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RADON MEASURES ON BANACH SPACES WITH THEIR WEAK TOPOLOGIES

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1. Introduction. The main concern of this paper is to present some improvements to results on the existence or non-existence of countably additive Borel measures that are not Radon measures on Banach spaces taken with their weak topologies, on the standard axioms (ZFC) of set-theory. However, to put the results in perspective we shall need to say something about consistency results concerning measurable cardinals.

We shall use the term *Borel measure* for a countably additive finite non-negative set function defined on the Borel sets of a topological space. A Borel measure μ on a topological space X is said to be a *Radon measure* if, for each Borel set B in X and each $\varepsilon > 0$, there is a compact set K contained in B with $\mu(K) > \mu(B) - \varepsilon$. A Borel measure μ on a topological space X will be said to be *inner regular* if, for each *open* set G in X and each $\varepsilon > 0$, there is a closed set F contained in G with $\mu(F) > \mu(G) - \varepsilon$.

A simple argument shows that *each Borel measure on a complete separable metric space is automatically a Radon measure.* We give a proof in Section 3. For a rather different proof see Royden [31], Proposition 18, page 411.

Following Marczewski and Sikorski [27], we shall say that a cardinal κ is of *measure zero*, if the only Borel measure on the discrete space with cardinal κ that assigns

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the value 0 to each individual point is the zero measure. Combining the elementary result italicised above with a theorem of Marczewski and Sikorski [27], we obtain the following theorem (see, for example, Gardner and Pfeffer [13], Theorem 11.10).

Theorem 1 (MS). *If X is a complete metric space, with a dense subset with cardinal of measure zero, then each Borel measure on X is a Radon measure.*

Note that when X is a metric space the condition that X has a dense subset with cardinal of measure zero is equivalent to the condition that X has a open base for its topology with cardinal of measure zero and to the condition that every discrete subset of X has cardinal of measure zero. In particular, we find that: *in a complete metric space X, every Borel measure is a Radon measure, if, and only if, each discrete subset of X has a cardinal of measure zero.*

This theorem has an immediate corollary (see, for example, Talagrand [34], Section 16-2-5).

Corollary 2 (MS). *Let E be a Banach space with a dense subset with cardinal of measure zero, and suppose that $(E, norm)$ and $(E, weak)$ have the same Borel sets. Then each Borel measure on $(E, weak)$ is also a Borel measure on $(E, norm)$ and is a Radon measure on both these spaces.*

We remark that there are many conditions that can be imposed on a Banach space to ensure that the weak Borel sets and the norm Borel sets coincide. We mention some of these conditions; full definitions for these and other concepts will be given in Section 2 below. If a Banach space is weakly compactly generated, then it is a K-analytic set in its weak topology. If a Banach space is K-analytic in its weak topology, then it has an equivalent locally uniformly convex norm. A locally uniformly convex norm is a Kadec norm. If a Banach space has a Kadec norm, then the weak Borel sets and the norm Borel sets coincide. There is an extensive renorming theory for Banach spaces that has yielded much information concerning Banach spaces that admit locally uniformly convex norms. For further details see the subsection on Banach spaces in Section 2 below.

The next result is well-known; Talagrand [34], see his Section (2-3-4), attributes it to Phillips and Grothendiek.

Theorem 3 (PG). *If E is a Banach space, a Radon measure on $(E, weak)$ extends from the weak Borel sets to the norm Borel sets to form a Radon measure on $(E, norm)$.*

Existing proofs of this result are rather complicated. We outline an alternative proof, based on the proof of Theorem 4.1 of [18], without claiming that it is particularly simple.

The following version of Choquet's capacitability theorem [2] is appropriate for our purposes.

Theorem 4 (C). *Let X be a K-analytic space. Then each inner regular Borel measure on X is a Radon measure.*

This yields immediately

Corollary 5 (C). *Let E be a Banach space that is K-analytic in its weak topology. Then each inner regular Borel measure on $(E, weak)$ is a Radon measure on $(E, weak)$ and also a Radon measure on $(E, norm)$.*

The absence of any condition on the cardinality of the dense subsets of the spaces in these results is partially explained by the fact that a K-analytic space is always a Lindelöf space and so contains no uncountable closed discrete subset. Dieudonné's example, see [15], Exercise 10 on page 231 (see acknowledgments at the beginning of the book), of a Borel measure that is not a Radon measure on the compact space of all ordinals up to ω_1, with the order topology, shows the necessity of the condition of inner regularity in Theorem 4 (C). The situation in Corollary 5 is illustrated by the example of the Banach space $c_0(\Gamma)$ with Γ a discrete space with uncountable cardinal. This space is weakly compactly generated (by the weakly compact set consisting of the origin together with the vectors of the standard basis) and so is weakly K-analytic. Thus $(c_0(\Gamma), weak)$ contains no *closed* uncountable discrete subset, but it does contain a discrete subset (not closed) of cardinal equal to that of Γ, and this set can be closed and discrete in $(c_0(\Gamma), norm)$. Further $(c_0(\Gamma), weak)$ contains no dense set of cardinal less than Γ. If it were possible to take Γ to be a discrete space with a cardinal not of measure zero, then there would be a Borel measure on $(c_0(\Gamma), norm)$ and on $(c_0(\Gamma), weak)$ that was neither a Radon measure on $(c_0(\Gamma), weak)$ nor on $(c_0(\Gamma), norm)$. See also some further remarks in Section 3.

