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Author's Information

Arkadij Zakrevskij – United Institute of Informatics Problems of the NAS of Belarus, Surganov Str. 6, 220012 Minsk, Belarus; e-mail: zakr@newman.bas-net.by

THE BOUNDARY DESCRIPTORS OF THE *n*-DIMENSIONAL UNIT CUBE SUBSET PARTITIONING[1]

Hasmik Sahakyan, Levon Aslanyan

Abstract: The specific class of all monotone Boolean functions with characteristic vectors of partitioning of sets of all true-vertices to be minimal is investigated. These characteristic vectors correspond to the column-sum vectors of special (0,1)-matrices – constructed by the interval bisection method.

Keywords: monotone Boolean functions, (0,1)-matrices.

ACM Classification Keywords: G.2.1 Discrete mathematics: Combinatorics

1. Introduction

The problem of general quantitative description of vertex subsets of n dimensional unit cube E^n, through their partitions, the existence problem and composing algorithms for vertex subsets by the given quantitative characteristics of partitions are considered. Each of these sub-problems has its own theoretical and practical significance. The existence and composing problems are studied intensively [BI, 1988; C, 1986; S, 1986], but the

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complexity issues are still open [B 1989]. Therefore, studying the problem in different restrictions, for particular type of subsets, and obtaining different necessary and sufficient conditions is essential.

Concerning the problem of general quantitative description - the complete description of the set of all integer-valued vectors - quantitative descriptors of subset partitions is obtained [S 1997]. The description is based on structures, where the characteristic vectors corresponding to the monotone Boolean functions of E^n have an important role. The main result of this research concerns with the partitioning-boundary-cases, which correspond to the special monotone Boolean functions having minimal characteristic partitioning vectors, which, in their turn, correspond to the column sum vectors of (0,1)-matrices constructed by the interval bisection method.

2. Structure

Let $M \subseteq E^n$ be a vertex subset of fixed size $|M| = m$, $0 \le m \le 2^n$. An integer, nonnegative vector $S = (s_1, s_2, \cdots, s_n)$ is called the **characteristic vector of partitions** of set M , if its coordinates equal to the partition-subsets sizes of M by coordinates $x_1, x_2, \cdots, x_n$ - the Boolean variables composing E^n . s_i equals the size of one of the partition-subsets of M by the i -th direction and then $m - s_i$ is the size of the complementary part of partition. To make this notation precise we will later assume that s_i is the size of the partition subset with $x_i = 1$.

The complete description of all integer-coordinate vectors, which serve as characteristic vectors of partitions for vertex subsets of size m sets is based on structures, where characteristic vectors corresponding to the monotone Boolean functions play an important role.

Let Ξ_{m+1}^n denotes the set of all vertices of n dimensional, $m+1$ valued discrete cube, i.e. the set of all integer-vectors $S = (s_1, s_2, \cdots, s_n)$ with $0 \le s_i \le m$, $i = 1, \cdots, n$. ψ_m denotes the set of all characteristic vectors of partitions of m -subsets of E^n . It is evident, that - $\psi_m \subseteq \Xi_{m+1}^n$. Below, the description of ψ_m has been given in terms of Ξ_{m+1}^n geometry: the vertices are distributed schematically on the $m \cdot n + 1$ layers of Ξ_{m+1}^n according to their weights – sums of all coordinates. The l -th layer contains all vectors $S = (s_1, s_2, \cdots, s_n)$ with $l = \sum_{i=1}^n s_i$.

Let $\hat{\psi}_m$ and $\breve{\psi}_m$ are subsets of ψ_m , consisting of all its upper and lower boundary vectors, correspondingly:

$\hat{\psi}_m$ is the set of all "upper" vectors $S \in \psi_m$, for which $R \notin \psi_m$ for all $R \in \Xi_{m+1}^n$, greater than S . $\breve{\psi}_m$ is the set of all "lower" vectors $S \in \psi_m$, for which $R \notin \psi_m$ for all vectors R from Ξ_{m+1}^n , less than S .

