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METHODS FOR SOLVING NECESSARY EQUIVALENCES

Frank Brown

Abstract: Nonmonotonic Logics such as Autoepistemic Logic, Reflective Logic, and Default Logic, are usually defined in terms of set-theoretic fixed-point equations defined over deductively closed sets of sentences of First Order Logic. Such systems may also be represented as necessary equivalences in a Modal Logic stronger than S5 with the added advantage that such representations may be generalized to allow quantified variables crossing modal scopes resulting in a Quantified Autoepistemic Logic, a Quantified Autoepistemic Kernel, a Quantified Reflective Logic, and a Quantified Default Logic. Quantifiers in all these generalizations obey all the normal laws of logic including both the Barcan formula and its converse. Herein, we address the problem of solving some necessary equivalences containing universal quantifiers over modal scopes. Solutions obtained by these methods are then compared to related results obtained in the literature by Circumscription in Second Order Logic since the disjunction of all the solutions of a necessary equivalence containing just normal defaults in these Quantified Logics, is equivalent to that system.

Keywords: Solving Necessary Equivalences, Modal Logic, Nonmonotonic Logic.

1. Introduction

Solving equations is an important aspect of Automated Deduction because the solutions to an equation can often be used in other parts of a theorem in order to prove the theorem or to derive consequences. Normally, we think of solving equations in some numeric valued algebra such as number theory, real algebra, complex algebra, or linear spaces; but there is no reason the process cannot be applied to radically different types of mathematical structures although the actual techniques for solving equations will depend on the nature of that structure's algebra. A mathematical structure of particular interest in Artificial Intelligence is the sets of sentences of First Order Logic (i.e. FOL) which are deductively complete by the laws of FOL because such sets form the basis of the fixed-point theories of nonmonotonic reasoning such as Autoepistemic Logic [Moore 1985], its kernel [Konolige 1987], Reflective Logic [Brown 1987], and Default Logic [Reiter 1980]. A set of sentences of FOL is said to be deductively complete if and only if all the theorems deducible from it by the laws of FOL are contained within it. Deriving properties of infinite sets of sentences would appear to involve a sophisticated automatic theorem prover for set theory and FOL syntax. However, we avoid this by noting that deductively complete sets of sentences may be represented by the proposition which is the meaning of all the sentences in that set. The modal sentence $[]((\wedge_i\Gamma_i)\rightarrow\alpha)$ where [] is the necessity symbol can then be used to represent the proof theoretic statement $`\alpha\in(\mathrm{fol}\{`\Gamma_i\})$ where $`\alpha$ is the name of a sentence $\alpha$, for each i, $`\Gamma_i$ is the name of the sentence Γ_i of FOL, and fol is the set of FOL sentences derivable from $\{`\Gamma_i\}$ by the laws of FOL. A nonmonotonic fixed-point equation defined in set theory such as:

$$K=\mathrm{fol}(\{`\Gamma_i\}\cup\{`\alpha: (`(\neg\alpha)\notin K)\})$$

is then represented as the necessary equivalence:

$$k\equiv(\Gamma\wedge((\neg[]\neg(k\wedge\alpha))\rightarrow\alpha))$$

This algebra for solving for k in a necessary equivalence is just FOL supplemented with propositional variables (e.g. k above) and the necessity operator of a particular modal quantificational logic called Z. Translations of Autoepistemic Logic, Reflective Logic, and Default Logic, to Z are proven, respectively, in [Brown 2003c], [Brown 2003a], and [Brown 2003b]. An Automatic Deduction System, based on Z, for Autoepistemic Logic and Reflective Logic is given in [Brown 2003e]. Besides providing an algebra for representing such nonmonotonic systems (where quantified variables are not allowed to cross modal scopes), this modal representation has the additional advantage of providing an explication of what it means to have quantified variables crossing modal scopes:

$$k\equiv(\Gamma\wedge\forall\xi((\neg[]\neg(k\wedge\alpha))\rightarrow\alpha))$$

where ξ may occur free in α. Quantified Autoepistemic Logic, its Quantified Kernel, and Quantified Reflective Logic and their relationships are discussed in [Brown 2004]. Quantified Reflective Logic and Quantified Default

Logic and their relationship is discussed in [Brown 2003d]. Herein, we exemplify necessary equivalence solving by solving an example involving normal defaults, similar to the example given above, which is expressible in all of these quantified systems.

Section 2 describes the Z Modal Quantificational Logic which is the algebra in which the necessary equivalence solving takes place. Section 3 discusses proving what is logically possible. Section 4 discusses necessary equivalence solving. Section 5 discusses deducing what is common to all solutions. Finally, some conclusions are drawn in Section 6.

2. The Modal Quantificational Logic Z

The syntax of Z Modal Quantificational Logic is an amalgamation of 3 parts:

(1) The first part is a First Order Logic (i.e. FOL) represented as the six tuple: ($\rightarrow$, #f, $\forall$, *vars*, *predicates*, *functions*) where $\rightarrow$, #f, $\forall$, are logical symbols, *vars* is a set of object variable symbols, *predicates* is a set of predicate symbols each of which has an implicit arity specifying the number of terms associated with it, and *functions* is a set of function symbols each of which has an implicit arity specifying the number of terms associated with it. Roman letters x, y, and z, possibly indexed with digits, are used as variables.

(2) The second part is an extension to allow Propositional Quantifiers. It consists of a set of propositional variables *propvars* and quantification over propositional variables (using $\forall$). Roman letters (other than x, y and z) possibly indexed with digits are used as propositional variables.

(3) The third part is Modal Logic [Lewis 1936] which adds the necessity symbol: [].

