A GEOMETRICAL INTERPRETATION TO DEFINE CONTRADICTION DEGREES BETWEEN TWO FUZZY SETS

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Abstract: *For inference purposes in both classical and fuzzy logic, neither the information itself should be contradictory, nor should any of the items of available information contradict each other. In order to avoid these troubles in fuzzy logic, a study about contradiction was initiated by Trillas et al. in [5] and [6]. They introduced the concepts of both self-contradictory fuzzy set and contradiction between two fuzzy sets. Moreover, the need to study not only contradiction but also the degree of such contradiction is pointed out in [1] and [2], suggesting some measures for this purpose. Nevertheless, contradiction could have been measured in some other way. This paper focuses on the study of contradiction between two fuzzy sets dealing with the problem from a geometrical point of view that allow us to find out new ways to measure the contradiction degree. To do this, the two fuzzy sets are interpreted as a subset of the unit square, and the so called contradiction region is determined. Specially we tackle the case in which both sets represent a curve in $[0,1]^2$. This new geometrical approach allows us to obtain different functions to measure contradiction throughout distances. Moreover, some properties of these contradiction measure functions are established and, in some particular case, the relations among these different functions are obtained.*

Keywords: *fuzzy sets, t-norm, t-conorm, fuzzy strong negations, contradiction, measures of contradiction.*

ACM Classification Keywords: *F.4.1 Mathematical Logic and Formal Languages: Mathematical Logic (Model theory, Set theory); I.2.3 Artificial Intelligence: Deduction and Theorem Proving (Uncertainty, "fuzzy" and probabilistic reasoning); I.2.4 Artificial Intelligence: Knowledge Representation Formalisms and Methods (Predicate logic, Representation languages).*

Introduction

One of the main problems tackled by fuzzy logic is how to deal with inferences that include imprecise information. So, several methods have been proposed within this field for inferring new knowledge from the original premises. In any inference process, however, we have to assure that the results yielded neither contradict each other nor the original information.

The concept of contradiction in fuzzy logic was introduced by Trillas *et al*. in [5] and [6]. These papers formalize the idea that a fuzzy set *P* associated with a vague predicate P is contradictory if it violates the principle of non-contradiction in the following sense: the statement "If *x* is P, then *x* is not P" holds with some degree of truth. So, they established that the fuzzy set P is contradictory regarding an involutive negation N if $\mu_P(x) \leq (N \circ \mu_P)(x)$ for all x, where $\mu_P(x)$, named the membership function of *P*, represents the degree in which *x* satisfies the predicate P. Contradiction between two fuzzy sets was also introduced in [5] and [6]. Analogously, two fuzzy sets *P* and *Q* are N-contradictory if the condition $\mu_P(x) \leq (N \circ \mu_Q)(x)$ holds for all x. The need to speak not only of contradiction but also of degrees of contradiction was later raised in [1] and [2], where a function was considered for the purpose of determining (or measuring) the contradiction degree of a fuzzy set. Also, in [2] the authors proposed a function that appears to be suited for measuring the degree of contradiction between two fuzzy sets. However, many functions could be constructed for these purposes, and it is useful to specify what conditions a function must meet to be used as a measure of contradiction. Specifically, some axioms are needed to be able to decide whether a function is suitable for measuring the degree of contradiction. These axioms were established in [3].

In this work, we retake the study of the contradiction between two fuzzy sets, focusing on the problem from a geometrical perspective that suggests new ways of defining measures of contradiction. Therefore, after a geometrical study to determine what we will name regions of contradiction and non-contradiction, we will then define some functions by analyzing some of its properties.

Preliminaries

Firstly, we will introduce a series of definitions and properties for their subsequent development in this article.

Definition 2.1 ([7]) A fuzzy set (FS) *P*, in the universe X ≠ Ø, is a set given as *P*={(x,μ (x)): x ∈ X} such that, for all x ∈ X, μ(x) ∈ [0,1], and where the function μ: X→ [0,1] is called membership function. We denote $\mathcal{F}$(X) the set of all fuzzy sets on X.

Definition 2.2 *P*∈$\mathcal{F}$(X) with membership function μ∈$[0,1]^X$ is to be said a normal fuzzy set if Sup{μ(x) : x∈X}=1.

Definition 2.3 A fuzzy negation (FN) is a non-increasing function N: [0,1] → [0,1] with N(0)=1 and N(1)=0. Moreover, N is a strong fuzzy negation if the equality N(N(y))=y holds for all y ∈ [0,1].

