REPRESENTING "RECURSIVE" DEFAULT LOGIC IN MODAL LOGIC

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Abstract: The "recursive" definition of Default Logic is shown to be representable in a monotonic Modal Quantificational Logic whose modal laws are stronger than S5. Specifically, it is proven that a set of sentences of First Order Logic is a fixed-point of the "recursive" fixed-point equation of Default Logic with an initial set of axioms and defaults if and only if the meaning of the fixed-point is logically equivalent to a particular modal functor of the meanings of that initial set of sentences and of the sentences in those defaults. This is important because the modal representation allows the use of powerful automatic deduction systems for Modal Logic and because unlike the original "recursive" definition of Default Logic, it is easily generalized to the case where quantified variables may be shared across the scope of the components of the defaults.

Keywords: Recursive Definition of Default Logic, Modal Logic, Nonmonotonic Logic.

1. Introduction

One of the most well known nonmonotonic logics [Antoniou 1997] which deals with entailment conditions in addition to possibility conditions in it's defaults is the so-called Default Logic [Reiter 1980]. The basic idea of Default Logic is that there is a set of axioms Γ and some non-logical default "inference rules" of the form:

$$\frac{\alpha : \beta_1 ... \beta_m}{\chi}$$

which is intended to suggest that χ may be inferred from α whenever each $\beta_1,...,\beta_m$ is consistent with everything that is inferable. Such "inference rules" are not recursive and are circular in that the determination as to whether χ is derivable depends on whether β_i is consistent which in turn depends on what was derivable from this and other defaults. Thus, tentatively applying such inference rules by checking the consistency of $\beta_1,...,\beta_m$ with only the current set of inferences produces a χ result which may later have to be retracted. For this reason inferences in a nonmonotonic logic such as Default Logic are essentially carried out not in the original nonmonotonic logic, but rather in some (monotonic) metatheory in which that nonmonotonic logic is monotonically defined. [Reiter 1980] explicated the above intuition by defining Default Logic "recursively" in terms of the set theoretic proof theory metalanguage of First Order Logic (i.e. FOL) with (more or less) the following fixed-point expression[1]:

$$'\kappa=(\text{dr}\ '\kappa\ '\Gamma\ '\alpha_i{:}'\beta_{ij}/'\chi_i)$$

where dr is defined as:

$(\text{dr}\ '\kappa\ '\Gamma\ '\alpha_i{:}'\beta_{ij}/'\chi_i) =_{df} \cup_{t=1,\omega}(\text{r}\ t\ '\kappa\ '\Gamma\ '\alpha_i{:}'\beta_{ij}/'\chi_i)$

$(\text{r}\ 0\ '\kappa\ '\Gamma\ '\alpha_i{:}'\beta_{ij}/'\chi_i) =_{df} (\text{fol}\ '\Gamma)$

$(\text{r}\ t{+}1\ 'k\ '\alpha_i{:}'\beta_{ij}/'\chi_i) =_{df} (\text{fol}((\text{r}\ t\ '\kappa\ '\Gamma\ '\alpha_i{:}'\beta_{ij}/'\chi_i)\cup\{'\chi_i{:}\ ('\alpha_i \varepsilon (\text{r}\ t\ '\kappa\ '\Gamma\ '\alpha_i{:}'\beta_{ij}/'\chi_i))\wedge\wedge_{j=1,mi}('(\neg\beta_{ij})\notin '\kappa)\}))$

where $'\alpha_i$, $'\beta_{ij}$, and $'\chi_i$ are the closed sentences of FOL occurring in the ith "inference rule" and $'\Gamma$ is a set of closed sentences of FOL. A closed sentence is a sentence without any free variables. fol is a function which produces the set of theorems derivable in FOL from the set of sentences to which it is applied. The quotations

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[1] [Reiter 1980] actually used a recursive definition whereby the r sets do not necessarily contain all their FOL consequences:

$(\text{dr}\ '\kappa\ '\Gamma\ '\alpha_i{:}'\beta_{ij}/'\chi_i) =_{df} \cup_{t=1,\omega}(\text{r}\ t\ '\kappa\ '\Gamma\ '\alpha_i{:}'\beta_{ij}/'\chi_i)$

$(\text{r}\ 0\ '\kappa\ '\Gamma\ '\alpha_i{:}'\beta_{ij}/'\chi_i) =_{df} '\Gamma$

$(\text{r}\ t{+}1\ '\kappa\ '\Gamma\ '\alpha_i{:}'\beta_{ij}/'\chi_i) =_{df} (\text{fol}(\text{r}\ t\ '\kappa\ '\Gamma\ '\alpha_i{:}'\beta_{ij}/'\chi_i))\cup\{'\chi_i{:}('\alpha_i \varepsilon (\text{r}\ t\ '\kappa\ '\Gamma\ '\alpha_i{:}'\beta_{ij}/'\chi_i))\wedge\wedge_{j=1,mi}('(\neg\beta_{ij})\notin '\kappa)\}$

If this definition were used then all the theorems in this paper should have $(\text{r}\ t\ '\kappa\ '\Gamma\ '\alpha_i{:}'\beta_{ij}/'\chi_i)$ replaced by $(\text{fol}(\text{r}\ t\ '\kappa\ '\Gamma\ '\alpha_i{:}'\beta_{ij}/'\chi_i))$ and $(\text{dr}\ '\kappa\ '\Gamma\ '\alpha_i{:}'\beta_{ij}/'\chi_i)$ replaced by $(\text{fol}(\text{dr}\ '\kappa\ '\Gamma\ '\alpha_i{:}'\beta_{ij}/'\chi_i))$.

appended to the front of these Greek letters indicate references in the metalanguage to the sentences of the FOL object language. Interpreted doxastically this fixed-point equation states:

The set of closed sentences which are believed is equal to
the union of all sets of closed sentences which are believed at any time.
That which is believed at time 0 is the set of closed sentences derived by the laws of FOL from 'Γ.
That which is believed at time t+1 is the set of closed sentences derived by the laws of FOL
from the union of both
the set of beliefs at time t
and the set of all 'χ_i for each i such that
the closed sentence 'α_i is believed at time t and for each j, the closed sentence 'β_{ij} is believable.