Greek letters are used as syntactic metavariables. $\pi,\pi_1...\pi_n,\rho,\rho_1...\rho_n$ range over the predicate symbols, ϕ, $\phi_1...\phi_n$ range over function symbols, $\delta,\delta_1...\delta_n$ range over terms, $\gamma,\gamma_1,...\gamma_n$, range over the object variables ξ, $\xi_1\ldots\xi_n,\zeta,\zeta_1\ldots\zeta_n$ range over a sequence of object variables of an appropriate arity, $f,f_1\ldots f_n$ range over predicate variables, and $\alpha,\alpha_1...\alpha_n$, $\beta,\beta_1...\beta_n$, $\chi,\chi_1...\chi_n$, Γ, κ range over sentences. Thus, terms are of the forms: γ and (ϕ $\delta_1...\delta_n$), and sentences are of the forms: ($\alpha \rightarrow\beta$), #f, ($\forall\gamma$ α), (π $\delta_1...\delta_n$), (f $\delta_1...\delta_n$), ($\forall f$ α), and ([]α). A zero arity predicate π, propositional variable f, or function ϕ is written as a sentence or term without parentheses, i.e., π instead of (π), f instead of (f), and ϕ instead of (ϕ). $\alpha\{\pi/\lambda\xi\beta\}$ is the sentence obtained from α by replacing all unmodalized occurrences of π by $\lambda\xi\beta$ followed by lambda conversion. $\alpha\{\pi_i/\lambda\xi\beta_i\}_{i=1,n}$, abbreviated as $\alpha\{\pi_i/\lambda\xi\beta_i\}$, represents simultaneous substitutions. $\wedge_{i=1,n}\beta_i$ represents ($\beta_1\wedge\ldots\wedge\beta_n$). The primitive symbols are listed in Figure 1. The defined symbols are listed in Figure 2.

Symbol	Meaning
$\alpha\rightarrow \beta$	if α then β.
#f	falsity
$\forall\gamma\ \alpha$	for all γ, α.
$\forall f\ \alpha$	for all f, α.
[] α	α is logically necessary

Figure 1: Primitive Symbols

Symbol	Definition	Meaning
$\neg\alpha$	$\alpha \rightarrow$ #f	not α
#t	$\neg$ #f	truth
$\alpha\vee\beta$	$(\neg\ \alpha)\rightarrow \beta$	α or β
$\alpha\wedge\beta$	$\neg(\alpha \rightarrow \neg\beta)$	α and β
$\alpha\leftrightarrow \beta$	$(\alpha\rightarrow \beta)\wedge(\beta\rightarrow \alpha)$	α if and only if β
$\exists\gamma\ \alpha$	$\neg\forall\gamma\ \neg\alpha$	some γ is α

$\exists f \alpha$	$\neg\forall f \neg\alpha$	some f is α
$<> \alpha$	$\neg[]\neg\alpha$	α is logically possible
$[\beta]\ \alpha$	$[](\beta\rightarrow\alpha)$	β entails α
$<\beta>\ \alpha$	$<>(\beta\wedge\alpha)$	α is possible with β
$\alpha\equiv\beta$	$[](\alpha\leftrightarrow\beta)$	α and β are synonymous
$\delta_1=\delta_2$	$(\pi\ \delta_1)\equiv(\pi\ \delta_2)$	δ_1 necessarily equals δ_2
$\delta_1\neq\delta_2$	$\neg(\delta_1=\delta_2)$	δ_1 does not necessarily equal δ_2
(det α)	$\forall f(([\alpha]f)\vee([\alpha]\neg f))$where f is not in α	α is deterministic
(world α)	$(<>\alpha)\wedge$(det α)	α is a world

Figure 2: Defined Symbols of Z

The laws of Z Modal Logic is an amalgamation of five parts:

(1) The first part, given in Figure 3, consists of the laws of a FOL [Mendelson 1964].

(2) The second part, which is given in Figure 4, consists of the additional laws needed for propositional quantification. The laws SOLR2, SOLA4 and SOLA5 are the analogues of FOLR2, FOLA4 and FOLA5 for propositional variables.

(3) The Third part, which is given in Figure 5, consists of the laws MR0, MA1, MA2 and MA3 which constitute an S5 modal logic [Hughes & Cresswell 1968] [Carnap 1956]. When added to parts 1 and 2, they form a fragment of a Second Order Modal Quantificational logic similar to a second order version of [Bressan 1972].

(4) The fourth part, which is given in Figure 6, consists of the Priorian World extension of S5 Modal Logic. The PRIOR law [Prior and Fine 1977] states that a proposition is logically true if it is entailed by every world. This law was implied in [Leibniz 1686]

(5) The fifth part, which is given in Figure 7, consists of laws axiomatizing what is logically possible. MA1a lets one derive theorems such as $<>\forall x((\pi\ x)\leftrightarrow(x=\phi))$ and therefore extends the work on propositional possibilities in [Hendry and Pokriefka 1985] to possibilities in FOL being simpler than the FOL approaches in [Brown 1987]. MA1b and MA4 allow one to derive $(\phi_1\ \xi_1)\neq(\phi_2\ \xi_2)$ when ϕ_1 and ϕ_2 are distinct function symbols. MA4 states that at least two things are not necessarily equal just as there are at least two propositions: #t and #f.

FOLA3: $\alpha \rightarrow (\beta\rightarrow \alpha)$

FOLA4: $(\alpha\rightarrow(\beta\rightarrow \rho)) \rightarrow ((\alpha\rightarrow \beta)\rightarrow (\alpha\rightarrow \rho))$

FOLA5: $((\neg \alpha)\rightarrow (\neg \beta))\rightarrow (((\neg \alpha)\rightarrow \beta)\rightarrow\alpha)$

FOLA6: $(\forall\gamma\ \alpha)\rightarrow \beta$ where β is the result of substituting an expression (which is free for the free positions of γ in α) for all the free occurrences of γ in α.