The strong negations were characterized by Trillas in [4]. He showed that N is a strong negation if and only if, there is an order automorphism g in the unit interval (that is, g:[0,1]→ [0,1] is an increasing continuous function with g(0)=0 and g(1)=1) such that $N(y)=g^{-1}(1-g(y))$, for all y ∈ [0,1]; from now on, let us denote $N_g=g^{-1}(1-g)$. Furthermore, the only fixed point of N_g is $n_g=g^{-1}(1/2)$.

Measuring N_g-contradiction between Two Fuzzy Sets

As mentioned above, μ and σ are said to be N_g-contradictory if $\mu(x)\leq N_g(\sigma(x))$ for all elements x in the universe of discourse, which is equivalent to $\mu(x)\leq g^{-1}(1-g(\sigma(x)))$ for all x, and also to $\text{Sup}\{g(\mu(x))+g(\sigma(x)) / x \in X\} \leq 1$. Here again ascertaining whether two sets are contradictory will fall short of the mark, and a distinction should be made between any differing degrees of contradiction occurring in such situations. This problem was addressed for the first time in [1] and [2].

In this section, in order to study the degree of N_g-contradiction between two fuzzy sets *P* and *Q* (with membership functions μ, σ ∈ $[0,1]^X$, respectively) we consider the set {(μ(x),σ(x)) : x ∈ X} as a subset of $[0,1]^2$ (we denote it by $X_{\mu\sigma}$ to be short) and we will firstly analyze in what regions of $[0,1]^2$ $X_{\mu\sigma}$ must remain provided that μ and σ are N_g-contradictory (see figure 1). The aim of this analysis is to find some relation suggesting the way of measuring the N_g-contradiction between two fuzzy sets. Secondly, we propose some possible functions in order to measure the degrees of contradiction, bearing in mind the mentioned analysis.

Regions of N_g-contradiction

As mentioned above, given μ, σ∈$[0,1]^X$ and a strong negation N_g, then μ and σ are N_g-contradictory if and only if

$$\mu(x)\leq N_g(\sigma(x))\ \forall\, x \in X \Leftrightarrow \sigma(x)\leq N_g(\mu(x))\ \forall\, x \in X \Leftrightarrow g(\mu(x))+g(\sigma(x))\leq 1\ \forall\, x \in X$$

The above inequalities determine a curve in the unit square, with equation $y_1=N_g(y_2)$ or $y_2=N_g(y_1)$ or $g(y_1)+g(y_2)=1$; this curve, called the *limit curve of N_g-contradiction,* is the border between two regions: the region in which contradictory sets remain and the region free of contradiction (see figure 1).

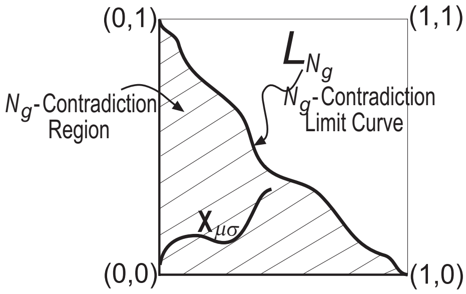

Figure 1: N_g-contradiction region and N_g-contradiction limit curve

Let us see these regions in several particular cases and after that, the general case will be discussed.

(a) N_s-contradiction with standard negation $N_s(y)=1-y$

Let N_s=1-id be the standard negation that is generated by g=id. Then μ and σ are N_s-contradictory if and only if $\mu(x)+\sigma(x)\leq 1$ for all x ∈ X, that is equivalent to $X_{\mu\sigma} \subset \{(y_1,y_2)\in [0,1]^2 : y_1+y_2\leq 1\}$ (see figure 2(a)).

(b) N_g-contradiction with $g(y)=y^2$

The order automorphism $g(y) = y^2$ determines the strong negation $N_g(y) = \sqrt{1-y^2}$, and the sets $\mu, \sigma \in [0,1]^X$ are N_g-contradictory if and only if $\mu(x) \le \sqrt{1-\sigma(x)^2}$, that is equivalent to $\mu(x)^2 + \sigma(x)^2 \le 1$. Therefore, μ and σ are N_g-contradictory if and only if

$$X_{\mu\sigma} = \{(\mu(x), \sigma(x)) : x \in X\} \subset \{(y_1, y_2) \in [0,1]^2 : y_1^2 + y_2^2 \le 1\}$$

Then, $X_{\mu\sigma}$ must remain inside or on the circumference with center (0,0) and radius 1 (see figure 2 (b)).