The purpose of this paper is to show that all this metatheoretic machinery including the formalized syntax of FOL, the proof theory of FOL, the axioms of set theory, and the set theoretic fixed-point equation is not needed and that the essence of the "recursive" definition of Default Logic in representable as a necessary equivalence in a simple (monotonic) Modal Quantificational Logic. Interpreted as a doxastic logic this necessary equivalence states:

That which is believed is logically equivalent to what is believed at any time.
That which is believed at time 0 is Γ.
That which is believed at time t+1 is
that which is believed at time t and for each i, if α_i is believed at time t and for each j, β_{ij} is believable then χ_i.

thereby eliminating all mention of any metatheoretic machinery.

The remainder of this paper proves that this modal representation is equivalent to the "recursive" definition of Default Logic. Section 2 describes a formalized syntax for a FOL object language. Section 3 describes the part of the proof theory of FOL needed herein (i.e. theorems FOL1-FOL10). Section 4 describes the Intensional Semantics of FOL which includes laws for meaning of FOL sentences: M0-M7, theorems giving the meaning of sets of FOL sentences: MS1, MS2, MS3, and laws specifying the relationship of meaning and modality to the proof theory of FOL (i.e. the laws R0, A1, A2 and A3 and the theorems C1, C2, C3, and C4). The modal version of the "Recursive" definition of Default Logic, called DR, is defined in section 5 and explicated with theorems MD1-MD8 and SS1-SS2. In section 6, this modal version is shown by theorems R1, DR1 and DR2 to be equivalent to the set theoretic fixed-point equation for Default Logic. Figure 1 outlines the relationship of all these theorems in producing the final theorems DR2, FOL10, and MD8.

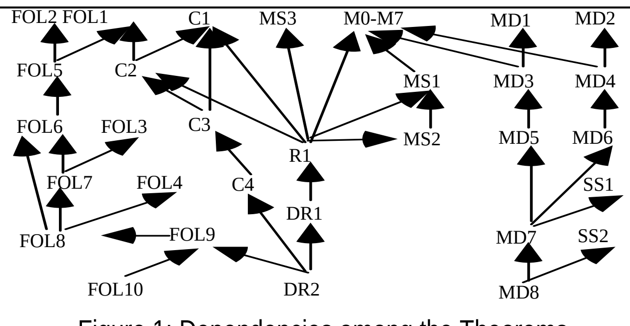

Figure 1: Dependencies among the Theorems

2. Formal Syntax of First Order Logic

We use a First Order Logic (i.e. FOL) defined as the six tuple: ($\rightarrow$, #f, $\forall$, *vars*, *predicates*, *functions*) where $\rightarrow$, #f, and $\forall$ are logical symbols, *vars* is a set of variable symbols, *predicates* is a set of predicate symbols each of which has an implicit arity specifying the number of associated terms, and *functions* is a set of function symbols each of which has an implicit arity specifying the number of associated terms. The sets of logical symbols, variables, predicate symbols, and function symbols are pairwise disjoint. Lower case Roman letters possibly indexed with digits are used as variables. Greek letters possibly indexed with digits are used as syntactic metavariables. γ, $\gamma_1,...\gamma_n$, range over the variables, ξ, $\xi_1...\xi_n$ range over sequences of variables of an appropriate arity, $\pi,\pi_1...\pi_n$ range over the predicate symbols, ϕ, $\phi_1...\phi_n$ range over function symbols, δ, $\delta_1...\delta_n$, σ range over terms, and α, $\alpha_1...\alpha_n$, β, $\beta_1...\beta_n$, χ, $\chi_1...\chi_n$, $\Gamma_1,...\Gamma_n$, φ range over sentences. The terms are of the forms γ and (ϕ δ1...δn), and the sentences are of the forms ($\alpha\rightarrow\beta$), #f, ($\forall\gamma$ α), and (π $\delta_1...\delta_n$). A nullary predicate π or function ϕ is written as a sentence or a term without parentheses. $\varphi\{\pi/\lambda\xi\alpha\}$ represents the replacement of all occurrences of π in φ by $\lambda\xi\alpha$ followed by lambda conversion. The primitive symbols are shown in Figure 2 with their intuitive interpretations.

Symbol	Meaning
$\alpha \rightarrow \beta$	if α then β.
#f	falsity
$\forall\gamma\ \alpha$	for all γ, α.

Figure 2: Primitive Symbols of First Order Logic

The defined symbols are listed in Figure 3 with their definitions and intuitive interpretations.

Symbol	Definition	Meaning		Symbol	Definition	Meaning
$\neg\alpha$	$\alpha \rightarrow$ #f	not α		$\alpha\wedge\beta$	$\neg(\alpha \rightarrow \neg\beta)$	α and β
#t	$\neg$ #f	truth		$\alpha\leftrightarrow\beta$	$(\alpha\rightarrow\beta) \wedge (\beta\rightarrow\alpha)$	α if and only if β
$\alpha\vee\beta$	$(\neg\alpha)\rightarrow\beta$	α or β		$\exists\gamma\ \alpha$	$\neg\forall\gamma\ \neg\alpha$	for some γ , α

Figure 3: Defined Symbols of First Order Logic

The FOL object language expressions are referred in the metalanguage (which also includes a FOL syntax) by inserting a quote sign in front of the object language entity thereby making a structural descriptive name of that entity. Generally, a set of sentences is represented as: {'Γ_i} which is defined as: {'Γ_i: #t} which in turn is defined as: {s: $\exists$i(s='Γ_i)} where i ranges over some range of numbers (which may be finite or infinite). With a slight abuse of notation we also write 'κ, 'Γ to refer to such sets.

3. Proof Theory of First Order Logic

FOL is axiomatized with a recursively enumerable set of theorems as its axioms are recursively enumerable and its inference rules are recursive. The axioms and inference rules of FOL [Mendelson 1964] are given in Figure 4.

