ON THE COHERENCE BETWEEN PROBABILITY AND POSSIBILITY MEASURES[1]

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Abstract: The purpose of this paper is to study possibility and probability measures in continuous universes, taking different line to the one proposed and dealt with by other authors. We study the coherence between the probability measure and the possibility measure determined by a function that is both a possibility density and distribution function. For this purpose, we first examine functions that satisfy this condition and then we anlyze the coherence in some notable probability distributions cases.

Keywords: Measure, possibility, probability.

ACM Classification Keywords: I.2.3 Artificial Intelligence: Deduction and Theorem Proving (Uncertainty, "fuzzy" and probabilistic reasoning); I.2.4 Artificial Intelligence: Knowledge Representation Formalisms and Methods (Predicate logic, Representation languages).

Introduction

Possibility distributions are, sometimes, good means for representing incomplete crisp information. It is precisely this incompleteness that often makes it impossible to determine a probability that could describe this information. Now, if the possibility distribution meets certain requirements, for example, it is either a density function or its graph "encloses" a finite area, it will always be possible to consider either the probability whose density function is this possibility distribution or an associated density function.

Fuzzy set-based possibility theory was introduced by L. Zadeh in 1978 (see [12]) and provided an alternative non-classical means, other than probability theory, of modeling and studying "uncertainty". Zadeh established in [12] the principle of consistency between possibility and probability, according to which "anything that is probable must be possible". This principle is expressed as "$\mathrm{P}(A) \leq \Pi(A)$", and the probability P could also be said to be coherent with the possibility Π. The finite case has been studied by M. Delgado and S. Moral in [4], where they characterize the probabilities that are coherent with a given possibility; also in [2] Castiñeira *et al.* deepened in that case defining a distance between possibility an probability measures, finding the closest probability to a given possibility and proving they are coherent. The case of continuous universes has been addressed by several authors, including Dubois *et al*, who, in [7], examined possibility/probability transformations taking into account the principle of insufficient reason from possibility to probability, and the principle of maximum specificity from probability to possibility. Although dealing with the same subject, the purpose of this paper is another. As density functions are to probabilities what possibility distributions are to possibility measures and, taking into account that a density function whose value is 1 at some point determines both a probability measure and a possibility measure, we set out to analyze the coherence between these probability and possibility measures.

This paper is organized as follows: After a background section, in section 2, we prove that a possibility generates a degenerated probability defined on a σ-algebra, as in the finite case where the coincident probabilities and possibilities were degenerated. In section 3, some functions are obtained which are both possibility distributions and density functions; particularly, some classic distributions have been considered, then we address the problem of coherence between possibilities and probabilities generated by the same function. Some counterexamples show that, even in these cases, the coherence between measures cannot be guaranteed. Finally, in section 4, we deal with the coherence between some classical probability distributions and their respective possibility measures, stressing the case of the normal law, where there exists coherence.

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1. Preliminaries

Let F(E) be the set of all fuzzy sets on $E \neq Ø$, with the partial order $\subset$ defined by: $A \subset B$ if and only if $\mu_A(x) \leq \mu_B(x)$ for all $x \in E$ being $\mu_A, \mu_B \in [0,1]^E$ the membership functions of A and B, respectively. We will consider the standard fuzzy sets theory (F(E), ∪, ∩, ^{c}) associated with the t-norm min, the t-conorm max and a strong negation N (see [11]), that is, $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$, $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$, and $\mu_{Pc}(x) = N(\mu(x))$. A family A$\subseteq$ F(E) is an algebra if it verifies: $Ø \in$ A, if $A \in$ A then $A^c \in$ A, and if $A,B \in$ A then $A \cup B \in$ A.

Definition 1.1. A $\subset$-measure in an algebra A$\subseteq$ F(E) is any function M: A $\rightarrow [0,1]$ such that: m$_1$) $M(Ø) = 0$; m$_2$) M(E)=1; and m$_3$) If $A \subset B$, then $M(A) \leq M(B)$.

Definition 1.2. A possibility (see [5] and [6]) in an algebra A$\subseteq$ F(E) is any mapping Π: A$\rightarrow [0,1]$ satisfying: p$_1$) $\Pi(E) = 1$; p$_2$) $\Pi(Ø) = 0$; and p$_3$) $\Pi(A \cup B) = \mathrm{Max}(\Pi(A), \Pi(B))$ for any $A,B \in$ A.