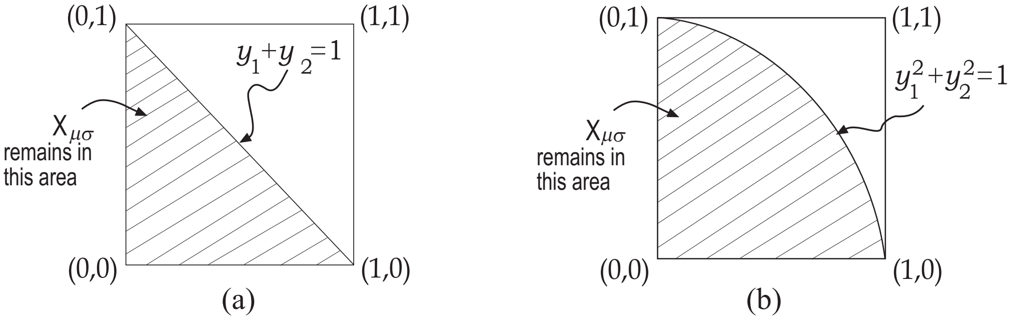

Figure 2: (a) N_s-contradiction area and (b) N_g-contradiction area with $g(y)=y^2$

(c) N_r-contradiction with N_r determined by $g(y)=y^r$, r>0

Let's consider the family of strong negations $\{N_r\}_{r>0}$, where for each r>0 N_r is determined by the automorphism $g_r(y)=y^r$. This family includes as particular cases the negations given in (a) and (b) and for each r>0 is $N_r(y)=(1-y^r)^{1/r}$ with a fixed point $y_{N_r} = \frac{1}{2^{1/r}}$. $\mu, \sigma \in [0,1]^X$ are N_r-contradictory if and only if

$$X_{\mu\sigma} \subset \{(y_1, y_2) \in [0,1]^2 : y_1^r + y_2^r \le 1\}$$

For each r>0 the curve $y_1^r + y_2^r = 1$ is the border that delimits the region of contradiction, and if $X_{\mu\sigma}$ takes some value $(\mu(x_0),\sigma(x_0))$ over the mentioned curve, then, they are not N_r-contradictory.

We must note that as r increases, curves $y_1^r + y_2^r = 1$ approach to the line $y_1=1$ (except in $y_2=1$) and to the line $y_2=1$ (except in $y_1=1$); more specifically, the family of functions $\{(1-y_1^r)^{1/r}\}_{r>0}$ converges punctually when $r \to \infty$, to the constant function 1 for all $y_1 \in [0,1)$ and in $y_1=1$ converges to 0 and the family of functions $\{(1-y_2^r)^{1/r}\}_{r>0}$ converges punctually when $r \to \infty$, to the constant function 1 for all $y_2 \in [0,1)$ and in $y_2=1$ converges to 0; therefore, the region of non N_r-contradiction between two FS decreases when r grows (see figure 3). Moreover, when $r \to 0$, the family of functions $\{(1-y_2^r)^{1/r}\}_{r>0}$ converges for each $y_2 \in (0,1]$ to the null function and for $y_2=0$ converges to 1; and the family of functions $\{(1-y_1^r)^{1/r}\}_{r>0}$ converges for each $y_1 \in (0,1]$ to the null function and for $y_1=0$ converges to 1. That is, as r decreases the curves that delimit the regions of contradiction get closer to the axes y_1 and y_2, and therefore, the region of non N_r-contradiction between two FS increases.

On the other hand, if 0<r<s then, the curve $y_1^s + y_2^s = 1$ is over curve $y_1^r + y_2^r = 1$ (see figure 3 for the representation of some of them) and so, if $\mu, \sigma \in [0,1]^X$ are N_r-contradictory, then they are N_s-contradictory for all s>r. In fact, if r<s it is $y_1^r > y_1^s$ for all $y_1 \in (0,1)$, and therefore taking into account that $g_{\frac{1}{r}}$ is increasing and that 1/s<1/r, is $(1-y_1^r)^{1/r} < (1-y_1^s)^{1/r} < (1-y_1^s)^{1/s}$ from where we follow that coordinate y_2 of the curve corresponding to s is bigger than the one corresponding to r. Finally, we observe that the family of curves mentioned above, practically fills the unit square $[0,1]^2$ (with the exception of the border of the unit square except point (0,1) and (1,0)), That is:

$$\bigcup_{r>0}\left\{(y_1,y_2)\in[0,1]^2 : y_1^r + y_2^r = 1\right\} = (0,1)^2 \cup \left\{(1,0),(0,1)\right\}$$