MA1: $\alpha \rightarrow (\beta\rightarrow \alpha)$

MA2: $(\alpha\rightarrow (\beta\rightarrow \rho)) \rightarrow ((\alpha\rightarrow \beta)\rightarrow (\alpha\rightarrow \rho))$

MA3: $((\neg \alpha)\rightarrow (\neg \beta))\rightarrow (((\neg \alpha)\rightarrow \beta)\rightarrow\alpha)$

MA4: $(\forall\gamma\ \alpha)\rightarrow \beta$ where β is the result of substituting an expression (which is free for the free positions of γ in α) for all the free occurrences of γ in α.

MA5: $((\forall\gamma(\alpha \rightarrow \beta)) \rightarrow (\alpha\rightarrow(\forall\gamma\ \beta)))$ where γ does not occur in α.

MR1: from α and $(\alpha\rightarrow \beta)$ infer β

MR2: from α infer $(\forall\gamma\ \alpha)$

Figure 4: Inferences Rules and Axioms of FOL

In order to talk about sets of sentences we include in the metatheory set theory symbolism as developed along the lines of [Quine 1969]. This set theory includes the symbols ε, $\not\in$, $\supseteq$, $=$, $\cup$ as is defined therein. The derivation operation (i.e. fol) of any First Order Logic obeys the Inclusion (i.e. FOL1), Idempotence (i.e. FOL2), Monotonic (i.e. FOL3) and Union (i.e. FOL4) properties:

FOL1: (fol 'Γ)$\supseteq$'Γ — Inclusion

FOL2: (fol 'κ)$\supseteq$(fol(fol 'κ)) — Idempotence

FOL3: ('$\kappa\supseteq$'Γ) $\rightarrow$ ((fol 'κ)$\supseteq$ (fol 'Γ)) — Monotonicity

FOL4: For any set ψ, if $\forall$t((ψ t)=(fol(ψ t))) and $\forall$t((ψ t+1)$\supseteq$(ψ t)) then ($\cup_{t=0,\omega}$(ψ t))=(fol($\cup_{t=0,\omega}$(ψ t))) — Union

From these four properties we prove the following theorems of the proof theory of First Order Logic:

FOL5: (fol 'κ)=(fol(fol 'κ)) proof: By FOL1 and FOL2.

FOL6: (r t 'κ 'Γ 'α_i:'β_{ij}/'χ_i)=(fol(r t 'κ 'Γ 'α_i:'β_{ij}/'χ_i)) proof: By induction on t it suffices to prove:

(1) (r 0 'κ 'Γ 'α_i:'β_{ij}/'χ_i)=(fol(r 0 'κ 'Γ 'α_i:'β_{ij}/'χ_i)) Unfolding r twice gives: (fol 'Γ)=(fol(fol 'Γ)) which is FOL5.

(2) (r t+1 'κ 'Γ 'α_i:'β_{ij}/'χ_i)=(fol(r t+1 'κ 'Γ 'α_i:'β_{ij}/'χ_i))

Unfolding r twice gives: (fol((r t 'κ 'Γ 'α_i:'β_{ij}/'χ_i)∪{'χ_i: ('α_iε(r t 'κ 'Γ 'α_i:'β_{ij}/'χ_i))∧∧$_{j=1,mi}$('(¬β_{ij})∉'κ)}))

= fol(fol((r t 'κ 'Γ 'α_i:'β_{ij}/'χ_i) ∪{'χ_i: ('α_iε(r t 'κ 'Γ 'α_i:'β_{ij}/'χ_i))∧∧$_{j=1,mi}$('(¬β_{ij})∉'κ)}))

which likewise is an instance of FOL5. QED.

FOL7: (r t+1 'κ 'Γ 'α_i:'β_{ij}/'χ_i)⊒(r t 'κ 'Γ 'α_i:'β_{ij}/'χ_i)

proof: By FOL6 this is equivalent to: (r t+1 'κ 'Γ 'α_i:'β_{ij}/'χ_i)⊒(fol(r t 'κ 'Γ 'α_i:'β_{ij}/'χ_i)). Unfolding r of t+1 gives:

(fol((r t 'κ 'Γ 'α_i:'β_{ij}/'χ_i) ∪{'χ_i: ('α_iε(r t 'κ 'Γ 'α_i:'β_{ij}/'χ_i))∧∧$_{j=1,mi}$('(¬β_{ij})∉'κ)})) ⊒(fol(r t 'κ 'Γ 'α_i:'β_{ij}/'χ_i))

By FOL3 it suffices to prove: ((r t 'κ 'Γ 'α_i:'β_{ij}/'χ_i)∪{'χ_i: ('α_iε(r t 'κ 'Γ 'α_i:'β_{ij}/'χ_i))∧∧$_{j=1,mi}$('(¬β_{ij})∉'κ)})⊒(r t 'κ 'Γ 'α_i:'β_{ij}/'χ_i) which holds in set theory. QED.

FOL8: (∪$_{t=0,\omega}$(r t 'κ 'Γ 'α_i:'β_{ij}/'χ_i)) =(fol(∪$_{t=0,\omega}$(r t 'κ 'Γ 'α_i:'β_{ij}/'χ_i)))

proof: ∀t((r t 'κ 'Γ 'α_i:'β_{ij}/'χ_i)=(fol(r t 'κ 'Γ 'α_i:'β_{ij}/'χ_i))) holds by FOL6. ∀t((r t+1 'κ 'Γ 'α_i:'β_{ij}/'χ_i)⊒(r t 'κ 'Γ 'α_i:'β_{ij}/'χ_i)) holds by FOL7. Instantiating the hypotheses in FOL4 to these theorems proves this theorem. QED.

FOL9: (dr 'κ 'Γ 'α_i:'β_{ij}/'χ_i)=(fol(dr 'κ 'Γ 'α_i:'β_{ij}/'χ_i)) proof: Unfolding dr twice gives: ∪$_{t=1,\omega}$(r t 'κ 'Γ 'α_i:'β_{ij}/'χ_i)= fol(∪$_{t=1,\omega}$(r t 'κ 'Γ 'α_i:'β_{ij}/'χ_i)) which holds by FOL8. QED.