FOLA7: $(\forall\gamma\ (\alpha \rightarrow \beta)) \rightarrow (\alpha \rightarrow(\forall\gamma\ \beta))$ where γ does not occur in α.

FOLR1: from α and $(\alpha\rightarrow \beta)$ infer β

FOLR2: from α infer $(\forall\gamma\ \alpha)$

Figure 3: The Laws of FOL

SOLR2: from α infer $(\forall f\ \alpha)$

SOLA6: $(\forall f\ \alpha)\rightarrow \beta$ where β is the result of substituting an expression (which is free for the free positions of f in α) for all the free occurrences of f in α.

SOLA7: $(\forall f(\alpha \rightarrow \beta)) \rightarrow (\alpha \rightarrow (\forall f\ \beta))$ where f does not occur in α.

Figure 4: Additional Laws for Propositional Quantifiers

MA1: $([]\alpha) \rightarrow \alpha$

MA2: $([\alpha]\beta) \rightarrow (([]\alpha)\rightarrow ([]\beta))$

MR0: from α infer $([]\ \alpha)$ provided α was derived only from logical laws

MA3: $([]\alpha) \vee ([]\neg[]\alpha)$

Figure 5: Additional Laws of S5 Modal logic

PRIOR: $(\forall w((\text{world } w) \rightarrow ([w]\alpha)))\rightarrow []\alpha$ where w does not occur free in α.

Figure 6: Additional Law of a Priorian World Logic

MA1a: $([]\alpha)\rightarrow\alpha\{\pi/\lambda\zeta\beta\}$ if no unmodalized occurrence of a higher order var is in α.

MA1b: $([]\alpha) \rightarrow \alpha\{\phi/\lambda\zeta\delta\}$ where α may not contain an unmodalized occurrence of a higher order variable, nor an unmodalized free object variable.

MA4: $\exists x\exists y(x\neq y)$

Figure 7: Additional Laws of Z Modal Logic

3. Deriving what is Logically Possible

The main problem with using axiom scheme MA1a to prove that something is logically possible lies in finding the appropriate substitution for the parameter π. The theorem ZP1 given below is the basis of a heuristic for finding such instances. Intuitively, we know that the conjunction of instances of a predicate and the conjunction of instances of the negation of a predicate are logically possible whenever the two do not coincide on any instance. For example: where $a\neq c$ and $b\neq c$ then $((\pi\ a)\wedge(\pi\ b)\wedge(\neg(\pi\ c)))$ is logically possible but $((\pi\ a)\wedge(\pi\ b)\wedge(\neg(\pi\ a)))$ is not. Thus if a sentence can be written in the form: $(\Gamma\wedge(\forall x(\alpha\rightarrow(\pi\ x)))\wedge(\forall x(\beta\rightarrow(\neg(\pi\ x)))))$ where π does not occur in α, β, and Γ, then it is logically possible if and only if $\{x:\alpha\}\cap\{x:\beta\}$ is empty, which is to say that $\exists x(\alpha\wedge\beta)$ does not follow from Γ or to say that Γ and $\neg\exists x(\alpha\wedge\beta)$ is logically possible. In this manner determining whether a sentence with n predicates is logically possible can sometimes be reduced to determining whether a sentence with n-1 predicates is logically possible without having to guess any instances of π in MA1a.

<u>Theorem ZP1: The Possibility of a Disjoint Predicate Definition:</u> [Brown 1989]

If Γ, α, and β do not contain any unmodalized occurrences of π nor of any higher order variable then:

$$(<>(\Gamma\wedge(\forall\xi(\alpha\rightarrow(\pi\ \xi)))\wedge(\forall\xi(\beta\rightarrow\neg(\pi\ \xi))))) \leftrightarrow(<>(\Gamma\wedge(\neg\exists\xi(\alpha\wedge\beta))))$$

ZP1 is applicable to any theory which can be put into a prenix conjunctive normal form such that no disjunct contains more than one unmodalized occurrence of π, since by the laws of classical logic such theories are equivalent to an expression of the form: $(\Gamma\wedge(\forall\xi(\alpha\rightarrow(\pi\ \xi)))\ \wedge(\forall\xi(\beta\rightarrow\neg(\pi\ \xi))))$. It is also applicable to any theory which can be put into a disjunction of a prenix conjunctive normal forms whose disjunctions contains no more than one unmodalized occurrence of π. The reason for this is that disjunction (and existential quantifiers) associate through possibility: $(<>(\alpha\vee\beta))\leftrightarrow((<>\alpha)\vee(<>\beta)$. For this reason, ZP1 is decidable for many important cases of FOL including propositional logic and the case of a theory with a finite number of intensional objects.

<u>Example 1</u>: Deducing a Logical Possibility thrice using ZP1. Let k be any sentence.