It is easy to check that any possibility Π is a $\subset$-measure. Furthermore, if $\mu \in [0,1]^E$ is such that $\sup\{\mu(x), x \in E\} = 1$, then the function Π_μ : F(E) $\rightarrow$ $[0,1]$ defined for all $A \in$ F(E) by $\Pi_\mu(A) = \sup\{Min(\mu(x), \mu_A(x)), x \in E\}$ is a possibility measure in F(E). The function μ is called *possibility distribution* of the Π_μ. Note that for all classic set $A \in$ P(E), where P(E) is the set of parts of E, the possibility measure given by the possibility distribution μ is defined by $\Pi_\mu(A) = \sup\{\mu(x), x \in A\}$.

Definition 1.3. Let M_1 and M_2 be two $\subset$-measures in an algebra A$\subseteq$ F(E), M_1 is *coherent* with M_2 if $M_1(A) \leq M_2(A)$ for all $A \in$ A.

As the purpose of this paper is to compare possibility and probability measures, we will consider the possibilities as being restricted to classic sets, that is, to σ–algebras A$\subseteq$P(E). Recall that A is a σ–algebra if for any $A \in$ A its complement $A^c \in$ A, and for any countable family $\{A_n\}_{n \in N} \subseteq$ A it is $\bigcup_{n \in N} A_n \in$ A. Moreover, the set function P: A $\rightarrow [0,1]$ is a probability measure if $P(E) = 1$ and P is σ -additive, that is, for any $\{A_n\}_{n \in N} \subseteq$ A such that $A_n \cap A_m = Ø$ if $n \neq m$, then $P\left(\bigcup_{n \in N} A_n\right) = \sum_{n \in N} P(A_n)$ holds.

From section 3 on we will consider Borel's σ-algebra in R, that is, the smallest σ-algebra that contains the semi-ring $\{[a,b); a,b \in R \text{ with } a < b\}$, or alternatively, the smallest σ-algebra that contains the open sets of R, and which is usually denoted by B. It is well known that every probability measure P: B $\rightarrow [0,1]$ is univocally determined by a distribution function, F: R $\rightarrow [0,1]$ ([10]), and if $F'(x) = f(x)$ exists for "almost any" point, then $P([a,b]) = \int_a^b f(x)dx$ (*). Generally, if f: R $\rightarrow$ R$^+$ is such that $\int_{-\infty}^{\infty} f(x)dx = 1$ (that is, f is a density function), f defines, as in (*), a probability measure on Borel's algebra of R. Note that, pursuant to the theorems of measure extension, every probability in (R,B) is determined by ascertaining its values in the intervals $[a,b]$.

2. Probability generated by a Possibility Measure

Let $\mu \in [0,1]^R$ such that $\sup\{\mu(x), x \in R\} = 1$ and let us consider the possibility measure generated by μ on the crisp sets of R, that is, Π_μ : P(R)$\rightarrow [0,1]$ defined for each $A \in$ P(R) by $\Pi_\mu(A) = \sup\{\mu(x), x \in A\}$, then Π_μ verifies: 1) $\Pi_\mu(Ø) = 0$; 2) Monotonicity: if $A \leq B$ then $\Pi_\mu(A) \leq \Pi_\mu(B)$; and 3) Subadditivity: $\Pi_\mu\left(\bigcup_{n \in N} A_n\right) \leq \sum_{n \in N} \Pi_\mu(A_n)$. Therefore, Π_μ is an exterior measure in R and also verifies that $\Pi_\mu(R) = 1$.

It is known that any exterior measure M generates a σ-additive measure on the σ-algebra of the M–measurable sets (see [9], [10]) according to:

Caratheodory's Theoreme: If M: $\mathrm{P}(E) \rightarrow [0,+\infty]$ is an exterior measure in a set $E \neq \emptyset$, then the family $\mathcal{A}=\{A \in \mathrm{P}(E);\ \forall X \in \mathrm{P}(E),\ \mathrm{M}(X)= \mathrm{M}\ (X \cap A\ )+ \mathrm{M}\ (X \cap A^c)\}$ is a σ-algebra and the restriction of M to $\mathcal{A}$ is a σ-additive measure.