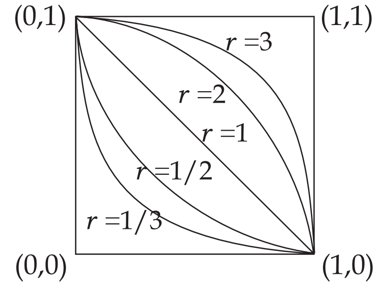

Figure 3: Curves $y_1^r + y_2^r = 1$

(d) General case of N-contradiction

If N is a strong FN, two sets $\mu, \sigma \in [0,1]^X$ are N-contradictory if and only if

$$X_{\mu\sigma} \subset \left\{(y_1,y_2)\in[0,1]^2 : y_1 \le N(y_2)\right\}$$

And the border curve that delimits the region exempt of contradiction is the curve of equation $y_1=N(y_2)$. Therefore, the border curve is determined by a strong negation and it will have the following properties of a strong negation:

1) It is decreasing in both variable y_1 and y_2.
2) It goes through (1,0) and through (0,1) since N(0)=1 y N(1)=0.
3) It is symmetric with respect to the line $y_1=y_2$ since $y_1=N(y_2)$ and $y_2=N(y_1)$ are the same curve, because $N(N(y))=y$ for all $y \in [0,1]$.

Then, the regions of contradiction are limited by all strong negations in [0,1].

Degrees of N-contradiction between Two Fuzzy Sets

As we discussed in the introduction, it is relevant to weight in which degree two sets are contradictory. In fact, $\mu_\varnothing$ and $\mu_\varnothing$ (where $\mu_\varnothing(x)=0$ for all $x \in X$) are N-contradictory for any strong FN N (see figure 4(a)). Nevertheless, if μ and σ are N-contradictory FS such that $\mu(x_0)=N(\sigma(x_0))$ for some x_0 (see figure 4(b)), and so $X_{\mu\sigma} \cap \left\{(y_1,y_2)\in[0,1]^2 : y_1 = N(y_2)\right\} \neq \varnothing$, then small disturbances over the value $(\mu(x_0),\sigma(x_0))$ could convert μ and σ into two sets very similar to the original ones but non N-contradictory. Meanwhile, small disturbances would never change the contradictory character of the empty set with itself. Thus, it seems adequate to assign 0 as the degree of N-contradiction for whichever μ and σ such that $X_{\mu\sigma} \cap \left\{(y_1,y_2)\in[0,1]^2 : y_1 \ge N(y_2)\right\} \neq \varnothing$ and a positive value, as much higher as $X_{\mu\sigma}$ is farther away from the limit curve ($y_1=N(y_2)$), in other case (see figure 4(c)).

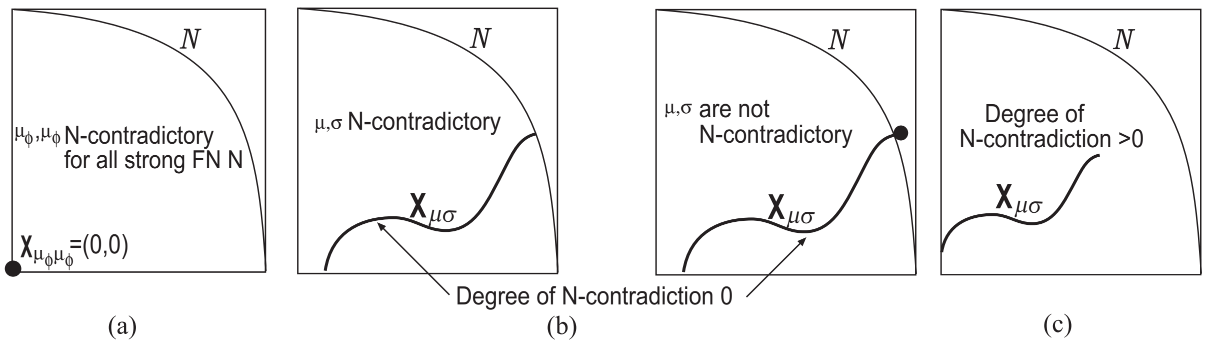

Figure 4: Geometrical interpretation of N-contradiction degree

Taking into account these observations, we are going to define different functions that could serve as a model to determine the different degrees of contradiction between two fuzzy sets