$<>((P\ b)\wedge(P\ a)\wedge(\neg(Q\ b))\ \wedge\forall x((\neg(AB\ x))\rightarrow((P\ x)\rightarrow(Q\ x)))$

$\quad\wedge\forall x((x\neq b)\rightarrow\neg(AB\ x))\wedge\forall x(((x\neq a)\wedge(x\neq b))\rightarrow((P\ x)\leftrightarrow([k](P\ x))))\ )$

This is equivalent to:

$<>((P\ b)\wedge(P\ a)\wedge\forall x\ ((x=b)\rightarrow\neg(Q\ x))\wedge\forall x(((\neg(AB\ x))\wedge(P\ x))\rightarrow(Q\ x))$

$\quad\wedge\forall x((x\neq b)\rightarrow\neg(AB\ x))\wedge\forall x(((x\neq a)\wedge(x\neq b))\rightarrow((P\ x)\leftrightarrow([k](P\ x))))\ )$

Instantiating ZP1 by letting α be $(\neg(AB\ x))\wedge(P\ x))$, β be x=b, and Γ be the sentences not containing Q gives:

$<>((P\ b)\wedge(P\ a)\wedge\neg\exists x((\neg(AB\ x))\wedge(P\ x)\wedge(x=b))\wedge\forall x((x\neq b)\rightarrow\neg(AB\ x))\wedge\forall x(((x\neq a)\wedge(x\neq b))\rightarrow((P\ x)\leftrightarrow([k](P\ x))))\ )$

which is equivalent to: $<>((P\ b)\wedge(P\ a)\wedge(AB\ b))\wedge\forall x((x\neq b)\rightarrow\neg(AB\ x))\wedge\forall x(((x\neq a)\wedge(x\neq b))\rightarrow((P\ x)\leftrightarrow([k](P\ x))))\ )$

which is: $<>((P\ b)\wedge(P\ a)\wedge\forall x((x=b)\rightarrow(AB\ x))\wedge\forall x((x\neq b)\rightarrow\neg(AB\ x))\wedge\forall x(((x\neq a)\wedge(x\neq b))\rightarrow((P\ x)\leftrightarrow([k](P\ x))))\ )$

Instantiating ZP1 by letting α be x=b, β be $x\neq b$, and Γ be the sentences not containing AB, gives:

$<>((P\ b)\wedge(P\ a)\wedge\neg\exists x((x=b)\wedge(x\neq b))\wedge\forall x(((x\neq a)\wedge(x\neq b))\rightarrow((P\ x)\leftrightarrow([k](P\ x))))\ )$

which is equivalent to: $<>((P\ b)\wedge(P\ a)\wedge\forall x(((x\neq a)\wedge(x\neq b))\rightarrow((P\ x)\leftrightarrow([k](P\ x))))\ )$

which is: $<>(\#t\wedge\forall x(((x{=}a)\vee(x{=}b)\vee((x\neq a)\wedge(x\neq b)\wedge([k](P\ x))))\rightarrow(P\ x))\wedge\forall x(((x\neq a)\wedge(x\neq b)\wedge(\neg([k](P\ x))))\rightarrow\neg(P\ x))$

Instantiating ZP1 by letting α be $((x{=}a)\vee(x{=}b)\vee((x\neq a)\wedge(x\neq b)\wedge([k](P\ x))))$, β be $((x\neq a)\wedge(x\neq b)\wedge(\neg([k](P\ x))))$ and Γ be #t gives: $<>(\#t\wedge\neg\exists x(((x{=}a)\vee(x{=}b)\vee((x\neq a)\wedge(x\neq b)\wedge([k](P\ x))))\wedge((x\neq a)\wedge(x\neq b)\wedge(\neg([k](P\ x))))\ ))$

which <>#t which is #t.

4. Solving Necessary Equivalences when Variables Cross Modal Scopes

A necessary equivalence has the form $k\equiv(\Phi\ k)$ where k is a propositional variable. The goal is to transform the initial necessary equivalence into a (possibly infinite) disjunction, $\exists i(k\equiv\beta_i)$ with each β_i free of k. If $k\equiv\beta_i$ implies the original equation then β_i is a solution to the original equation. The following procedure solves necessary equivalences when no variable crosses modal scopes:

Procedure for Solving Modal Equivalences [Brown 1986]:

Step 1: First, one by one, each subformula α which contains k and is equivalent to $([]\alpha)$ is pulled out of the necessary equivalence causing it to be split into two cases. This is done by using the following theorem schema replacing any instance of the left side by the corresponding instance of the right side:

$(k\equiv(\phi\ \alpha))\leftrightarrow((\alpha\wedge(k\equiv(\phi\ \#t)))\vee((\neg\alpha)\wedge(k\equiv(\phi\ \#f))))$

Step 2: Second, the resulting equivalences are simplified by the laws of the modal logic.

Step 3: Third, on each disjunct the simplified value for k is back substituted into each such α or $(\neg\alpha)$ sentence thereby eliminating k from them.

Step 4: Fourth, the α and $(\neg\alpha)$ sentences are simplified using the modal laws giving a disjunction of necessary equivalences.

When no variables cross modal scopes, for any decidable case of First Order Logic, this method is an algorithm resulting in a finite disjunction of solutions as is illustrated in Example 2 below:

Example 2: Solving a Modal Equation: $k\equiv(((<k>\neg A)\rightarrow B)\wedge((\neg<k>\neg A)\rightarrow A))$

Step 1 gives: $((<k>\neg A)\wedge(k\equiv((\#t\rightarrow B)\wedge((\neg\#t)\rightarrow A))))\vee((\neg(<k>\neg A))\wedge(k\equiv((\#f\rightarrow B)\wedge((\neg\#f)\rightarrow A))))$

Step 2 gives: $((<k>\neg A)\wedge(k\equiv B))\vee((\neg(<k>\neg A))\wedge(k\equiv A))$

Step 3 gives: $((<B>\neg A)\wedge(k\equiv B))\vee((\neg(<A>\neg A))\wedge(k\equiv A))$

Step 4 gives: $(\#t\wedge(k\equiv B))\vee(\#t\wedge(k\equiv A))$ which is: $(k\equiv B)\vee(k\equiv A)$