FOL10: ('k=(dr 'κ 'Γ 'α_i:'β_{ij}/'χ_i))→('κ=(fol 'κ)) proof: From the hypothesis and FOL9 'k=(fol(dr 'κ)) is derived. Using the hypothesis to replace (dr 'κ) by 'κ in this result gives: ('κ=(fol 'κ)) QED.

4. Intensional Semantics of FOL

The meaning (i.e. mg) [Brown 1978, Boyer & Moore 1981] or rather disquotation of a sentence of FOL is defined in Figure 5 below[1]. mg is defined in terms of mgs which maps each FOL object language sentence and an association list into a meaning. mgn maps each FOL object language term and an association list into a meaning. An association list is a list of pairs consisting of an object language variable and the meaning to which it is bound.

M0: (mg 'α) =df (mgs '(∀γ1...γn α)'()) where 'γ1...'γn are all the free variables in 'α

M1: (mgs '(α → β)a) ↔ ((mgs 'α a)→(mgs 'β a))

M2: (mgs '#f a) ↔ #f

M3: (mgs '(∀ γ α)a) ↔ ∀x(mgs 'α(cons(cons 'γ x)a))

M4: (mgs '(π δ_1...δ_n)a) ↔ (π(mgn 'δ_1 a)...(mgn 'δ_n a)) for each predicate symbol 'π.

M5: (mgn '(φ δ_1...δ_n)a) = (φ(mgn 'δ_1 a)...(mgn 'δ_n a)) for each function symbol 'φ.

M6: (mgn 'γ a) = (cdr(assoc 'γ a))

M7: (assoc v L)= (if(eq? v(car(car L)))(car L)(assoc v(cdr L))) where: cons, car, cdr, eq?, and if are as in Scheme.

Figure 5: The Meaning of FOL Sentences

The meaning of a set of sentences is defined in terms of the meanings of the sentences in the set as:

(ms 'κ) =df ∀s((sε'κ)→(mg s)).

[1] The laws M0-M7 are analogous to Tarski's definition of truth except that finite association lists are used to bind variables to values rather than infinite sequences. M4 is different since mg is interpreted as being meaning rather than truth.

MS1: $(ms\{'\alpha{:}\ \Gamma\}) \leftrightarrow \forall\xi((\Gamma\{s/'\alpha\})\rightarrow\alpha)$ where ξ is the sequence of all the free variables in $'\alpha$ and where Γ is any sentence of the intensional semantics. proof: $(ms\{'\alpha{:}\Gamma\})$ Unfolding ms and the set pattern abstraction symbol gives: $\forall s((s\varepsilon\{s{:}\ \exists\xi((s='\alpha)\wedge\Gamma)\})\rightarrow(mg\ s))$ where ξ is a sequence of the free variables in 'a. This is equivalent to: $\forall s((\exists\xi((s='\alpha)\wedge\Gamma))\rightarrow(mg\ s))$ which is: $\forall s\forall\xi\ (((s='\alpha)\wedge\Gamma)\rightarrow(mg\ s))$ which is: $\forall\xi(\Gamma\{s/'\alpha\}\rightarrow(mg\ '\alpha))$. Unfolding mg using M0-M7 then gives: $\forall\xi((\Gamma\{s/'\alpha\})\rightarrow\alpha)$ QED

The meaning of a set is the meaning of all the sentences in the set (i.e. MS2):

MS2: $(ms\{'\Gamma_i\}) \leftrightarrow \forall i\forall\xi_i\Gamma_i$ proof: $(ms\{'\Gamma_i\})$ Unfolding the set notation gives: $(ms\{'\Gamma_i{:}\ \#t\})$. By MS1 this is equivalent to: $\forall_i\forall\xi_i((\#t\{s/'\alpha\})\rightarrow\Gamma_i)$ which is equivalent to: $\forall i\forall\xi_i\Gamma_i$ QED.

The meaning of the union of two sets of FOL sentences is the conjunction of their meanings (i.e. MS3):

MS3: $(ms('\kappa\cup'\Gamma)) \leftrightarrow ((ms\ '\kappa)\wedge(ms\ '\Gamma))$ proof: Unfolding ms and union in: $(ms('\kappa\cup'\Gamma))$ gives:

$\forall s((s\varepsilon\{s{:}\ (s\varepsilon'\kappa)\vee(s\varepsilon'\Gamma)\})\rightarrow(mg\ s))$ or rather: $\forall s(((s\varepsilon'\kappa)\vee(s\varepsilon'\Gamma))\rightarrow(mg\ s))$ which is logically equivalent to: $(\forall s((s\varepsilon'\kappa)\rightarrow(mg\ s)))\wedge(\forall s((s\varepsilon'\Gamma)\rightarrow(mg\ s)))$. Folding ms twice then gives: $((ms\ '\kappa)\wedge(ms\ '\Gamma))$ QED.

The meaning operation may be used to develop an Intensional Semantics for a FOL object language by axiomatizing the modal concept of necessity so that it satisfies the theorem:

C1: $('\alpha\varepsilon(fol\ '\kappa)) \leftrightarrow ([]\ ((ms\ '\kappa)\rightarrow(mg\ '\alpha)))$

for every sentence $'\alpha$ and every set of sentences $'\kappa$ of that FOL object language. The necessity symbol is represented by a box: []. C1 states that a sentence of FOL is a FOL-theorem (i.e. fol) of a set of sentences of FOL if and only if the meaning of that set of sentences necessarily implies the meaning of that sentence. One modal logic which satisfies C1 for FOL is the Z Modal Quantificational Logic described in [Brown 1987; Brown 1989] whose theorems are recursively enumerable. Z has the metatheorem: $(<>\Gamma)\{\pi/\lambda\xi\alpha\}\rightarrow(<>\Gamma)$ where Γ is a sentence of FOL and includes all the laws of S5 Modal Logic [Hughes & Cresswell 1968] whose modal axioms and inference rules are in Figure 6. Therein, κ and Γ are arbitrary sentences of the intentional semantics.