These sets of all "upper" and "lower" boundary vectors have symmetric structures - for each upper vector there exists a corresponding (opposite) lower vector, and vice versa; so that also the numbers of these vectors are equal:

$$\hat{\psi}_m = \{ \hat{S}_1, \cdots, \hat{S}_r \} \text{ and } \breve{\psi}_m = \{ \breve{S}_1, \cdots, \breve{S}_r \}.$$

Let $\hat{S}_j$ and $\breve{S}_j$ be an arbitrary pair of opposite vectors from $\hat{\psi}_m$ and $\breve{\psi}_m$ correspondingly. $I(\hat{S}_j)$ (equivalently $I(\breve{S}_j)$) will denote the minimal sub-cube of Ξ_{m+1}^n , passing through this pair of vectors. Then,

$$I(\hat{S}_j) = \{ Q \in \Xi_{m+1}^n / \hat{S}_j \le Q \le \breve{S}_j \} \text{ (the coordinate-wise comparison is used).}$$

The following Theorem states that the minimal sub-cubes passing the pairs of corresponding opposite vectors of the boundary subsets are continuously and exactly filling the vector area ψ_m .

Theorem 1 [S 1997]: $\psi_m = \bigcup_{j=1}^r I(\hat{S}_j)$.

3. Boundary Cases

Boundary vectors of ψ_m can be described by the monotone Boolean functions, defined on E^n: the set of all characteristic boundary vectors is a subset of the set of all characteristic vectors of partitions of monotone Boolean functions. This fact is confirmed by the following theorem.

Theorem 2 [S 1997]: $\widehat{\psi}_m \subseteq M_m^1$ and $\breve{\psi}_m \subseteq M_m^0$,

where M_m^1 and M_m^0 are the sets of characteristic vectors of those m-subsets of E^n, which correspond to the sets of all one-valued vertices and all zero-valued vertices defined by monotone Boolean functions correspondingly.

Let $\wp_{min}(m,n)$ is the class of monotone Boolean functions for which the partitioning characteristic vectors of $\widehat{\psi}_m$ are placed on the minimal possible layer of Ξ_{m+1}^n, - denote this layer by L_{min}. A similar class of functions is related to $\breve{\psi}_m$.

The structure of functions of class $\wp_{min}(m,n)$ is related to the linear ordering of vertices of E^n by decreasing of numeric values of the vectors expressed as $x_n, x_{n-1}, \cdots, x_1$. Call this sequence of vertices "decreasing", denoting it by D_n. D_n has some useful properties. The first 2^{n-1} vertices of D_n compose an $n-1$ dimensional sub-cube (interval), incident to the vertex $x_i = 1, i = \overline{1,n}$. The reminder 2^{n-1} vertices are in the "shadow" of that interval by the variable x_n. The vertex order in both sub-cubes correspond to the orderings in D_{n-1}. In general, for an arbitrary k the first 2^k vertices of the sequences compose k dimensional intervals, where variables $x_n, x_{n-1}, \cdots, x_{k+1}$ accept fixed values; and then the next 2^k vertices are in the "shadow" of that interval by the direction x_{k+1}, and continuing the same way we will receive the recursive structure of the series D_n.

The given construction provides that the initial segments by m of the "decreasing" sequence D_n correspond to the sets of all one-valued-vectors of some monotone Boolean function. We denote this functions by $\mu(m,n)$.

Next theorem confirms that the constructed functions are the required monotone Boolean functions.

Theorem 3: $\mu(m,n) \in \wp_{min}(m,n)$.

The general technique to prove this Theorem consists of partitioning of E^n by one or two directions, the inductions by these parameters and the partitioning of all possible cases into the several sub cases.