The Caratheodory´s method applied to the exterior measure Π_μ generates a degenerated probability as follows:

Theoreme 2.1. Let $\mu \in [0,1]^E$ such that $\sup\ \{\mu(x), x\in \mathrm{R}\} =1$, then the family of Π_μ-measurable sets is $\mathcal{A} = \{ A\in \mathrm{P}(\mathrm{R}), \mathrm{supp}(\mu)\subset A$ or $A\subset (\mathrm{supp}(\mu))^c\}$, where $\mathrm{supp}(\mu)= \{x\in \mathrm{R},\ \mu(x)\neq 0\}$ is the support of μ, and the possibility measure Π_μ restricted to the Π_μ -measurable sets is a degenerated probability defined for each $A\in \mathcal{A}$ by $\Pi_\mu(A)=0$, if $A\subset (\mathrm{supp}(\mu))^c$, and $\Pi_\mu(A)=1$ if $\mathrm{supp}(\mu)\subset A$.

Proof: Let us see that $\mathcal{A}$ is the σ-algebra constructed by Caratheodory's method.

$\mathcal{A}$ is a σ-algebra trivially. The elements of $\mathcal{A}$ are Π_μ-measurable; indeed, if $\mathrm{supp}(\mu)\subset A$, for each $X\subset \mathrm{R}$,

$$\sup_{x\in X}\mu(x) = \sup_{x\in (A\cap X)\cup(A^c\cap X)}\mu(x) = \max\left(\sup_{x\in A\cap X}\mu(x),\ \sup_{x\in A^c\cap X}\mu(x)\right) = \sup_{x\in A\cap X}\mu(x) = \Pi_\mu(A\cap X) = \Pi_\mu(A\cap X)+\Pi_\mu(A^c\cap X)$$

holds, as $\Pi_\mu(A^c \cap X) = 0$. Similarly, if $A\subset(\mathrm{supp}(\mu))^c$, we could prove that A is Π_μ-measurable.

Furthermore, we will prove that the only Π_μ-measurable elements are elements of $\mathcal{A}$: If $A\subset \mathrm{R}$ is Π_μ-measurable, then, in particular, $1= \Pi_\mu(\mathrm{R})=\Pi_\mu(A)+ \Pi_\mu(A^c)$ (•) holds, and two options can be analyze:

1) There exists $x_0\in \mathrm{R}$ such that $\mu(x_0)=1$. If, moreover, $x_0\in A$ it follows from (•) that $\Pi_\mu(A^c)=0$, which means that $A^c\subset (\mathrm{supp}(\mu))^c$ and, therefore, $\mathrm{supp}(\mu)\subset A$ and $A\in \mathcal{A}$. Similarly, if $x_0\in A^c$, it is $A\subset (\mathrm{supp}(\mu))^c$ and $A\in \mathcal{A}$.
2) For all $x\in \mathrm{R}$, $\mu(x)<1$. In this case, μ reaches its supreme value at $+\infty$ or $-\infty$, and this point of infinity is an accumulation point of A, $x\in A'$, or of A^c. Let us suppose that $x\in A'$, then $\Pi_\mu(A)=1$, and it follows from (•) that $\Pi_\mu(A^c)=0$, which means that, again, $\mathrm{supp}(\mu)\subset A$ and $A\in \mathcal{A}$. If the point of infinity at which μ reaches the supreme is an accumulation point of A^c, it follows, similarly, that $A^c\subset (\mathrm{supp}(\mu))^c$ and $A\in \mathcal{A}$.

Finally, the values of Π_μ on elements of $\mathcal{A}$ follow from the definition of Π_μ. □

3. Possibility and Probability Measures generated by a Density Function and their Coherence

We will address the coherence of measures in a continuous universe when the possibility and probability are determined by the same function, that is, a possibility distribution in the first instance and a density function in the second one. For this purpose, a first section analyzes how this type of functions can be derived from a given density function and, then, from a given possibility distribution. The second section deals with the coherence between a possibility and a probability both generated by a given density function.

3.1. Possibility Distributions and Density Functions

In this section, some conditions for a function to be a density function and a possibility distribution at the same time are stated; moreover the cases of some notable distributions are analyzed.

Lemma 3.1. If f: $\mathrm{R} \rightarrow [0,+\infty]$ is a bounded density function, then the function $\mu_f : \mathrm{R} \rightarrow [0,1]$ defined for each $x\in \mathrm{R}$ by $\mu_f(x) = kf(kx)$, where $k=1/\sup\{f(x), x\in \mathrm{R}\}$, is a density function and a possibility distribution function. Additionally, if f is continuous, then there exists $y_0\in \mathrm{R}$ such that $\mu_f(y_0) = 1$.