R0: from κ infer $([]\ \kappa)$	A2: $([](\kappa\rightarrow\Gamma))\rightarrow(([]\kappa)\rightarrow([]\Gamma))$
A1: $([]\kappa)\rightarrow\kappa$	A3: $([]\kappa)\vee([]\neg[]\kappa)$

Figure 6: The Laws of S5 Modal Logic

These S5 modal laws and the laws of FOL given in Figure 6 constitute an S5 Modal Quantificational Logic similar to [Carnap 1946; Carnap 1956], and a FOL version [Parks 1976] of [Bressan 1972] in which the Barcan formula: $(\forall\gamma([]\kappa))\rightarrow([]\forall\gamma\kappa)$ and its converse hold. The R0 inference rule implies that anything derivable in the metatheory is necessary. Thus, in any logic with R0, contingent facts would never be asserted as additional axioms of the metatheory. The defined Modal symbols are in Figure 7 with their definitions and interpretations.

Symbol	Definition	Meaning		Symbol	Definition	Meaning
$<>\kappa$	$\neg\ []\ \neg\kappa$	α is logically possible		$[\kappa]\ \Gamma$	$[]\ (\kappa\rightarrow\Gamma)$	β entails α
$\kappa\equiv\Gamma$	$[]\ (\kappa\leftrightarrow\Gamma)$	α is logically equivalent to β		$<\kappa>\ \Gamma$	$<>\ (\kappa\wedge\Gamma)$	α and β is logically possible

Figure 7: Defined Symbols of Modal Logic

From the laws of the Intensional Semantics we prove that the meaning of the set of FOL consequences of a set of sentences is the meaning of that set of sentences (C2), the FOL consequences of a set of sentences contain the FOL consequences of another set if and only if the meaning of the first set entails the meaning of the second set (C3), and the sets of FOL consequences of two sets of sentences are equal if and only if the meanings of the two sets are logically equivalent (C4):

C2: $(ms(fol\ '\kappa))\equiv(ms\ '\kappa)$ proof: The proof divides into two cases:

(1) $[(ms\ '\kappa)](ms(fol\ '\kappa))$ Unfolding the second ms gives: $[(ms\ '\kappa)]\forall s((s\varepsilon(fol\ '\kappa))\rightarrow(mg\ s))$

By the soundness part of C1 this is equivalent to: $[(ms\ '\kappa)]\forall s(([(ms\ '\kappa)](mg\ s))\rightarrow(mg\ s))$

By the S5 laws this is equivalent to: $\forall s(([(ms\ '\kappa)](mg\ s))\rightarrow[(ms\ '\kappa)](mg\ s))$ which is a tautology.

(2) $[(ms(fol\ '\kappa))](ms\ '\kappa)$ Unfolding ms twice gives: $[\forall s((s\varepsilon(fol\ '\kappa))\rightarrow(mg\ s))]\forall s((s\varepsilon'\kappa)\rightarrow(mg\ s))$

which is: $[\forall s((s\varepsilon(fol\ '\kappa))\rightarrow(mg\ s))]((s\varepsilon'\kappa)\rightarrow(mg\ s))$ Backchaining on the hypothesis and then dropping it gives: $(s\varepsilon'\kappa)\rightarrow(s\varepsilon(fol\ '\kappa))$. Folding $\supseteq$ gives an instance of FOL1. QED.

C3: $(fol\ '\kappa)\supseteq(fol\ '\Gamma) \leftrightarrow ([(ms\ '\kappa)](ms\ '\Gamma))$ proof: Unfolding $\supseteq$ gives: $\forall s((s\varepsilon(fol\ '\Gamma))\rightarrow(s\varepsilon(fol\ '\kappa)))$

By C1 twice this is equivalent to: $\forall s(([(ms\ '\Gamma)](mg\ s))\rightarrow([(ms\ '\kappa)](mg\ s)))$

By the laws of S5 modal logic this is equivalent to: $([(ms\ '\kappa)]\forall s(([(ms\ '\Gamma)](mg\ s))\rightarrow(mg\ s)))$

By C1 this is equivalent to: $[(ms\ '\kappa)]\forall s((s\varepsilon(fol\ '\Gamma))\rightarrow(mg\ s))$. Folding ms then gives: $[(ms\ '\kappa)](ms(fol\ '\Gamma))$

By C2 this is equivalent to: $[(ms\ '\kappa)](ms\ '\Gamma)$. QED.

C4: $((fol\ '\kappa)=(fol\ '\Gamma)) \leftrightarrow ((ms\ '\kappa)\equiv(ms\ '\Gamma))$ proof: This is equivalent to $(((fol\ '\kappa)\supseteq(fol\ '\Gamma))\wedge((fol\ '\Gamma)\supseteq(fol\ '\kappa))) \leftrightarrow ([(ms\ '\kappa)](ms\ '\Gamma))\wedge([(ms\ '\Gamma)](ms\ '\kappa))$ which follows by using C3 twice.

5. "Recursive" Default Logic Represented in Modal Logic

The fixed-point equation for Default Logic may be expressed as a necessary equivalence in an S5 Modal Quantificational Logic using a recursive definition, as follows:

$$\kappa\equiv(DR\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)$$

where DR is defined as:

$(DR\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i) =df\ \forall t(R\ t\ k\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)$

$(R\ 0\ k\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i) =df\ \Gamma$

$(R\ t{+}1\ k\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i) =df\ (R\ t\ k\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i) \wedge\forall_i((([(R\ t\ k\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)]\alpha_i)\wedge\wedge_{j=1,mi}(<\kappa>\beta_{ij}))\rightarrow\chi_i)$

When the context is obvious $\Gamma\ \alpha_i{:}\beta_{ij}/\chi_i$ is omitted and just $(DR\ \kappa)$ and $(R\ t\ \kappa)$ are written. Given below are some properties of DR. The first two theorems state that DR entails Γ and any conclusion χ_i of a default whose entailment condition holds in DL and whose possible conditions are possible with κ.

MD1: $[(DR\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)]\Gamma$ proof: Unfolding DR gives: $[\forall t(R\ t\ k\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)]\Gamma$

Letting t be 0 shows that it suffices to prove: $[(R\ 0\ k\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)]\Gamma$. Unfolding R gives a tautology. QED.