The minimal layer L_{min} might be presented by the following formula:

$$L_{min} = \sum_{i=1}^{n} s_i = \sum_{i=1}^{p} \left((n - k_i - (i-1)) \cdot 2^{k_i} + k_i \cdot 2^{k_i - 1} \right),$$

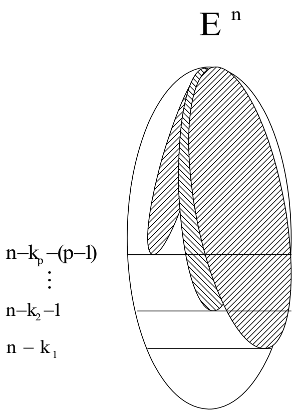
where parameters k_i correspond to the binary presentation of number m:

$$m = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_p}.$$

Numerical expressions of coordinates of characteristic vector $S = (s_1, s_2, \cdots, s_n)$ of function $\mu(m,n)$ are also calculable:

$$s_{k_j+1} = \left(\sum_{l=1}^{j-1} 2^{k_l - 1} \right) + 2^{k_j}, \quad j \in \overline{1,p}$$

for digits $k_1, k_2, ..., k_p$ of binary presentation, and



$$s_i = \left(\sum_{l=1}^{j} 2^{k_l - 1} \right) + \left(\sum_{l=j+1}^{p} 2^{k_l} \right) \text{ for } i, k_{j+1} + 2 \le i \le k_j ,\ \ j \in \overline{1, p-1}$$

and

$$s_i = \sum_{l=1}^{p} 2^{k_l - 1} = m/2 \text{ for } i, 1 \le i \le k_p$$

$$s_i = \sum_{l=1}^{p} 2^{k_l} = m \text{ for } i, k_1 + 2 \le i \le n$$

4. Synthesis by Bisections

Subsets $M \subseteq E^n$ might be presented (coded) by $(0,1)$-matrices with m rows and n columns, where the rows correspond to the vertices-elements of M. All the rows are different, and the quantities of 1's in columns (column sums) correspond to the sizes of (one of the two) partition-subsets of M in corresponding directions.

Thus the existence and synthesis issues of vertex subsets by given quantitative characteristics of their partitions might be reformulated as the corresponding existence and synthesis problems for $(0,1)$-matrices.

Let it is given an integer vector $S = (s_1, \cdots, s_n)$, $0 \le s_i \le m$, $i = 1, \cdots, n$. If there exists an $m \times n$ $(0,1)$-matrix with all different rows and with s_i column sums, then by the finite number of row pairs replacements it can be transformed into the equivalent matrix of "canonical form" – where each column consists of continuous intervals of 1's and (then - below) 0's such that they compose bisections of intervals of the previous stage (column). Therefore the problem of synthesis of the given $(0,1)$-matrices in supposition of the existence might be solved, in particular, by the algorithms which compose the matrices in column-by-column bisection fashion.

The first column is being constructed by allocating of s_1 1's to the first s_1 rows-positions followed by the $m - s_1$ 0's in others. Two intervals is the result: – the s_1 interval of 1's, and the $m - s_1$ interval of 0's. Within each of these intervals we have the same row, and pairs of (i, j) rows with i and j from different intervals are different.

The second column has been constructed by a similar bi-partitioning of intervals of the first column - situating first 1's and then 0's on these intervals such that the summary length of 1-intervals is equal to s_2, and the summary length of 0-intervals is equal to $m - s_2$. The current k-th column has been constructed by consecutive and continuous bi-partitioning of intervals of the $k-1$ column – providing only that the summary length of all 1's-intervals is equal to s_k.

When in some column we get all 1 length intervals, then all the rows of matrix become different by the set of constructed columns, and the remainder columns might be constructed arbitrarily.

Partitioning of the intervals in each step can be performed by different ways - following different goals. Let assume that the partitioning of intervals aims to maximize some quantitative characteristics, which might lead to the matrices with different rows. One of such characteristics - the number of pairs of different rows – is considered in [S, 1995], where is proven that the optimal partitioning is achieved when intervals are partitioned by the equi-difference principle.

In an analogue sophisticated situation when we are not restricted by column sums (or when we allow the whole set of descriptions) and the aim is only in minimization of number of columns for differentiations of rows or in maximization of the number of different row pairs, the best partitioning is known by the “interval bisection” method, which requires the number of columns - $n = \lceil log_2 m \rceil$ [K, 1998].