We now turn our attention to the study of Banach spaces E for which $(E, weak)$ admits the construction of a Borel measure that is not Radon. Such Banach spaces are easy to find, if one has a cardinal that is not of measure zero. However, there may be no such cardinals, and in their absence it is difficult to find such Banach spaces. The first example is due to Talagrand [33] and [34], Section (16-1-2).

Theorem 6 (T). *Let Γ be an uncountable discrete space and let $\ell_c^\infty(\Gamma)$ be the Banach space of all bounded real-valued functions of countable support on Γ with the* *supremum norm. Then there is a Borel measure on* $(\ell_c^\infty(\Gamma), weak)$ *that is not a Radon measure.*

A second example is due to de Maria and Rodriguez-Salinas [3]. We write $\mathbb{N} = \{1, 2, \ldots\}$ and denote the Stone-Čech compactification of $\mathbb{N}$ by $\beta\mathbb{N}$.

Theorem 7 (dMR-S). *Let* $C(\beta\mathbb{N}\backslash\mathbb{N})$ *be the Banach space of all continuous functions on the compact set* $\beta\mathbb{N}\backslash\mathbb{N}$ *with the supremum norm. Then there is a Borel measure on* $(C(\beta\mathbb{N}\backslash\mathbb{N}), weak)$ *that is not a Radon measure.*

De Maria and Rodriguez-Salinas obtain their result by combining some new ideas with Talagrand's method.

The main aim of this paper is to give some general criteria on a compact Hausdorff space K that ensure that $(C(K), weak)$ admits a Borel measure that is not a Radon measure. We prove three theorems.

Theorem 8. *Let* K *be a compact Hausdorff space with a non-empty family* $\mathcal{D}$ *of non-empty proper clopen subsets with the two following properties.*

(a) The union of any increasing sequence of members of $\mathcal{D}$ *is properly contained in a member of* $\mathcal{D}$.

(b) If $S_1, S_2, \ldots$ *and* $T_1, T_2, \ldots$ *are two increasing sequences of clopen sets, all contained in a fixed set of* $\mathcal{D}$*, with*

$$S_n \cap T_n = \emptyset, \quad \textit{for} \quad n \geq 1,$$

then there are disjoint clopen sets S_0 *and* T_0 *with*

$$S_n \subset S_0 \quad \textit{and} \quad T_n \subset T_0, \quad \textit{for} \quad n \geq 1.$$

Then $(C(K), weak)$ *admits a Borel measure that is not a Radon measure.*

This theorem is proved by use of Talagrand's method. We show that Talagrand's Theorem 6 (T) can be obtained as a consequence. We obtain a second consequence of Theorem 8.

Theorem 9. *Let* K *be an infinite compact Hausdorff space that is a totally disconnected* F*-space with the property that each non-empty zero set in* K *contains some infinite open set. Then* $(C(K), weak)$ *admits a Borel measure that is not a Radon measure.*

We verify that Theorem 7 (dMR-S) is a consequence of Theorem 9. We also obtain a generalisation of de Maria and Rodriguez-Salinas' theorem.

Theorem 10. *Let X be a locally compact Hausdorff space that is not pseudo-compact. Then $C(\beta X \backslash X)$ admits a Borel measure that is not a Radon measure.*

Before we embark on the difficult proofs of Theorems 6 to 11 we give in Section 4 a simple proof of the following analogue of Theorem 6.

Theorem 11. *Let Γ be an uncountable discrete space and let $\ell_c^\infty(\Gamma)$ be the Banach space of all bounded real-valued functions of countable support on Γ with the supremum norm. Then there is a Borel measure on $\ell_c^\infty(\Gamma)$ taken with the topology of pointwise convergence on Γ that is not a Radon measure.*

In fact, all the Borel measures that are not Radon measures constructed in Theorems 6 to 11, are Borel measures taking only the values 0 and 1, assigning the value 0 to each point and the value 1 to the whole space.

We remark that, *if a Hausdorff space X contains a discrete subset D (not necessarily closed) whose cardinal is not of measure zero, then there is a Borel measure μ on X that is not a Radon measure.* The measure μ can be obtained by extending to X the non-zero Borel measure on D assigning measure 0 to each individual point of D, provided by the assumption that the cardinal of D is not of measure zero.