The process described above provides an algorithm [Brown & Araya 1991] for solving necessary equivalences in the modal representations of Reflective Logic [Brown 2003a] and Autoepistemic Logic [Brown 2003c], and with some additional details of [Default Logic 2003b]. These modal representations can be generalized to allow for universally quantified variables crossing modal scopes (at the top level of the right side of the equation). We now address the problem of solving necessary equivalences when quantified variables cross modal scopes as in:

$$k\equiv(\Gamma\wedge\wedge_{i}\forall\xi_{i}((([k]\alpha_{i})\wedge(\wedge_{j=1,mi}(<k>\beta_{ij})))\rightarrow\chi_{i}))$$

We want to eliminate the modal expressions containing k from the right side of the necessary equivalence. The difficulty lies in the fact that in general we cannot apply Step 1 in the above algorithm because the modal expressions may contain the ξ_i variables which are captured by quantifiers inside the right side of the necessary equivalence. The solution to this dilemma is to allow quantified statements such as: $\forall\xi((([k]\alpha)\wedge(\wedge_{j=1,m}(<k>\beta_j)))\rightarrow\chi)$ to be divided into a finite number of instances for each particular formula, which may or may not hold in a particular solution, leaving all the remaining instances in the quantified statement:

Step 0: Divide a quantified expression over a modal scope into parts using the schema:

$$(k\equiv(\Psi\wedge\forall\xi((([k]\alpha)\wedge(\wedge_{j=1,m}(<k>\beta_j)))\rightarrow\chi_i)))$$

$$\leftrightarrow(k\equiv(\Psi\wedge\forall\xi((\phi\wedge([k]\alpha)\wedge(\wedge_{j=1,m}(<k>\beta_j)))\rightarrow\chi_i))\wedge\forall\xi((\neg\phi\wedge([k]\alpha)\wedge(\wedge_{j=1,m}(<k>\beta_j)))\rightarrow\chi)))$$

where ϕ specifies a finite number of instances thereby allowing the quantifier above it in the resulting expression to be eliminated. The parts remaining under the quantifier that are consistent with the solution may then sometimes be eliminated by theorems PR or NPR. We call this application <u>Step 5</u>:

<u>Theorem PR: Reduction of Possible Reflections</u>: If Γ, α, and β are sentences of Z, and if γ is not free in Γ then:

$(\forall\gamma <>(\Gamma\wedge(\forall\gamma\ \beta)\wedge\alpha)) \rightarrow((k\equiv(\Gamma\wedge\forall\gamma((<k>\alpha)\rightarrow\beta)))\leftrightarrow(k\equiv(\Gamma\wedge\forall\gamma\ \beta)))$

proof: $(k\equiv(\Gamma\wedge\forall\gamma\ ((<k>\alpha)\rightarrow\beta)))\leftrightarrow (k\equiv(\Gamma\wedge\forall\gamma\ \beta))$ is true iff:

$((k\equiv(\Gamma\wedge\forall\gamma\ ((<k>\alpha)\rightarrow\beta))) \rightarrow (k\equiv(\Gamma\wedge\forall\gamma\ \beta))) \wedge ((k\equiv(\Gamma\wedge\forall\gamma\ \beta)) \rightarrow (k\equiv(\Gamma\wedge\forall\gamma\ ((<k>\alpha)\rightarrow\beta))))$

Using the hypothesis in each case we get:

$(((k\equiv(\Gamma\wedge\forall\gamma\ ((<k>\alpha)\rightarrow\beta)))\rightarrow((\Gamma\wedge\forall\gamma((<k>\alpha)\rightarrow\beta))\equiv(\Gamma\wedge\forall\gamma\ \beta)))$

$\wedge((k\equiv(\Gamma\wedge\forall\gamma\ \beta))\rightarrow ((\Gamma\wedge\forall\gamma\ \beta)\equiv(\Gamma\wedge\forall\gamma\ ((<k>\alpha)\rightarrow\beta)))))$

which would be true if: $(((k\equiv(\Gamma\wedge\forall\gamma\ ((<k>\alpha)\rightarrow\beta)))\rightarrow\forall\gamma(<k>\alpha)) \wedge ((k\equiv(\Gamma\wedge\forall\gamma\ \beta))\rightarrow\forall\gamma\ (<k>\alpha)))$

which is implied by: $(\forall\gamma\ (<(\Gamma\wedge\forall\gamma((<k>\alpha)\rightarrow\beta))>\alpha))\wedge\forall\gamma(<(\Gamma\wedge\forall\gamma\ \beta)>\alpha)$

which is equivalent to just: $\forall\gamma\ (<>(\Gamma\wedge(\forall\gamma\ \beta)\wedge\alpha))$ which is the hypothesis of the theorem. QED.

<u>Theorem NPR: Reduction of Normal Possible Reflections</u>:

If Γ and α are sentences of Z, and if γ is not free in Γ then:

$(\forall\gamma <>(\Gamma\wedge(\forall\gamma\ \alpha))) \rightarrow((k\equiv(\Gamma\wedge\forall\gamma\ ((<k>\alpha)\rightarrow\alpha)))\leftrightarrow(k\equiv(\Gamma\wedge\forall\gamma\ \alpha)))$

proof: Letting β be α in PR gives: $(\forall\gamma <>(\Gamma\wedge(\forall\gamma\ \alpha)\wedge\alpha)) \rightarrow ((k\equiv(\Gamma\wedge\forall\gamma((<k>\alpha)\rightarrow\alpha)))\leftrightarrow(k\equiv(\Gamma\wedge\forall\gamma\ \alpha)))$

Since $(\forall\gamma\ \alpha)$ implies α this is proves the theorem. QED.