MD2: $([(DR\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)]\ ((([(R\ t\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)]\alpha_i)\wedge\wedge_{j=1,mi}(<\kappa>\beta_{ij}))\rightarrow \chi_i)$

proof: By R0 it suffices to prove: $(DR\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)\rightarrow ((([(R\ t\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)]\alpha_i)\wedge(\wedge_{j=1,m_i}(<\kappa>\beta_{ij})))\rightarrow\chi_i)$

Unfolding DR gives: $\forall t(R\ t\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\ \chi_i)\rightarrow((([(R\ t\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)]\alpha_i)\wedge(\wedge_{j=1,m_i}(<\kappa>\beta_{ij})))\rightarrow\chi_i)$

Letting the quantified t be t+1, it suffices to prove:

$(R\ t{+}1\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\ \chi_i)\rightarrow((([(R\ t\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)]\alpha_i)\wedge(\wedge_{j=1,m_i}(<\kappa>\beta_{ij})))\rightarrow\chi_i)$. Unfolding R t+1 gives:

$((R\ t\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)\wedge(\forall_i((([(R\ t\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\ \chi_i)]\alpha_i)\wedge(\wedge_{j=1,m_i}(<\kappa>\beta_{ij})))\rightarrow\chi_i))$

$\wedge([(R\ t\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)]\alpha_i)\wedge(\wedge_{j=1,m_i}(<\kappa>\beta_{ij})))\rightarrow\chi_i$

Letting the quantified i be i gives a tautology. QED.

The concept (i.e. ss) of the combined meaning of all the sentences of the FOL object language whose meanings are entailed by a proposition is defined as follows:

$$(ss\ \kappa) =df\ \forall s(([\kappa](mg\ s))\rightarrow(mg\ s))$$

SS1 shows that a proposition entails the combined meaning of the FOL object language sentences that it entails. SS2 shows that if a proposition is necessarily equivalent to the combined meaning of the FOL object language sentences that it entails, then there exists a set of FOL object language sentences whose meaning is necessarily equivalent to that proposition:

SS1: $[\kappa](ss\ \kappa)$ proof: By R0 it suffices to prove: $\kappa\rightarrow(ss\ \kappa)$. Unfolding ss gives: $\kappa\rightarrow\forall s(([\kappa](mg\ s))\rightarrow(mg\ s))$ which is equivalent to: $\forall s(([\kappa](mg\ s))\rightarrow(\kappa\rightarrow(mg\ s)))$ which is an instance of A1. QED.

SS2: $(\kappa\equiv(ss\ \kappa))\rightarrow\exists s(\kappa\equiv(ms\ s))$ proof: Letting s be {s: $([\kappa](mg\ s))$} gives: $(\kappa\equiv(ss\ \kappa))\rightarrow(\kappa\equiv(ms\{s: ([\kappa](mg\ s))\}))$. Unfolding ms and lambda conversion gives: $(\kappa\equiv(ss\ \kappa))\leftrightarrow(\kappa\equiv\forall s(([\kappa](mg\ s))\rightarrow(mg\ s)))$. Folding ss gives a tautology. QED.

The theorems MD3 and MD4 are analogous to MD1 and MD2 except that DR is replaced by the combined meanings of the sentences entailed by DR.

MD3: $[ss(DR\ \kappa(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i)]\forall i\Gamma_i$ proof: By R0 it suffices to prove: $(ss(DR\ \kappa(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i))\rightarrow\forall i\Gamma_i$ which is equivalent to: $(ss(DR\ \kappa(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i))\rightarrow\Gamma_i$. Unfolding ss gives: $(\forall s(([(DR\ \kappa(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i)](mg\ s))\rightarrow(mg\ s)))\rightarrow\Gamma_i$ which by the laws M0-M7 is equivalent to: $(\forall s(([(DR\ \kappa(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i)](mg\ s))\rightarrow(mg\ s)))\rightarrow(mg\ '\Gamma_i)$. Backchaining on $(mg\ '\Gamma_i)$ with s in the hypothesis being $'\Gamma_i$ in the conclusion shows that it suffices to prove: $([(DR\ \kappa(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i)](mg\ '\Gamma_i)$ which by the meaning laws: M0-M7 is equivalent to: $[(DR\ \kappa(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i)]\Gamma_i)$ which by S5 Modal Logic is equivalent to: $([(DR\ \kappa(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i)]\forall i\Gamma_i)$ which is an instance of theorem MD1. QED.

MD4: $[ss(DR\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)]\ ((([(R\ t\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow\chi_i)$

proof: By R0 it suffices to prove: $(ss(DR\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i))\rightarrow((([(R\ t\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)]\alpha_i)\wedge(\wedge_{j=1,m_i}(<\kappa>\beta_{ij})))\rightarrow\chi_i)$

Unfolding ss: $(\forall s(([(DR\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)](mg\ s))\rightarrow(mg\ s)))\rightarrow((([(R\ t\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\ \chi_i)]\alpha_i)\wedge(\wedge_{j=1,m_i}(<\kappa>\beta_{ij})))\rightarrow\chi_i)$

Instantiating s in the hypothesis to $'\chi_i$ and then dropping the hypothesis gives:

$(([(DR\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)](mg\ '\chi_i))\rightarrow(mg\ '\chi_i))\rightarrow((([(R\ t\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\ \chi_i)]\alpha_i)\wedge(\wedge_{j=1,m_i}(<\kappa>\beta_{ij})))\rightarrow\chi_i)$

Using the meaning laws M0-M7 gives:

$(([(DR\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)]\chi_i)\rightarrow\chi_i)\rightarrow((([(R\ t\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\ \chi_i)]\alpha_i)\wedge(\wedge_{j=1,m_i}(<\kappa>\beta_{ij})))\rightarrow\chi_i)$

Backchaining on χ_i shows that it suffices to prove:

$(([(R\ t\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)]\alpha_i)\wedge(\wedge_{j=1,m_i}(<\kappa>\beta_{ij})))\rightarrow([(DR\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/(mg\ '\chi_i))]\chi_i)$

By the laws of S5 modal logic this is equivalent to:

$[(DR\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)]\ ((([(R\ t\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)]\alpha_i)\wedge(\wedge_{j=1,m_i}(<\kappa>\beta_{ij})))\rightarrow\chi_i)$ which is MD2. QED.