Proof: μ_f is a density function. Indeed, $\int_{-\infty}^{\infty}\mu_f(x)dx = \int_{-\infty}^{\infty} f(kx)d(kx) = 1$. It is also a possibility distribution, since $0 \le \mu_f(x) \le \sup\{\ \mu_f(x),\ x\in \mathrm{R}\}=k\ \sup\{f(kx),\ x\in \mathrm{R}\}= 1$.

Finally, if f is continuous, there exists $x_0\in \mathrm{R}$ such that $f(x_0) = \sup\{f(x),\ x\in \mathrm{R}\}=1/k$; hence, it suffices to consider $y_0= x_0/k$, since then $\mu_f(x_0/k) = 1$. □ μ_f will be said to be the possibility distribution associated with f.

Some examples: The possibility distributions associated with some well-known probability distributions are listed below (for more details about these distributions, see [3]).

(a) Normal distribution of parameters α, σ, $N(\alpha,\sigma)$: Its density function is $f(x) = (1/\sigma\sqrt{2\pi})e^{-(x-\alpha)^2/2\sigma^2}$ with maximum $f(\alpha)=1/(\sigma\sqrt{2\pi})$, then $\mu_f(x)=\sigma\sqrt{2\pi}f(\sigma\sqrt{2\pi}x)=e^{-(\sigma\sqrt{2\pi}x-\alpha)^2/2\sigma^2}$. In particular, when $\sigma=1/\sqrt{2\pi}$, $\mu_f(x) = f(x) = e^{-\pi(x-\alpha)^2}$ which is a density function for the normal distribution $N(\alpha,1/\sqrt{2\pi})$.

(b) Cauchy distribution of parameters a,b: Its density function is $f(x)=\dfrac{a}{\pi(a^2+(x-b)^2)}$ whose maximum, reached in b, is $f(b)=1/(a\pi)$; hence, its associated possibility distribution is

$$\mu_f(x) = a\pi f(a\pi x) = \frac{a^2}{a^2+(a\pi x - b)^2}$$

If $b=0$, then $\mu_f(x) = 1/(1+\pi^2x^2)$, and its probability distribution is a Cauchy distribution with $a = 1/\pi$.

(c) Gamma distribution of parameters $p>0, a>0, \Gamma(p,a)$: Its density function is $f(x)=\dfrac{a^p}{\Gamma(p)}x^{p-1}e^{-ax}$ if $x>0$, and $f(x)=0$ if $x \le 0$, where $\Gamma(p) = \int_0^{+\infty} e^{-x}x^{p-1}dx$ is the second-class Euler's function.

Note, firstly, that if $p \in (0,1)$, then f is not bounded and, therefore, there is no associated possibility distribution. When $p=1$, it is also a particular case of the exponential distribution that will be dealt with in the following example. If $p>1$, the function is bounded, reaching its maximum value in $x = \dfrac{p-1}{a}$, and its associated possibility distribution can be ascertained. It will be calculated for two particular cases so as to avoid tedious calculations. If $p=2$, then $f(x)=a^2xe^{-ax}$ if $x>0$, and $f(x)=0$ if $x \le 0$, and its maximum is $f(1/a)=a/e$; therefore, the associated possibility distribution is $\mu_f(x)=e^2xe^{-ex}$ if $x>0$, and $\mu_f(x)=0$, if $x \le 0$, which is also a density function for the distribution $\Gamma(2,e)$. If $p=3$, we get the law $\Gamma(3,e^2/2)$.

(d) Exponential Distribution of parameter θ: Its density function is $f(x)=\theta e^{-\theta x}$ if $x \ge 0$, and $f(x)=0$ if $x<0$, whose maximum is $f(0)=\theta$. Hence, the associated possibility distribution is $\mu_f(x)=e^{-x}$ if $x \ge 0$, and $\mu_f(x)=0$, if $x<0$, which is a density function for the exponential distribution with $\theta=1$ or also for $\Gamma(1,1)$. ◁

The inverse problem of getting a density function that is also a possibility distribution from another possibility distribution is easily solved if this distribution "encloses" a finite area, as shown in the following result.

Lemma 3.2. Let $\mu \in [0,1]^{\mathbb{R}}$ such that $\int_{\mathbb{R}} \mu(x)dx = A < +\infty$ and let us suppose that there exists $x_0 \in \mathbb{R}$ such that $\mu(x_0)=1$, then $f_\mu(x) = \mu(A(x-x_0)+x_0)$ with $x \in \mathbb{R}$ is a density function and also a possibility distribution.