<u>Example 3: Partially Solving a Modal Equation with Quantifiers over Modal Scopes</u>

$k\equiv((P\ a)\wedge(\neg(Q\ b))\wedge\forall x((\neg(AB\ x))\rightarrow((P\ x)\rightarrow(Q\ x)))\ \wedge\forall x((<k>\neg(AB\ x))\rightarrow\neg(AB\ x))\ \wedge\forall x((P\ x)\leftrightarrow([k](P\ x)))\ )$

Step 0: Dividing $\forall x((P\ x)\leftrightarrow([k](P\ x)))$ on a and b and simplifying by noting that [k]Pa gives :

$(k\equiv((P\ a)\wedge(\neg(Q\ b))\ \wedge\forall x((\neg(AB\ x))\rightarrow((P\ x)\rightarrow(Q\ x)))\ \wedge\forall x((<k>\neg(AB\ x))\rightarrow\neg(AB\ x))$

$\quad\wedge((P\ b)\leftrightarrow([k](P\ b)))\wedge\forall x(((x\neq a)\wedge(x\neq b))\rightarrow((P\ x)\leftrightarrow([k](P\ x))))\ ))$

Steps 1&2: Splitting on ([k](P b)) gives:

$((([k](P\ b))\wedge(k\equiv((P\ b)\wedge(P\ a)\wedge(\neg(Q\ b))\ \wedge\forall x((\neg(AB\ x))\rightarrow((P\ x)\rightarrow(Q\ x)))\ \wedge\forall x((<k>\neg(AB\ x))\rightarrow\neg(AB\ x))$

$\qquad\wedge\forall x(((x\neq a)\wedge(x\neq b))\rightarrow((P\ x)\leftrightarrow([k](P\ x))))\ )))$

$\vee(\neg([k](P\ b))\wedge(k\equiv((\neg(P\ b))\wedge(P\ a)\wedge(\neg(Q\ b))\ \wedge\forall x((\neg(AB\ x))\rightarrow((P\ x)\rightarrow(Q\ x)))$

$\qquad\wedge\forall x((<k>\neg(AB\ x))\rightarrow\neg(AB\ x))\ \wedge\forall x(((x\neq a)\wedge(x\neq b))\rightarrow((P\ x)\leftrightarrow([k](P\ x))))\ ))))$

Step 0: In the first necessary equivalence $(P\ b)\wedge(\neg(Q\ b))$ implies $(AB\ b)$. Thus dividing the AB default $\forall x((<k>\neg(AB\ x))\rightarrow\neg(AB\ x))$ on b gives: $\forall x(((x\neq b)\wedge(<k>\neg(AB\ x)))\rightarrow\neg(AB\ x))$ which is equivalent to: $(<k>((x\neq b)\rightarrow\neg(AB\ x)))\rightarrow((x\neq b)\rightarrow\neg(AB\ x))$ resulting in:

$((([k](P\ b))\wedge(k\equiv((P\ b)\wedge(P\ a)\wedge(\neg(Q\ b))\ \wedge\forall x((\neg(AB\ x))\rightarrow((P\ x)\rightarrow(Q\ x)))$

$\qquad\wedge\forall x((<k>((x\neq b)\rightarrow\neg(AB\ x)))\rightarrow((x\neq b)\rightarrow\neg(AB\ x)))\wedge\forall x(((x\neq a)\wedge(x\neq b))\rightarrow((P\ x)\leftrightarrow([k](P\ x))))\ )))$

$\vee(\neg([k](P\ b))\wedge(k\equiv((\neg(P\ b))\wedge(P\ a)\wedge(\neg(Q\ b))\ \wedge\forall x((\neg(AB\ x))\rightarrow((P\ x)\rightarrow(Q\ x)))$

$\qquad\wedge\forall x((<k>\neg(AB\ x))\rightarrow\neg(AB\ x))\ \wedge\forall x(((x\neq a)\wedge(x\neq b))\rightarrow((P\ x)\leftrightarrow([k](P\ x))))\ ))))$

In the second necessary equivalence, if a=b then k≡#f which implies that $\neg([k](P\ b))$ is equivalent to $\neg([\#f](P\ b))$ which is equivalent to #f. Thus $a\neq b$ on the second case giving:

$((([k](P\ b))\wedge(k\equiv((P\ b)\wedge(P\ a)\wedge(\neg(Q\ b))\ \wedge\forall x((\neg(AB\ x))\rightarrow((P\ x)\rightarrow(Q\ x)))$

$\qquad\wedge\forall x((<k>((x\neq b)\rightarrow\neg(AB\ x)))\rightarrow((x\neq b)\rightarrow\neg(AB\ x)))\wedge\forall x(((x\neq a)\wedge(x\neq b))\rightarrow((P\ x)\leftrightarrow([k](P\ x))))\ )))$

$$\vee(\neg([k](P\,b))\wedge(a\neq b)\wedge(k\equiv((\neg(P\,b))\wedge(P\,a)\wedge(\neg(Q\,b))\wedge\forall x((\neg(AB\,x))\rightarrow((P\,x)\rightarrow(Q\,x)))$$
$$\wedge\forall x((\langle k\rangle\neg(AB\,x))\rightarrow\neg(AB\,x))\wedge\forall x(((x\neq a)\wedge(x\neq b))\rightarrow((P\,x)\leftrightarrow([k](P\,x))))))))$$