Definition 3.1 Given μ, σ ∈ $[0,1]^X$ and N a strong FN , we define the following contradiction measure functions:

i) $C_1^N(\mu,\sigma) = \text{Max}\left(0, \underset{x\in X}{\text{Inf}}\left(N(\sigma(x)) - \mu(x)\right)\right)$

ii) $C_2^N(\mu,\sigma) = \text{Max}\left(0, \underset{x\in X}{\text{Inf}}\left(N(\mu(x)) - \sigma(x)\right)\right)$

iii) $C_3^N(\mu,\sigma) = \text{Max}\left(0, 1 - \underset{x\in X}{\text{Sup}}\left(g(\mu(x)) + g(\sigma(x))\right)\right)$

iv) $C_4^N(\mu,\sigma) = 0$ if μ and σ are not N-contradictory, and in the other case $C_4^N(\mu,\sigma) = \dfrac{d(X_{\mu\sigma}, L_N)}{d((0,0), L_N)}$

where *d* is the Euclidean distance and $L_N = \{(y_1, y_2) \in [0,1]^2 : N(y_1) = y_2\}$ is the limit curve, and therefore, $d(X_{\mu\sigma}, L_N) = \text{Inf}\ \{d((\mu(x), \sigma(x)),\ (y_1, y_2)) : x \in X, (y_1, y_2) \in L_N\}$ and $d((0,0), L_N)$ $d((0,0), L_N) = \text{Inf}\ \{d((0,0), (y_1, y_2)) : (y_1, y_2) \in L_N\}$.

Remark: The four previous functions take values in [0,1] and it is satisfied that all of them are zero or all are strictly positive. The functions C_1^N and C_2^N come motivated by the characterization of contradiction "μ and σ are N_g-contradictory if and only if μ(x)≤ N_g(σ(x)) ∀ x ∈ X ⇔ σ(x)≤ N_g(μ(x)) ∀ x ∈ X", while C_3^N is based on the characterization "μ and σ are N_g-contradictory if and only if g(μ(x))+g(σ(x))≤1 ∀x∈X". Although both characterizations are equivalent, C_1^N, C_2^N and C_3^N they do not coincide, as we will show it at a later example. On the other hand, C_4^N represents a relative distance: the Euclidean distance of the set $X_{\mu\sigma}$ to the limit curve relative to the distance of the "most contradictory" sets to the same curve. While C_1^N represents the infimum of the distances between the abscises of the values (μ(x),σ(x)) and the corresponding of the limit curve (see figure 5), C_2^N represents the infimum of the distances between the ordinates of the values (μ(x),σ(x)) and the corresponding of the limit curve (see figure 5). As far as C_3^N is concerned, some geometrical interpretations can be found in some particular cases.

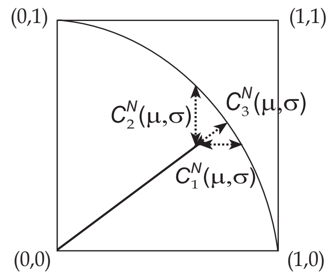

Figure 5: Geometrical interpretation of different contradiction degrees

Proposition 3.2 Let N_{id} be the standard FN; for all μ, σ ∈ $[0,1]^X$ the degrees of contradiction between μ and σ by means of the formula in definition 3.1 satisfy that $C_1^{N_{Id}}(\mu,\sigma) = C_2^{N_{Id}}(\mu,\sigma) = C_3^{N_{Id}}(\mu,\sigma) = C_4^{N_{Id}}(\mu,\sigma)$ (fig.6).

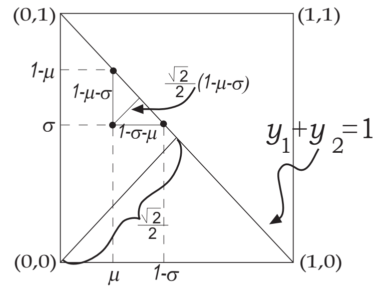

Figure 6: Geometrical interpretation of the proposition 3.2

However, generally, the four measures are different as the following examples show.