Proof: Let $\alpha(x) = A(x-x_0)+x_0$, then $\int_{-\infty}^{+\infty} f_\mu(x)dx = \int_{-\infty}^{+\infty} \mu(\alpha(x))dx = \frac{1}{A}\int_{-\infty}^{+\infty}\mu(\alpha(x))d(\alpha(x)) = 1$ holds.

Therefore, μ is a density function. Additionally, $f_\mu(x_0) = \mu(x_0) = 1$, which means that f_μ is also a possibility distribution. □ f_μ will be said to be a density function associated with μ.

Note that there are many density functions associated with a function μ under the above conditions. Indeed, $f_\mu(x) = Ax$ and all its translations would also be density functions. The fact that we considered the translation to x_0 is really a practical matter, as if μ. reaches the value 1 at a single point x_0, then the graph of f_μ is obtained by "squashing" the graph of μ. and leaving the fixed point $(x_0,1)$, which would mean that it would be "most like" the original μ.

Example: The function $\mu(x)=e^{-|x|}$, with $x\in\mathbb{R}$, is a possibility distribution, since $\mu\in[0,1]^{\mathbb{R}}$ and $\mu(0)=1$, but it is not a density function, as $\int_{-\infty}^{+\infty}e^{-|x|}dx=2$. However, associated density functions can indeed be found: $f_\mu(x)=e^{-2|x|}$ and its translations.

3.2 Coherence between Possibility and Probability

Let $\mu\in[0,1]^{\mathbb{R}}$ such that $\int_{\mathbb{R}}\mu(x)dx=1$ and $\sup_{x\in\mathbb{R}}\mu(x)=1$. Let Π_μ be the generated possibility by μ and P_μ the probability with density function μ. Our aim is to study when $P_\mu\le\Pi_\mu$, that is, when P_μ is coherent with Π_μ. The following result shows that there is "local coherence" with the possibility for "small" subsets.

Proposition 3.3. Let $A\in\mathcal{B}$ such that $\mathcal{L}^1(A)\le 1$, where $\mathcal{L}^1$ designates the Lebesgue measure in $\mathbb{R}$; then for any $\mu\in[0,1]^{\mathbb{R}}$ such that $\int_{\mathbb{R}}\mu(x)dx=1$ and $\sup_{x\in\mathbb{R}}\mu(x)=1$, it is $P_\mu(A)\le\Pi_\mu(A)$.

Proof: $P_\mu(A)=\int_A\mu(x)dx\le\sup_{x\in A}\mu(x)\cdot\mathcal{L}^1(A)\le\Pi_\mu(A)$. □

Generally, it cannot be guaranteed that $P_\mu(A)\le\Pi_\mu(A)$ for any $A\in\mathcal{B}$, as shown by the following examples.

Pareto distribution of parameters a,x_0: Its density function is $f(x)=\dfrac{a}{x_0}\left(\dfrac{x_0}{x}\right)^{a+1}$ if $x\ge x_0$, and $f(x)=0$ if $x<x_0$, and its associated possibility function taking $x_0=a$ is $\mu_f(x)=(a/x)^{a+1}$ if $x>a$, and $\mu_f(x)=0$ if $x<a$. Then, for each $b>a$, $P_{\mu_f}((b,+\infty])=\int_b^{+\infty}\left(\dfrac{a}{x}\right)^{a+1}dx=\left(\dfrac{a}{b}\right)^{a}>\left(\dfrac{a}{b}\right)^{a+1}=\Pi_{\mu_f}((b,+\infty])$.