Theorems MD5 and MD6 show that R is entailed by the meanings of the sentences entailed by DR:

MD5: $[ss(DR\ \kappa(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i)](R\ 0\ k(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i))$

proof: Unfolding R 0 gives: $(ss(DR\ \kappa(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i))\rightarrow(\forall i\Gamma_i)$ which holds by MD3. QED.

MD6: $([ss(DR\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)](R\ t\ k\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i))\rightarrow([ss(DR\ \kappa\ \alpha_i{:}\beta_{ij}/\chi_i)](R\ t{+}1\ k\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i))$

proof: Unfolding R t+1 in the conclusion gives:

$([ss(DR\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)]((R\ t\ k\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)\wedge\forall_i((([(R\ t\ k\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)]\alpha_i)\wedge\wedge_{j=1,mi}(<\kappa>\beta_{ij}))\rightarrow\chi_i)))$

Using the hypothesis gives: $[ss(DR\ \kappa\ \alpha_i{:}\beta_{ij}/\chi_i)]\ \forall_i((([(R\ t\ k\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)]\alpha_i)\wedge\wedge_{j=1,mi}(<\kappa>\beta_{ij}))\rightarrow\chi_i)$

which holds by MD4. QED.

Finally MD7 and MD8 show that talking about the meanings of sets of FOL sentences in the modal representation of Default Logic is equivalent to talking about propositions in general.

MD7: $(ss(DR\ \kappa(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i))\equiv(DR\ \kappa(\forall i\Gamma_i)\ \alpha_i{:}\beta_{ij}/\chi_i)$

proof: In view of SS1, it suffices to prove: $[ss(DR\ \kappa(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i)](DR\ \kappa(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i)$

Unfolding the second occurrence of DR gives: $[ss(DR\ \kappa(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i)]\forall t(R\ t\ k\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)$

which is equivalent to: $\forall t([ss(DR\ \kappa(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i)](R\ t\ k\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i))$

By induction on t the proof divides into a base case and an induction step:

(1)Base Case: $([ss(DR\ \kappa(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i)](R\ 0\ k\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i))$ which holds by theorem MD5.

(2)Induction Step: $([ss(DR\ \kappa(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i)](R\ t\ k\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)) \rightarrow([ss(DR\ \kappa(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i)](R\ t{+}1\ k\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i))$ which holds by theorem MD6. QED.

MD8: $(\kappa\equiv(DR\ \kappa(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i))\rightarrow\exists s(\kappa\equiv(ms\ s))$ proof: From the hypothesis and MD7 $\kappa\equiv(ss(DR\ \kappa(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i))$ is derived. Using the hypothesis to replace $(DR\ \kappa(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i)$ by κ in this result gives: $\kappa\equiv(ss\ \kappa)$ By SS2 this implies the conclusion. QED.

6. Conclusion: The Relationship between "Recursive" Default Logic and the Modal Logic

The relationship between the "recursive" set theoretic definition of Default Logic [Reiter 1980] and the modal representation of it is proven in two steps. First theorem R1 shows that the meaning of the set r is the proposition R. Theorem DR1 shows that the meaning of the set dr is the proposition DR. DL2 shows that a set of FOL sentences which contains its FOL theorems is a fixed-point of the fixed-point equation of Default Logic with an initial set of axioms and defaults if and only if the meaning (or rather disquotation) of that set of sentences is logically equivalent to DR of the meanings of that initial set of sentences and those defaults.

R1: $(ms(r\ t(fol\ '\kappa)\{'\Gamma i\}'\alpha_i{:}'\beta_{ij}/'\chi_i))\equiv(R\ t(ms\ '\kappa)(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i)$

proof: Inducting on the numeric variable t gives a base case and an induction step:

(1) The Base Case: $(ms(r\ 0(fol\ '\kappa)\{'\Gamma i\}'\alpha_i{:}'\beta_{ij}/'\chi_i))\equiv(R\ 0(ms\ '\kappa)(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i)$. Starting from $(ms(r\ 0(fol\ '\kappa)\{'\Gamma i\}'\alpha_i{:}'\beta_{ij}/'\chi_i))$ unfolding r gives: $(ms(fol\{'\Gamma i\}))$. By C2 this is equivalent to: $(ms\{'\Gamma i\})$. By MS2 this is equivalent to: $(\forall i\Gamma_i)$. Folding R then gives: $(R\ t(ms\ '\kappa)(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i)$ which proves the base case.

(2) The Induction Step: $((ms(r\ t(fol\ '\kappa)\{'\Gamma i\}'\alpha_i{:}'\beta_{ij}/'\chi_i))\equiv(R\ t(ms\ '\kappa)(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i))$

$\rightarrow((ms(r\ t{+}1(fol\ '\kappa)'\{'\Gamma i\}'\alpha_i{:}'\beta_{ij}/'\chi_i))\equiv(R\ t{+}1(ms\ '\kappa)(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i))$

Setting aside the induction hypothesis, we start from: $(ms(r\ t{+}1(fol\ '\kappa)\{'\Gamma i\}'\alpha_i{:}'\beta_{ij}/'\chi_i))$

Unfolding r gives: $(ms(fol((r\ t\ '\kappa\{'\Gamma i\}'\alpha_i{:}'\beta_{ij}/'\chi_i)\cup\{'\chi_i{:}\ ('\alpha_i\varepsilon(r\ t\ '\kappa\{'\Gamma i\}'\alpha_i{:}'\beta_{ij}/'\chi_i))\wedge\wedge_{j=1,mi}('(\neg\beta_{ij})\notin'\kappa)\})))$

By C2 this is equivalent to: $(ms((r\ t\ '\kappa\{'\Gamma i\}\ '\alpha_i{:}'\beta_{ij}/'\chi_i)\cup\{'\chi_i{:}\ ('\alpha_i\varepsilon(r\ t\ '\kappa\{'\Gamma i\}'\alpha_i{:}'\beta_{ij}/'\chi_i))\wedge\wedge_{j=1,mi}('(\neg\beta_{ij})\notin'\kappa)\}))$