Example 3.3 Given $X_{\mu\sigma}$ = {(0.25,0.75), (0.65,0.65), (0.75,0.5), (0.79,0.3)} and the strong negation $N_3(y) = \left(1 - y^3\right)^{1/3}$, with g(y)=y³. Then (see figure 7):

$$C_1^{N_3}(\mu,\sigma) = \operatorname*{Inf}_{x \in X}\left(\left(1 - \sigma(x)^3\right)^{1/3} - \mu(x)\right) = 0.2009$$ reaching the infimum at point (0.79, 0.3)

$$C_2^{N_3}(\mu,\sigma) = \operatorname*{Inf}_{x \in X}\left(\left(1 - \mu(x)^3\right)^{1/3} - \sigma(x)\right) = 0.2448$$ reaching the infimum at point (0.25, 0.75)

$$C_3^{N_3}(\mu,\sigma) = 1 - \operatorname*{Sup}_{x \in X}\left(\mu(x)^3 + \sigma(x)^3\right) = 0.4507$$ reaching the supremum at point (0.65, 0.65)

Since μ and σ are N_3-contradictory and since $L_N = \left\{(y_1, y_2) \in [0,1]^2 : {y_1}^3 + {y_2}^3 = 1\right\}$, then

$$C_4^{N_3}(\mu,\sigma) = \frac{d(X_{\mu\sigma}, L_N)}{d((0,0), L_N)} = d\left((0.79, 0.06), L_N\right) = 0.1969$$

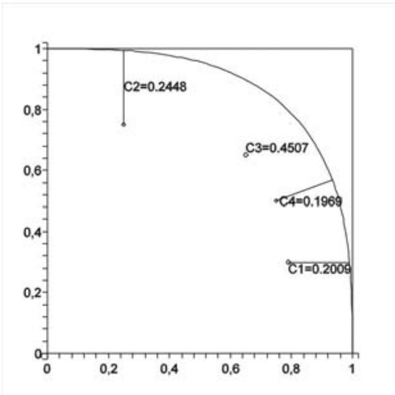

Figure 7: Geometrical interpretation of the example 3.3

Proposition 3.4 Let N_g be the strong FN with g(y)=y², i.e. $N_g(y) = \sqrt{1 - y^2}$, then for all μ, $\sigma \in [0,1]^X$ the degrees of contradiction $C_3^{N_g}$ and $C_4^{N_g}$ between μ and σ verify that $C_3^{N_g}(\mu,\sigma) = 1 - \left(1 - C_4^{N_g}(\mu,\sigma)\right)^2$ (see figure 8).

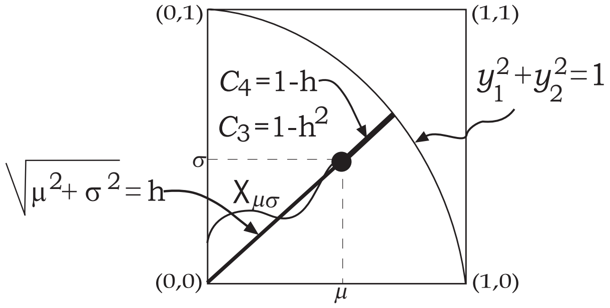

Figure 8: Geometrical interpretation of the proposition 3.4

Let us observe that, in general, the relation of the proposition 3.3 is not satisfied as the example 3.3 shows:

$$C_3^{N_3}(\mu,\sigma) = 0.4507 \neq 1 - \left(1 - C_4^{N_3}(\mu,\sigma)\right)^2 = 1 - (1 - 0.1969)^2 = 0.3550$$

The following properties of the above measure of N-contradiction functions between two fuzzy sets can be proved.

Proposition 3.5 For each i=1,2,3,4 function C_i^N : $[0,1]^X$ x $[0,1]^X \to [0,1]$ defined for every two μ, $\sigma \in [0,1]^X$ as definition 3.1 verifies:

i) $C_i^N(\mu_\varnothing, \mu_\varnothing) = 1$.

ii) $C_i^N(\mu,\sigma) = 0$ if μ or σ normal.

iii) Symmetry: $C_i^N(\mu,\sigma) = C_i^N(\sigma,\mu)$ for i=3,4. For i=1,2 is verified that $C_1^N(\mu,\sigma) = C_2^N(\sigma,\mu)$.

iv) Given $\{\mu_\alpha\}_{\alpha\in I} \subset [0,1]^X$, it holds that: $\operatorname{Inf}_{\alpha\in I} C_i^N(\mu_\alpha,\sigma) = C_i^N\left(\operatorname{Sup}_{\alpha\in I}\mu_\alpha,\sigma\right)$. As a particular case of (iv) it is verified that given μ_1, $\mu_2,\sigma\in[0,1]^X$ if $\mu_1 \le \mu_2$, then $C_i^N(\mu_1,\sigma) \ge C_i^N(\mu_2,\sigma)$ (Anti-Monotonicity).