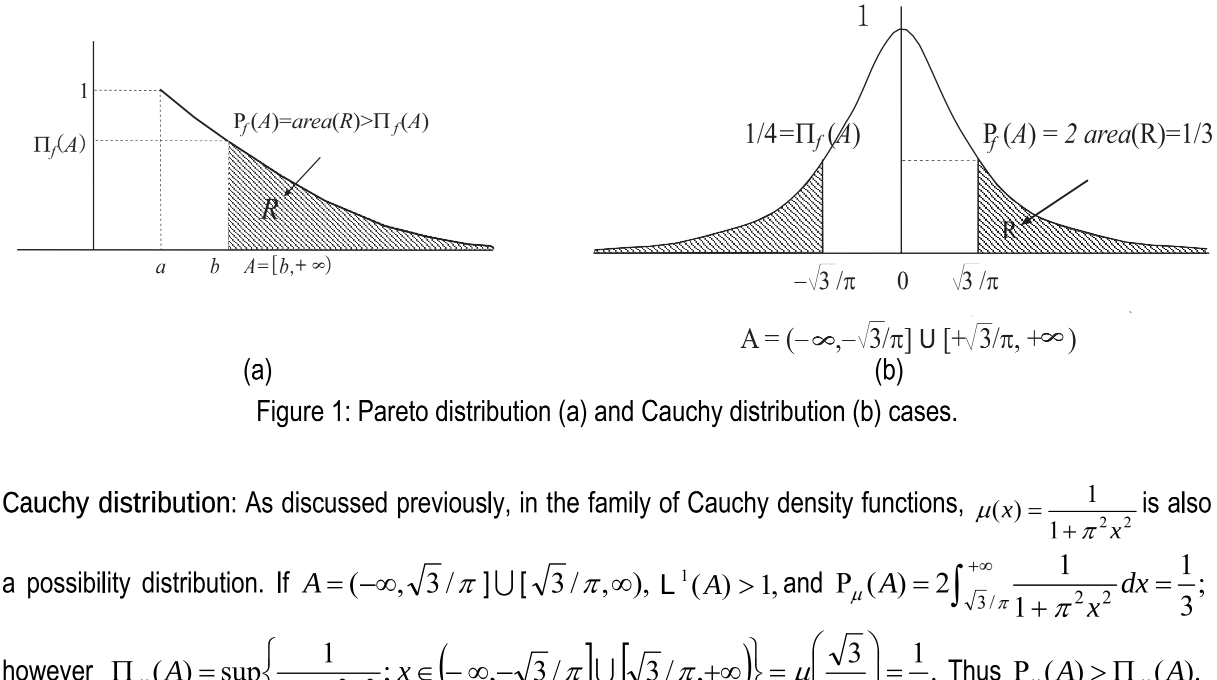


Figure 1: Pareto distribution (a) and Cauchy distribution (b) cases.

Cauchy distribution: As discussed previously, in the family of Cauchy density functions, $\mu(x)=\dfrac{1}{1+\pi^2x^2}$ is also a possibility distribution. If $A=(-\infty,\sqrt{3}/\pi]\cup[\sqrt{3}/\pi,\infty)$, $\mathcal{L}^1(A)>1$, and $P_\mu(A)=2\int_{\sqrt{3}/\pi}^{+\infty}\dfrac{1}{1+\pi^2x^2}dx=\dfrac{1}{3}$; however $\Pi_\mu(A)=\sup\left\{\dfrac{1}{1+\pi^2x^2};x\in\left(-\infty,-\sqrt{3}/\pi\right]\cup\left[\sqrt{3}/\pi,+\infty\right)\right\}=\mu\left(\dfrac{\sqrt{3}}{\pi}\right)=\dfrac{1}{4}$. Thus $P_\mu(A)>\Pi_\mu(A)$.

4. A Survey of the Coherence in some notable Distributions Cases

In this section, we deal with the coherence between some notable distributions and the possibility measures generated by the density functions of the above distributions.

4.1. Coherence and Normal Distribution

Bearing in mind how important the normal distribution is, this section is given over to studying the coherence between the probability and possibility generated by its density function.

As discussed in section 3.1, it holds that the density functions of the distributions $N(\alpha, 1/\sqrt{2\pi})$, with $\alpha \in \mathrm{R}$, are also possibility distributions; furthermore, they are the only ones within the normal family, as it should hold that

$$\sup_{x \in \mathrm{R}} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\alpha)^2}{2\sigma^2}} = \frac{1}{\sigma\sqrt{2\pi}} = 1$$

then necessarily has to be $\sigma = 1/\sqrt{2\pi}$.

Theorem 4.1. Let f be the density function of the normal distribution $N(\alpha, 1/\sqrt{2\pi})$, if Π_f and P_f are, respectively, the possibility and probability measures generated by f, then $\mathrm{P}_f(A) \le \Pi_f(A)$ for all $A \in \mathrm{B}$.

Proof: It can be proven, without loss of generality, for $f(x) = e^{-\pi x^2}$ which corresponds to $N(0, 1/\sqrt{2\pi})$, since any of the others is a translation of this one, and the relationship between probability and possibility will be the same.