For the given m let’s describe all the possible column sum vectors of matrices composed by the interval bisection method.

In case of $m = 2^n$ it is evident, that $s_i = m - s_i = 2^{n-1}$ for $i = 1, \cdots, n$.

For an arbitrary m, ($m > 1$), an odd length interval may appear during the separate column partitioning stage. Partitioning the odd length interval takes an extra 1 or 0, which leads to the different column sums. Factually it is satisfactory to consider the case when in each stage an extra 1 is allocated on each odd length interval during the partitioning and let this leads to the column sums $s_1, s_2, \cdots, s_n$. Column sums corresponding to all the other allocations of extra 1’s and 0’s can be received through the inversions and monotone transformation of $s_1, s_2, \cdots, s_n$ [S, 1995].

So the number of odd length intervals of each column is equivalent to the difference of 1’s and 0’s used in the next column, and the goal becomes to calculate the number of odd length intervals (denote it by d_i for i-th column).

Let $2^{n-1} \le m < 2^n$ and $m = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_p}$ is the binary presentation of number m, where $n - 1 = k_1 > k_2 > \cdots > k_p \ge 0$.

Since $m \ge 2^{n-1}$, then the i-th column for $i = 1, \cdots, n-1$ will create 2^i intervals. These intervals will have the length approximately equal to

$$2^{k_1 - i} + 2^{k_2 - i} + \cdots + 2^{k_p - i}. \tag{1}$$

The same time, the n-th column constructs the m intervals of length 1.
Consider the following cases:

- $k_p - i > 0$

  the i-th column has no intervals of odd length, therefore $s_{i+1} = m - s_{i+1} = m/2$.

- $k_p - i = 0$

  the i-th column will have 2^i odd length intervals, therefore $s_{i+1} - (m - s_{i+1}) = 2^i$ and $s_{i+1} = \frac{m + 2^i}{2}$.

- $k_1, k_2, \cdots, k_j \ge i$ and $k_{j+1}, \cdots, k_p < i$

$$\underbrace{2^{k_1 - i} + \cdots + 2^{k_j - i}}_{\substack{k_j - i \ge 0 \\ (I)}} + \underbrace{2^{k_{j+1} - i} + \cdots + 2^{k_p - i}}_{\substack{k_{j+1} - i < 0 \\ (II)}} \tag{2}$$

1. $k_j - i = 0$,

   Component (I) in (2) provides that each of 2^i intervals of i-th stage is at least of length $2^{k_1 - i} + \cdots + 2^{k_j - i} = q$ (which is odd) and the component (II) - that the length of $2^{k_{j+1} - i} + \cdots + 2^{k_p - i}$ share of 2^i intervals is $q + 1$. Therefore the number of even intervals equals

$\left(\sum_{l=j+1}^{p} 2^{k_l - i}\right) \cdot 2^i = \sum_{l=j+1}^{p} 2^{k_l}$ and the number of odd length intervals is equal to $d_i = 2^i - \sum_{l=j+1}^{p} 2^{k_l} = 2^{k_j} - \sum_{l=j+1}^{p} 2^{k_l}$

Hence: $$s_{i+1} = \frac{\sum_{l=1}^{p} 2^{k_l} + 2^{k_j} - \sum_{l=j+1}^{p} 2^{k_l}}{2} = \left(\sum_{l=1}^{j-1} 2^{k_l - 1}\right) + 2^{k_j} \quad (3)$$

for $i = k_j$ and the same formula holds for each k_j, $j = 1,2,\cdots,p$.

2. $k_j - i > 0$,

in this case component (I) in (2) provides that each of 2^i intervals of i-th stage is of even length and the component (II) - that the $2^{k_{j+1} - i} + \cdots + 2^{k_p - i}$ share of 2^i intervals is of odd length.