Property (ii) is stronger than the second axiom given in [3] ($C(\mu,\mu)=0$ for all normal $\mu \in [0,1]^X$) to define measures of contradiction. Moreover, C_i^N for each i=1,2,3,4, is a positive or strict measure of contradiction as defined in [3] since $C_i^N(\mu,\sigma) = 0$ provided that $\operatorname{Sup}_{x\in X}\left(g(\mu(x)) + g(\sigma(x))\right) \ge 1$.

Example 3.6 Given *P, Q* $\in \mathcal{F}([0,1])$ with membership functions μ, σ such that $\mu(x)= -2x+1$ if $x\le 1/2$ and 0, if $x>1/2$ and $\sigma(x)= x$, if $x\le 1/2$ and 1/2, if $x>1/2$. As μ is normal, for all strong negation N the degree of contradiction is zero, $C_i^N(\mu,\sigma) = 0$ with i=1,2,3,4. However, there are strong negations for which sets P, Q are contradictory. For instance, for all negations N_g such that $g(y)=y^p$ with $p\ge 1$.

Measuring Contradiction between Two Fuzzy Sets

In this section, we will deal with the case of contradiction without depending on a prefixed negation. The previous section establishes the contradiction between two fuzzy sets related to a chosen strong negation. We now address contradiction more generally, without depending on any specific FN. In [5] and [6] two FS *P, Q* $\in \mathcal{F}(X)$ with membership functions μ, σ were defined contradictory if they were N-contradictory regarding some strong FN N. The following result was proved in [2].

Proposition 4.1 ([2]) If *P, Q* $\in \mathcal{F}(X)$ with membership functions μ, σ are contradictory, then: for all $\{x_n\}_{n\in N}\subset X$, if $\lim_{n\to\infty}\{\mu(x_n)\} = 1$, then $\lim_{n\to\infty}\{\sigma(x_n)\} = 0$, and if $\lim_{n\to\infty}\{\sigma(x_n)\} = 1$, then $\lim_{n\to\infty}\{\mu(x_n)\} = 0$. In particular, if $\mu(x)=1$, for some $x \in X$, then $\sigma(x)=0$ and vice-versa.

With the intention of measuring how contradictory two FS are, we will define some functions motivated in the previous section, being of interest, for one of them, to consider the following corollary also given in [2].

Corollary 4.2 ([2]) If μ, $\sigma \in [0,1]^X$ are contradictory, then $\operatorname{Sup}_{x\in X}\left(\mu(x) + \sigma(x)\right) < 2$.

Definition 4.3 Given μ, $\sigma \in [0,1]^X$, we define the following contradiction measure functions:

i) $C_1(\mu,\sigma)=0$ if there exists $\{x_n\}_{n\in N} \subset X$ such that $\lim_{n\to\infty}\{\mu(x_n)\} = 1$ or $\lim_{n\to\infty}\{\sigma(x_n)\} = 1$, and, in other case $C_1(\mu,\sigma) = \operatorname{Min}\left(\operatorname{Inf}_{x\in X}(1-\mu(x)), \operatorname{Inf}_{x\in X}(1-\sigma(x))\right)$.

ii) $C_2(\mu,\sigma)=0$ if there exists $\{x_n\}_{n\in N} \subset X$ such that $\lim_{n\to\infty}\{\mu(x_n)\} = 1$ or $\lim_{n\to\infty}\{\sigma(x_n)\} = 1$, and, in other case

$$C_2(\mu,\sigma) = 1 - \frac{\operatorname{Sup}_{x\in X}\left(\mu(x)+\sigma(x)\right)}{2}.$$