Firstly, it is $\mathrm{P}_f((-\infty,-a] \cup [a,+\infty)) \le \Pi_f((-\infty,-a] \cup [a,+\infty))$ for all $a \ge 0$: if $a \ge 1$, $e^{-\pi x^2} \le x e^{-\pi x^2}$ for any $x \ge a$, then $\mathrm{P}_f((-\infty,-a] \cup [a,+\infty)) = 2\int_a^{+\infty} e^{-\pi x^2} dx \le 2\int_a^{+\infty} x e^{-\pi x^2} dx = \frac{e^{-\pi a^2}}{\pi} < f(a) = \Pi_f((-\infty,-a] \cup [a,+\infty))$.

If $a \in [0,1)$, the function $G(a) = f(a) - \mathrm{P}_f((-\infty,-a] \cup [a,+\infty)) = e^{-\pi a^2} - 2\int_a^{+\infty} e^{-\pi x^2} dx$ is non-negative. Indeed, from $G'(a) = -2a\pi e^{-\pi a^2} - 2\frac{d}{da}\left(\int_0^{+\infty} e^{-\pi x^2} dx - \int_0^a e^{-\pi x^2} dx\right) = 2e^{-\pi a^2}(-a\pi + 1)$ it follows that G is increasing in $[0, 1/\pi)$ and decreasing in $(1/\pi, 1]$, moreover as $G(0) = 0$ and

$G(1) = e^{-\pi} - 2\int_1^{+\infty} e^{-\pi x^2} dx \ge e^{-\pi} - 2\int_1^{+\infty} e^{-\pi x} dx = e^{-\pi}\left(1 - \frac{2}{\pi}\right) > 0,$ then $G(a) \ge 0$ for all $a \in [0,1]$.

Finally, let us see that $\mathrm{P}_f(A) \le \Pi_f(A)$ for any $A \in \mathrm{B}$. If 0 is an accumulation point of A, then $\mathrm{P}_f(A) \le 1 = f(0) = \Pi_f(A)$. If 0 is not an accumulation point of A, then there exists $a > 0$ such that $A \subset (-\infty,-a] \cup [a,+\infty)$ and a or $-a$ is either an element of A or an accumulation point of A. Therefore,

$$\mathrm{P}_f(A) \le \mathrm{P}_f((-\infty,-a] \cup [a,+\infty)) \le \Pi_f((-\infty,-a] \cup [a,+\infty)) = f(a) = \Pi_f(A). \square$$

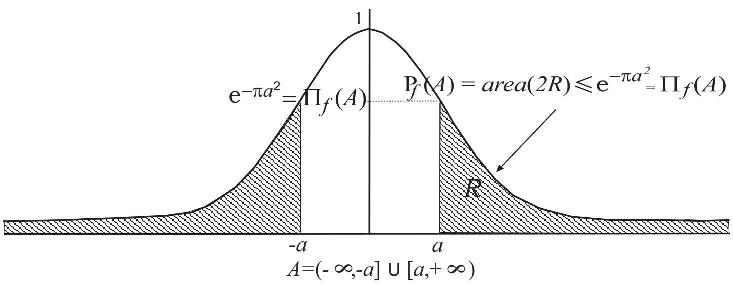


Figure 2: Density function of the normal distribution.

4.2. Coherence and other Distributions

Even though important distributions, like the Cauchy distribution, do not generate coherent probabilities and possibilities, we can find other common distributions, apart from the important case of the normal distribution, which also generate coherent probabilities and possibilities. Let us take a look at some of these.

1. Uniform distribution, with density function $f(x) = 1$ if $|x - a| \le 1/2$ and $f(x) = 0$ if $|x - a| > 1/2$. Trivially, $P_f(A) \le \Pi_f(A)$ is satisfied for any $A \in \mathcal{B}$, since $\int_A f(x)dx = \mathcal{L}^1(A \cap [a - 1/2, a + 1/2])$.

2. Simpson's distribution, with density function $f(x) = 1 - |x - a|$ if $|x - a| \le 1$ and $f(x) = 0$ if $|x - a| > 1$. Let $A \in \mathcal{B}$, if a is an accumulation point of A, then $\Pi_f(A) = 1 \ge P_f(A)$. If a is not an accumulation point of A, there exists $\varepsilon \in (0,1)$ such that $A \subset (-\infty, a - \varepsilon] \cup [a + \varepsilon, +\infty)$ and $\Pi_f(A) = f(a + \varepsilon) = f(a - \varepsilon) = 1 - \varepsilon$. Therefore, $P_f(A) \le P_f((-\infty, a - \varepsilon] \cup [a + \varepsilon, +\infty)) = (1 - \varepsilon)^2 < 1 - \varepsilon = \Pi_f(A)$.