Step 5 twice: NPR is used to eliminate $(\langle k\rangle((x\neq b)\rightarrow\neg(AB\,x)))$ from the first necessary equivalence and $(\langle k\rangle\neg(AB\,x))$ from the second necessary equivalence. On the first necessary equivalence the hypothesis to NPR is: $\langle\rangle((P\,b)\wedge(P\,a)\wedge(\neg(Q\,b))\wedge\forall x((\neg(AB\,x))\rightarrow((P\,x)\rightarrow(Q\,x)))\wedge\forall x((x\neq b)\rightarrow\neg(AB\,x))\wedge\forall x(((x\neq a)\wedge(x\neq b))\rightarrow((P\,x)\leftrightarrow([k](P\,x)))))$ and on the second necessary equivalence the hypothesis to NPR is: $\langle\rangle((\neg(P\,b))\wedge(P\,a)\wedge(\neg(Q\,b))\wedge\forall x((\neg(AB\,x))\rightarrow((P\,x)\rightarrow(Q\,x)))\wedge\forall x(\neg(AB\,x))\wedge\forall x(((x\neq a)\wedge(x\neq b))\rightarrow((P\,x)\leftrightarrow([k](P\,x)))))$. The first possibility is deduced to be #t by three applications of ZP1 (See Example 1 herein). The second possibility is deduced to be $a\neq b$ by three applications of ZP1. Since $a\neq b$ is a hypothesis of this case it is true. Applying NPR in both cases then gives:

$$((([k](P\,b))\wedge(k\equiv((P\,b)\wedge(P\,a)\wedge(\neg(Q\,b))\wedge\forall x((\neg(AB\,x))\rightarrow((P\,x)\rightarrow(Q\,x)))$$
$$\wedge\forall x((x\neq b)\rightarrow\neg(AB\,x))\wedge\forall x(((x\neq a)\wedge(x\neq b))\rightarrow((P\,x)\leftrightarrow([k](P\,x))))))))$$
$$\vee(\neg([k](P\,b))\wedge(a\neq b)\wedge(k\equiv((\neg(P\,b))\wedge(P\,a)\wedge(\neg(Q\,b))\wedge\forall x((\neg(AB\,x))\rightarrow((P\,x)\rightarrow(Q\,x)))$$
$$\wedge\forall x(\neg(AB\,x))\wedge\forall x(((x\neq a)\wedge(x\neq b))\rightarrow((P\,x)\leftrightarrow([k](P\,x))))))))$$

Steps 3&4 twice: Since (P b) is in the first necessary equivalence, the entailment on the first case holds. Likewise since $(\neg(P\,b))$ is in the second necessary equivalence $(\neg([k](P\,b)))$ is $(\neg([k]\#f))$ which is $(\langle\rangle k)$ which holds since #f is not a solution if $a\neq b$. Thus we get:

$$(((k\equiv((P\,b)\wedge(P\,a)\wedge(\neg(Q\,b))\wedge\forall x((\neg(AB\,x))\rightarrow((P\,x)\rightarrow(Q\,x)))$$
$$\wedge\forall x((x\neq b)\rightarrow\neg(AB\,x))\wedge\forall x(((x\neq a)\wedge(x\neq b))\rightarrow((P\,x)\leftrightarrow([k](P\,x)))))))$$
$$\vee((a\neq b)\wedge(k\equiv((\neg(P\,b))\wedge(P\,a)\wedge(\neg(Q\,b))\wedge\forall x((\neg(AB\,x))\rightarrow((P\,x)\rightarrow(Q\,x)))$$
$$\wedge\forall x(\neg(AB\,x))\wedge\forall x(((x\neq a)\wedge(x\neq b))\rightarrow((P\,x)\leftrightarrow([k](P\,x))))))))$$

Since (AB b) is derivable in the first necessary equivalence, and since in either equation (P a) and (P b) hold if and only if each is entailed in k this is equivalent to:

$$((k\equiv((P\,b)\wedge(P\,a)\wedge(\neg(Q\,b))\wedge\forall x((\neg(AB\,x))\rightarrow((P\,x)\rightarrow(Q\,x)))\wedge\forall x((AB\,x)\leftrightarrow(x=b))\wedge\forall x((P\,x)\leftrightarrow([k](P\,x)))))$$
$$\vee((a\neq b)\wedge(k\equiv((\neg(P\,b))\wedge(P\,a)\wedge(\neg(Q\,b))\wedge\forall x((\neg(AB\,x))\rightarrow((P\,x)\rightarrow(Q\,x)))\wedge\forall x(\neg(AB\,x))\wedge\forall x((P\,x)\leftrightarrow([k](P\,x)))))))$$

The result is essentially a disjunction of two necessary equivalences (which partially solve for k. Further derivation along these lines gives solutions making P hold for any subset of $\{x: \#t\}-\{a\ b\}$. However, if all we want is what is common to all solutions then eliminating ([k](P x)) is not required as is shown in the next section.

5. Aggregating the Solutions

Parallel Circumscription [McCarthy 1986] in Second Order Logic with circumscribed π_i, variable, and fixed predicates ρ_j is equivalent to the "infinite disjunction" of all the solutions of a necessary equivalence:

$$\exists k(k\wedge(k\equiv(\Gamma\wedge\wedge_i\forall\xi_i((\langle k\rangle\neg(\pi_i\,x))\rightarrow\neg(\pi_i\,x))\wedge\wedge_j\forall\xi_j((\rho_j\,x)\leftrightarrow([k](\rho_j\,x))))))$$

For this reason, in some cases, such as in Example 3 above we can compare the disjunction of the solutions with results obtained by Circumscription. The key theorem for this comparison is the following theorem which allows fixed predicates to be ignored after the Circumscribed predicates are eliminated.

The Fixed Predicate Lemma [Brown 1989]: $\exists k(k\wedge(k\equiv(\Gamma\wedge\wedge_i\forall\xi((\rho_i\,x)\leftrightarrow([k](\rho_i\,x))))))\equiv\Gamma$

The necessary equivalences related to Circumscription are a small subclass of the necessary equivalences that are expressible. The main goal herein is to solve necessary equivalences, rather than to compute the "infinite disjunction" of solutions which is all that Circumscription does. However, the solutions in Example 3 are aggregated and compared in Example 4 with Circumscription for the case where $a\neq b$. This case encompasses the case where a and b are distinct 0-arity function symbols.[1]

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[1] Herein, = is necessary equality as defined in Figure 2. It should not be confused with an extensional equality predicate which only provides substitution properties through nonmodal contexts.