Remark: It is evident that the function C_1 measures the minimum between distance (Euclidean) of $X_{\mu\sigma}$ to the line $y_1=1$ (that we will note L_1) and the distance of $X_{\mu\sigma}$ to the line $y_2=1$ (that we will note L_2): (see figure 9(a))

$$C_1(\mu,\sigma) = \operatorname{Min}\left(d\left(X_{\mu\sigma},L_1\right), d\left(X_{\mu\sigma},L_2\right)\right) = \frac{\operatorname{Min}\left(d\left(X_{\mu\sigma},L_1\right), d\left(X_{\mu\sigma},L_2\right)\right)}{d((0,0), L_1 ó L_2)}.$$

On the other hand, $C_2(\mu,\sigma) = \dfrac{d_1\left(X_{\mu\sigma},(1,1)\right)}{2} = \dfrac{d_1\left(X_{\mu\sigma},(1,1)\right)}{d_1((0,0),(1,1))}$ that is, the function C_2 measures the reticular distance between $X_{\mu\sigma}$ and (1,1), relative to the reticular distance from (0,0) to (1,1) (let us remind that $d_1\left((y_1,y_2),(z_1,z_2)\right) = |y_1-z_1| + |y_2-z_2|$) (see figure 9(b)). These geometrical interpretations of the measures C_1 and C_2 suggest another way of measuring the contradiction degree: $C_3(\mu,\sigma)=0$ if there exists

$\{x_n\}_{n\in N}\subset X$ such that $\lim_{n\to\infty}\{\mu(x_n)\}=1$ or $\lim_{n\to\infty}\{\sigma(x_n)\}=1$, and, in other case $C_3(\mu,\sigma)=\dfrac{d(X_{\mu\sigma},(1,1))}{d((0,0),(1,1))}$ (see figure 9(c)).

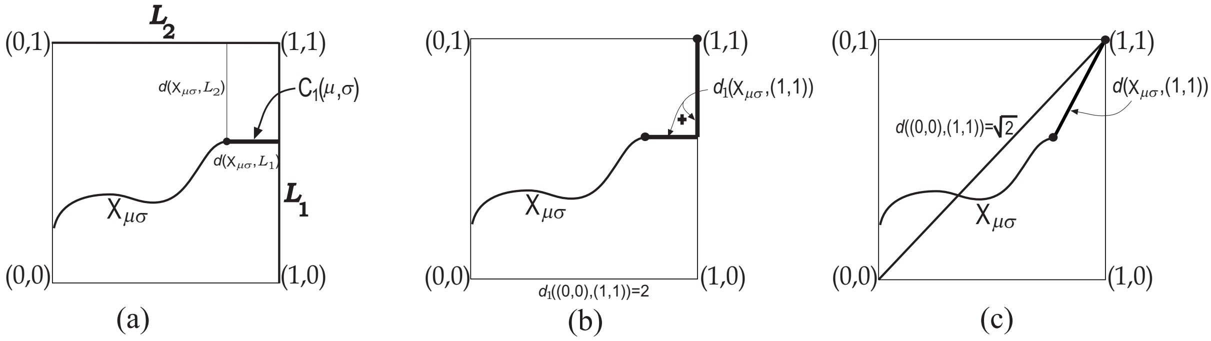

Figure 9: Geometrical interpretation of the measures C_1, C_2 and C_3

In the same way that happened with measures of N-contradiction between two fuzzy sets, the following result can be demonstrated.

Proposition 4.4 For each i=1,2,3 function C_i: $[0,1]^X \times [0,1]^X \to [0,1]$ defined for each pair $\mu, \sigma\in[0,1]^X$ as the above definition verifies:

i) $C_i(\mu_\varnothing,\mu_\varnothing)=1$.

ii) $C_i(\mu,\sigma)=0$ if μ or σ normal.

iii) Symmetry: $C_i(\mu,\sigma)=C_i(\sigma,\mu)$.

iv) Anti-Monotonicity: given μ_1, μ_2, $\sigma\in[0,1]^X$ if $\mu_1\le\mu_2$, then $C_i(\mu_1,\sigma)\ge C_i(\mu_2,\sigma)$. Besides, for the case i=1 axiom of the infimum given in [3] is also verified. That is, given $\{\mu_\alpha\}_{\alpha\in I}\subset[0,1]^X$, it holds that: $\operatorname{Inf}_{\alpha\in I} C_i(\mu_\alpha,\sigma)=C_i\left(\operatorname{Sup}_{\alpha\in I}\mu_\alpha,\sigma\right)$.

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