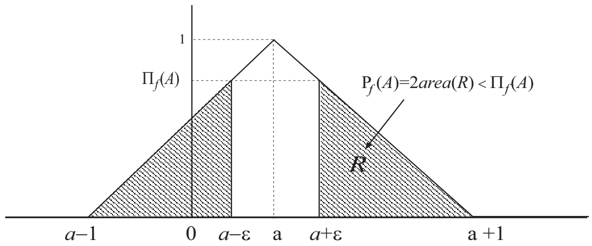
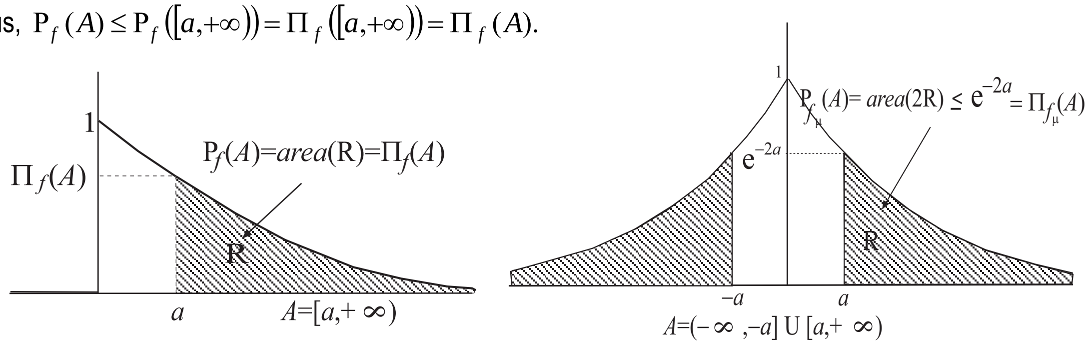


Figure 3: Density function of Simpson's distribution.

3. Exponential distribution, with density function $f(x) = e^{-x}$ if $x \ge 0$, and $f(x) = 0$ if $x < 0$. For each $a \in \mathbb{R}$:

- If $a \ge 0$, it is $P_f([a, +\infty)) = \int_a^{+\infty} e^{-x}dx = e^{-a} = \Pi_f([a, +\infty))$.
- If $a < 0$, it is $P_f([a, +\infty)) = \int_a^{+\infty} e^{-x}dx = 1 = \Pi_f([a, +\infty))$.

For each $A \in \mathcal{B}$, there exists $a \in \mathbb{R}$ such that $A \subset [a, +\infty)$ and $a \in A$ or a is an accumulation point of A; thus, $P_f(A) \le P_f([a, +\infty)) = \Pi_f([a, +\infty)) = \Pi_f(A)$.



(a) (b)

Figure 4: (a) Density function of the exponential distribution, and (b) density function $f_\mu(x) = e^{-2|x|}$

4. Finally, going back to the example in section 3.1, let $f_\mu(x) = e^{-2|x|}$ be the density function associated with the possibility distribution $\mu(x) = e^{-|x|}$. The probability and possibility measures generated by f_μ are also coherent. Indeed, for all $a \ge 0$, it is $P_{f_\mu}((-\infty, a] \cup [a, +\infty)) = 2\int_a^{+\infty} e^{-2x}dx = e^{-2a} = f_\mu(a) = \Pi_{f_\mu}((-\infty, a] \cup [a, +\infty))$, from which we can deduce, just as we did for the normal law, that for all $A \in \mathcal{B}$, $P_{f_\mu} A) \le \Pi_{f_\mu}(A)$.

Conclusions and further Works

In this paper, we have discussed the topic of the coherence between probability and possibility measures in the continuous case, that is, when these measures are defined on σ-algebras in the set $\mathbb{R}$ of real numbers. For this purpose, we have firstly found functions that are density functions and possibility distributions at the same time and, then we have studied the coherence between probability and possibility measures generated by the same density function. Moreover, the case of some significant distributions has been analysed.

The problem of finding the closest probability to a given possibility is an interesting open problem, technically more complex than in the finite case which was successfully accomplished in [2].

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