}\ \Omega$, *i.e.* $x = Lu$, $u = Lx$, $y = Lv$, $v = Ey$. *Let* $f \in \mathcal{I}(X) : \text{dom}\ \Omega \to \text{dom}\ \Omega$.

(i) If $f = L$, *then* L *of the exponential type:* $L(uv) = Lu + Lv$.

(ii) If $f = E$, *then* $E(x+y) = (Ex)(Ey)$.

(iii) If f *is multiplicative:* $f(xy) = f(x)(f(y)$, *then solutions of this functional equation are power elements* $x^a = E(aLx)$, *where* $(x, a) \in \Upsilon(\Omega)$ (cf. Definition 2.3.

(iv) If f *is multiplicative, then*

$$L'(uv) = L'u + L'v, \quad \text{where } L' = Lf, \tag{4.12}$$

i.e. L*' is of the exponential type.*

(v) If f *is additive, then*

$$L''(uv) = L''u + L''v, \quad \text{where } L'' = fL, \tag{4.13}$$

i.e. L*" is of the exponential type.*

(vi) If f is additive, then

$$L'''(uv) = L'''u + L'''v, \quad \text{where } L''' = fLf, \tag{4.14}$$

i.e. L''' is also additive.

P r o o f. (i) and (ii) are consequences of the Leibniz condition (cf. PR[2]).

(iii) follows from Theorem 2.2(ii).

(iv) Since f is multiplicative, by (i) we have $L'(uv) = Lf(uv) = L[f(u)f(v)] = Lf(u) + Lf(v) = L'(u) + L'(v)$.

(v) Since f is additive, by (i) we find $L''(uv) = fL(uv) = f(Lu + Lv) = fLu + fLv = L''u + L''v$.

(vi) Since f is additive, again by (i) (as in the proof of (iv)), $L'''(uv) = fLf(uv) = f(Lfu + Lfv) = L'''u + L'''v$. ■

It is easy to verify the following

Corollary 4.1. *Suppose that all assumptions of Theorem 4.1 are satisfied. Let $h = f^{-1}$.*

(i) If $f = L$, then $h = E$.

(ii) If $f = E$, then $h = L$.

(iii) If f is multiplicative, then h is also multiplicative.

(iv) If f is multiplicative, then $hE = (Lf)^{-1}$, $hE(x + y) = (hEx)(hEy)$ and the last equation has solutions of the form $h^{-1}Ex = fEx$.

(v) If f is additive, then $Eh = (fL)^{-1}$, $Eh(x + y) = (Ehx)(Ehy)$ and the last equation has solutions of the form $Eh^{-1}x = Efx$.

(vi) If f is additive, then hEh is also additive.

Similar results can be obtained in Leibniz algebras with left invertible operators.

Example 4.2. (cf. DP[1]) Let X be a complex Banach space. Denote by $B(X)$ the set of all bounded operators mapping X into itself. A strongly continuous family of operators $\{W(t)\}_{t\geq 0} \subseteq B(X)$ is a *C-regularized semigroup* if $W(0) = C$ and $W(t)W(s) = W(t+s)C$ for all $s, t \geq 0$. This family is *nondegenerate*, if $W(t)x = 0$ implies $x = 0$. A C-regularized semigroup is nondegenerate if and only if C is injective. An operator A generates a nondegenerate C-regularized semigroup $\{W(t)\}_{t\geq 0}$ if

$$Bx = C^{-1}\Big[\frac{\mathrm{d}}{\mathrm{d}t}W(t)x\Big|_{t=0}\Big]$$

with the maximal domain.

If there is a nondegenerate C-regularized semigroup $\{W(t)\}_{t\geq 0}$ such that $A = C^{-1}W(1)$, then its generator is, by definition, $\log Ax \equiv Bx$.

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