The number of odd length intervals is equal to

$$d_i = \left(\sum_{l=j+1}^{p} 2^{k_l - i}\right) \cdot 2^i = \sum_{l=j+1}^{p} 2^{k_l} \text{ and } s_{i+1} = \frac{\sum_{l=1}^{p} 2^{k_l} + \sum_{l=j+1}^{p} 2^{k_l}}{2} = \left(\sum_{l=1}^{j} 2^{k_l - 1}\right) + \left(\sum_{l=j+1}^{p} 2^{k_l}\right) \quad (4)$$

for $i \neq k_j$ and the same formula holds for each k_j, $j = 1,2,\cdots,p.$.

Thus (3) and (4) describe column sums of matrices composed by interval bisection method by the given m. Comparing this column sums to the coordinates of characteristic vector of monotone Boolean function $\mu(m,n)$ points out that the $(0,1)$-matrix A, which corresponds to the set of all true-vectors of monotone Boolean function $\mu(m,n)$ has the same set of column sums as the matrix – constructed by the bisection method.

Conclusion

Resuming, - the n-cube subsets and partitioning (Set Systems and characterization by inclusion of a particular element) in specific boundary cases, and the bisection strategy characterization are strongly similar having the same characteristic numerical descriptors, simply related to the binary representations of the set sizes.

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Authors' Information

Hasmik Sahakyan – Institute for Informatics and Automation Problems, NAS Armenia, P.Sevak St. 1, Yerevan-14, Armenia; e-mail: hasmik@ipia.sci.am

Levon Aslanyan – Institute for Informatics and Automation Problems, NAS Armenia, P.Sevak St. 1, Yerevan-14, Armenia; e-mail: lasl@sci.am

OPTIMAL CONTROL OF A SECOND ORDER PARABOLIC HEAT EQUATION

Mahmoud Farag, Mainouna Al-Manthari

***Abstract**: In this paper, we are concerned with the optimal control boundary control of a second order parabolic heat equation. Using the results in [Evtushenko, 1997] and spatial central finite difference with diagonally implicit Runge-Kutta method (DIRK) is applied to solve the parabolic heat equation. The conjugate gradient method (CGM) is applied to solve the distributed control problem. Numerical results are reported.*

***Keywords**: Distributed control problems, Second order parabolic heat equation, Runge-Kutta method, CGM.*

***ACM Classification Keywords**: F.2.1 Numerical Algorithms and Problems; G.4 Mathematical Software*

Introduction

In the recent years, optimal control of systems governed by partial differential equations have been extensively studied. We refer for instance to [Lions, 1971], [Farag, 2004] for parabolic problems and to [Wu,2003], [Borzi, 2002] for numerical studies. In this paper, we are concerned with the optimal control boundary control of a second order parabolic heat equation. Using the results in [Evtushenko, 1997] and spatial central finite difference with diagonally implicit Runge-Kutta method of order 2 in 2 stages is applied to solve the parabolic heat equation. The conjugate gradient method (CGM) is applied to solve the distributed control problem. Numerical results are reported.

Consider the second order heat equation

$(1)\qquad \dfrac{\partial y(x,t)}{\partial t} = a^2 \dfrac{\partial^2 y(x,t)}{\partial x^2} + u(x,t),(x,t) \in \Omega = (0,l)\, x\, (0,T)$ where $y(x,t)$ is the temperature at time t and at a point x and $u(x,t)$ is a distributed control.

The initial and boundary conditions are given by

$(2)\qquad y(x,0) = \varphi(x),\ x \in [0,l],$

$(3)\qquad \dfrac{\partial y(0,t)}{\partial x} = 0,\ \dfrac{\partial y(l,t)}{\partial x} = \nu\,[g(t) - y(l,t)],\ t \in (0,T)$ where $g(t)$ is a boundary control.

The problem is to find control functions $u(x,t)$ and $g(t)$ that minimize the cost functional

$$(4)\qquad J = \int_0^l \Phi(y(s,T))\, ds$$

where Φ is continuously differentiable with respect to its argument.

DIRK Method

In this section we present some basic results about the Runge-Kutta methods, the diagonally implicit Runge-Kutta method of order 2 in 2 stages (DIRK). The reader is referred to [Alexander, 1977], [Shamardan, 1998].