Example 4: Aggregating the Solutions when $a\neq b$ to $(P\ a)\wedge(\neg(Q\ b))\wedge\forall x((\neg(AB\ x))\rightarrow((P\ x)\rightarrow(Q\ x)))$ by AB with P fixed and Q variable:

$$\exists k(k\wedge(k\equiv((P\ a)\wedge(\neg(Q\ b))\wedge\forall x((\neg(AB\ x))\rightarrow((P\ x)\rightarrow(Q\ x)))$$
$$\wedge\forall x((\langle k\rangle\neg(AB\ x))\rightarrow\neg(AB\ x))\wedge\forall x((P\ x)\leftrightarrow([k](P\ x)))\ )))$$

From Example 3 and assuming $a\neq b$ we get:

$$\exists k(k\wedge((k\equiv((P\ b)\wedge(P\ a)\wedge(\neg(Q\ b))\ \wedge\forall x((\neg(AB\ x))\rightarrow((P\ x)\rightarrow(Q\ x)))\wedge\forall x((AB\ x)\leftrightarrow(x=b))\wedge\forall x((P\ x)\leftrightarrow([k](P\ x)))\ ))$$
$$\vee(k\equiv((\neg(P\ b))\wedge(P\ a)\wedge(\neg(Q\ b))\wedge\forall x((\neg(AB\ x))\rightarrow((P\ x)\rightarrow(Q\ x)))\wedge\forall x(\neg(AB\ x))\ \wedge\forall x((P\ x)\leftrightarrow([k](P\ x)))\ ))\ ))$$

Pushing $\exists k$ to lowest scope gives:

$$((\exists k(k\wedge(k\equiv((P\ b)\wedge(P\ a)\wedge(\neg(Q\ b))\wedge\forall x((\neg(AB\ x))\rightarrow((P\ x)\rightarrow(Q\ x)))\wedge\forall x((AB\ x)\leftrightarrow(x=b))\wedge\forall x((P\ x)\leftrightarrow([k](P\ x)))))))$$
$$\vee(\exists k(k\wedge(k\equiv((\neg(P\ b))\wedge(P\ a)\wedge(\neg(Q\ b))\ \wedge\forall x((\neg(AB\ x))\rightarrow((P\ x)\rightarrow(Q\ x)))\wedge\forall x(\neg(AB\ x))\wedge\forall x((P\ x)\leftrightarrow([k](P\ x)))))\ ))\ )$$

By the Fixed Predicate Lemma (twice) this is equivalent to:

$$((P\ b)\wedge(P\ a)\wedge(\neg(Q\ b))\ \wedge\forall x((\neg(AB\ x))\rightarrow((P\ x)\rightarrow(Q\ x)))\wedge\forall x((AB\ x)\leftrightarrow(x=b)))$$
$$\vee((\neg(P\ b))\wedge(P\ a)\wedge(\neg(Q\ b))\wedge\forall x((\neg(AB\ x))\rightarrow((P\ x)\rightarrow(Q\ x)))\wedge\forall x(\neg(AB\ x)))$$

which is equivalent to:

$$(P\ a)\wedge(\neg(Q\ b))\wedge\forall x((\neg(AB\ x))\rightarrow((P\ x)\rightarrow(Q\ x)))\wedge(\ ((P\ b)\wedge\forall x((AB\ x)\leftrightarrow(x=b)))\vee((\neg(P\ b))\wedge\forall x(\neg(AB\ x)))\ )$$

Since $a\neq b$, It follows that $\neg(AB\ a)$ and therefore that $(Q\ a)$ holds. This is exactly what one would expect as is suggested by the following quote from [Konolige 1989]: "Consider the simple abnormality theory (see [McCarthy 1986]), with $W=\{\forall x.Px\wedge\neg AB(x)\rightarrow Qx, Pa, \neg Qb\}$ (this is a variation of an example in [Perlis, 1986].) We would expect Qa to be a consequence of Circum(W;ab;Q), but it is not."[1] "The reason is that there are ab-minimal models of W in which b and a refer to the same individual and $\neg Qa$ is true."[2]

6. Conclusion

The Z Modal Quantificational Logic provides an interesting algebra for deriving fixed-point solutions to necessary equivalences where universally quantified variables cross modal scope. Herein, some specific methods for solving some simple classes of problems have been described and exemplified. The presented methods do not solve all problems, but the Z logic provides a framework for developing more general solution methods generalizing the ones herein presented. Since many quantified nonmonotonic logics are representable in the Z Modal Logic, including Quantified Autoepistemic Logic, Quantified Autoepistemic Kernels, Quantified Reflective Logic, and Quantified Default Logic, such deduction techniques could be applicable to a wide range of quantified generalizations of most of the well known nonmonotonic logics.

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[1] If $a\neq b$, such as is the case where a and b are two distinct 0-arity function symbols, it is a consequence of Circumscription as defined in Second Order Logic. In Second Order Logic $x=y$ is defined to be $\forall f((f\ x)\leftrightarrow(f\ y))$.

[2] The obvious definition of minimal model does not give the desired properties. The definition of minimal model could be changed to being minimal not with respect to all models but with respect to those models giving function symbols the same interpretation.

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Author Information

Frank M. Brown – University of Kansas, Lawrence, Kansas, 66045, e-mail: brown@ku.edu.