By MS3 this is equivalent to: $((ms(r\ t\ '\kappa\{'\Gamma i\}'\alpha_i{:}'\beta_{ij}/'\chi_i))\wedge(ms\{'\chi_i{:}\ ('\alpha_i\varepsilon(r\ t\ '\kappa\{'\Gamma i\}'\alpha_i{:}'\beta_{ij}/'\chi_i))\wedge\wedge_{j=1,mi}('(\neg\beta_{ij})\notin'\kappa)\}))$

By MS2 this is : $(ms(r\ t\ '\kappa\{'\Gamma i\}'\alpha_i{:}'\beta_{ij}/'\chi_i))\wedge\forall_i((('\alpha_i\varepsilon(r\ t\ '\kappa\{'\Gamma i\}\alpha_i{:}'\beta_{ij}/'\chi_i))\wedge\wedge_{j=1,mi}('(\neg\beta_{ij})\notin'\kappa))\rightarrow(mg\ '\chi_i))$

Using C1 twice gives and folding $<\kappa>$ gives:

$(ms(r\ t\ '\kappa\{'\Gamma i\}'\alpha_i{:}'\beta_{ij}/'\chi_i))\wedge\forall_i((([ms(r\ t\ '\kappa\{'\Gamma i\}'\alpha_i{:}'\beta_{ij}/'\chi_i)](mg\ '\alpha_i))\wedge\wedge_{j=1,mi}(<(ms\ '\kappa)>(mg\ '\beta_{ij})))\rightarrow(mg\ '\chi_i))$

Using the M0-M7 gives: $(ms(r\ t\ '\kappa\{'\Gamma i\}''\alpha_i{:}'\beta_{ij}/'\chi_i))\wedge\forall_i((([ms(r\ t\ '\kappa\{'\Gamma i\}''\alpha_i{:}'\beta_{ij}/'\chi_i)]\alpha_i)\wedge\wedge_{j=1,mi}(<(ms\ '\kappa)>\beta_{ij}))\rightarrow\chi_i)$

Using the induction hypothesis twice gives:

$(R\ t(ms\ '\kappa)(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i)\wedge\forall_i((([(R\ t(ms\ '\kappa)(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i)]\alpha_i)\wedge\wedge_{j=1,mi}(<(ms\ '\kappa)>\beta_{ij}))\rightarrow\chi_i)$

Folding R then gives: $((R\ t{+}1(ms\ '\kappa)(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i)$ which proves the Induction Step. QED.

DR1: $(ms(dr(fol\ '\kappa)\{'\Gamma i\}'\alpha_i{:}'\beta_{ij}/'\chi_i))\equiv(DR(ms\ '\kappa)(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i)$

proof: $(ms(dr(fol\ '\kappa)'\{'\Gamma i\}'\alpha_i{:}'\beta_{ij}/'\chi_i))$ Unfolding the definition of dr gives: $ms(\cup_{t=1,\omega}(r\ t(fol\ '\kappa)'\{'\Gamma i\}'\alpha_i{:}'\beta_{ij}/'\chi_i))$

Unfolding $\cup$ gives: $ms\{s{:}\ \exists t(s\varepsilon(r\ t(fol\ '\kappa)\{'\Gamma i\}'\alpha_i{:}'\beta_{ij}/'\chi_i))\}$. Unfolding ms gives: $\forall s((s\varepsilon\{s{:}\ \exists t(s\varepsilon(r\ t(fol\ '\kappa)\{'\Gamma i\}'\alpha_i{:}'\beta_{ij}/'\chi_i))\})\rightarrow(mg\ s))$ which is equivalent to: $\forall s((\exists t(s\varepsilon(r\ t(fol\ '\kappa)\{'\Gamma i\}'\alpha_i{:}'\beta_{ij}/'\chi_i)))\rightarrow(mg\ s))$ which is equivalent to: $\forall t\forall s((s\varepsilon(r\ t(fol\ '\kappa)\{'\Gamma i\}'\alpha_i{:}'\beta_{ij}/'\chi_i))\rightarrow(mg\ s))$. Folding ms gives: $\forall t(ms(r\ t(fol\ '\kappa)\{'\Gamma i\}'\alpha_i{:}'\beta_{ij}/'\chi_i))$

By R1 this is equivalent to: $\forall t(R\ t(ms\ '\kappa)(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i)$. Folding DR then gives $(DR(ms\ '\kappa)(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i)$ QED.

DR2: $((fol\ '\kappa)=(dr(fol\ '\kappa)\{'\Gamma i\}\ '\alpha_i{:}'\beta_{ij}/'\chi_i))\leftrightarrow((ms\ '\kappa)\equiv(DR(ms\ '\kappa)(ms\ '\Gamma)\alpha_i{:}\beta_{ij}/\chi_i))$

proof: By FOL9 (fol 'κ)=(dr(fol 'κ){'Γi}'α_i:'β_{ij}/'χ_i) is: (fol 'κ)=(fol(dl(fol 'κ){'Γi}'α_i:'β_{ij}/'χ_i)). By C4 this is equivalent to: (ms 'κ)≡(ms(dr(fol 'κ){'Γi}'α_i:'β_{ij}/'χ_i)). By DR1 this is equivalent to: (ms 'κ)≡(DR(ms 'κ)(∀iΓ_i)α_i:β_{ij}/χ_i) QED.

Theorem DR2 shows that the set of theorems: (fol 'κ) of a set 'κ is a fixed-point of a fixed-point equation of Default Logic if and only if the meaning (ms 'κ) of 'κ is a solution to the necessary equivalence. Furthermore, by FOL10 there are no other fixed-points (such as a set not containing all its theorems) and by MD8 there are no other solutions (such as a proposition not representable as a sentence in the FOL object language). Therefore, the Modal representation of Default Logic (i.e. DR), faithfully represents the set theoretic description of the "recursive" definition of Default Logic (i.e. dr). Finally, we note that (∀iΓ_i) and (ms 'κ) may be generalized to be arbitrary propositions Γ and κ giving the more general modal representation: κ≡(DR κ Γ α_i:β_{ij}/χ_i).

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