An open question concerns the relationship, if any, between the concept of σ-fragmentability of a Banach space, a concept that we have recently studied with Namioka, see [16], [18], [19], [20], [21], [22], [23], [24], and the existence on the Banach space of weak Borel measures that are not weak Radon measures. Until recently our ignorance of these two concepts matched exactly and left open the possibility that a Banach space is σ-fragmentable in some sense, if, and only if, all the weak Borel measures are Radon measures. Now Holický and Pelant [17] have given an example, on the assumption that there is no real meassurable cardinal, of a Banach space, which is not σ-fragmented by any metric, but all the weak Borel measures are Radon meassures.

The above results leave unanswered difficult questions concerning the Borel measures on $(\ell^\infty, weak)$. Since the points of ℓ^∞ are separated by a countable family of weakly continuous functions it is easy to show that there can be no such measure taking only the values 0 or 1, the points having measure 0 and the whole space having measure 1. On the other hand, ℓ^∞ contains a norm discrete family of points of cardinal $2^{\aleph_0}$, and so, if $2^{\aleph_0}$ is not of measure zero, then $(\ell^\infty, norm)$ and so also $(\ell^\infty, weak)$ admits a Borel measure that is not a Radon measure. Indeed it may be possible to construct a Borel measure on $(\ell^\infty, weak)$ that is not a Radon measure, without any measurable cardinal assumption. In this direction, Talagrand [34], Theorem 16-3-3, see also [11], has constructed a Baire measure on $(\ell^\infty, weak)$ that is not a Radon measure. It is not known if this Baire measure can be extended to a Borel measure.

We conclude this introduction by quoting some results concerning cardinals of measure zero and the two types of measurable cardinals. A cardinal $\kappa > \aleph_0$ is said to be *real-valued measurable*, if there exists a Borel measure on the discrete space $\Gamma(\kappa)$ of cardinal κ, that assigns the value 0 to each point of $\Gamma(\kappa)$ assigns the value 1 to $\Gamma(\kappa)$ and is additive over any disjoint family of cardinal less than κ, of sets of $\Gamma(\kappa)$, see, for example, Drake [6], page 177. It is known, see [6], Theorem 1.4, page 176, that if there is a cardinal that is not of measure zero, then the smallest such cardinal is a real-valued measurable cardinal. A cardinal $\kappa > \aleph_0$ is said to be *measurable*, if there is a Borel measure on the discrete space $\Gamma(\kappa)$ of cardinal κ, that takes only the values 0 and 1, and assigns the value 0 to each point of $\Gamma(\kappa)$ and assigns the value 1 to $\Gamma(\kappa)$ and is additive over any disjoint family of cardinal less than κ of sets of $\Gamma(\kappa)$, see [6], page 177.

As is well-known, Gödel proved that the assertion "$V = L$" is consistent with the axioms (ZFC). Further, it is known that "$V = L$" implies that there is no measurable cardinal, see, for example, [6], Theorem 2.10, page 184. Again, "$V = L$" implies that there are no real-valued measurable cardinals, so that all cardinals are of measure zero, see, for example, [26], Lemma 27.7, page 303, and [6], Theorem 1.3, page 174. Thus it is consistent with ZFC to assume that all cardinals are of measure zero.

However the status of the question of whether or not the existence of a measurable cardinal is consistent with ZFC is rather different. Note that the existence of a measurable cardinal implies the existence of an inaccessible cardinal, see [26], Lemma 27.2, page 298. Now it follows that Gödel's second incompleteness theorem implies that it can not be shown by methods that are formalizable in ZFC that the existence of inaccessible cardinals is consistent with ZFC, see [26], Theorem 27 and its explanation on page 86.

We are grateful to a referee, who made many penetrating comments that have enabled us to improve the presentation of this article.

2. Notation and Conventions. In this section we give a brief summary of our notation and conventions.

2.1. Measure Theory. We have already in the introduction given the definitions of *Borel measures*, *inner regular* Borel measures and *Radon* measures. A Borel measure μ is said to be *τ-additive* if, whenever $\mathcal{U}$ is a family of open sets with the property

'if U_1, U_2 belong to $\mathcal{U}$ there is a U_3 in $\mathcal{U}$ with $U_1 \cup U_2 \subset U_3$'.

then

$$\sup\{\mu(U) : U \in \mathcal{U}\} = \mu\left(\bigcup\{U : U \in \mathcal{U}\}\right).$$

It is easy to verify that a Radon measure is automatically τ-additive. A Borel measure on a Hausdorff space that takes only the values 0 and 1 and assigns the value 0 to each individual point and the value 1 to the whole space is necessarily non-τ-additive.

If μ is a Borel measure on a Hausdorff space X, a set S in X is said to be *a support* for μ if $\mu(S) = \mu(X)$. If there is a minimal element in the family of closed supports for μ, this minimal element is unique and is called *the support* of μ. If μ is τ-additive, it has a minimal closed support.

We remark that many results that hold for Borel measures on metric spaces, fail for general topological spaces, but do hold for τ-additive Borel measures on topological spaces (see, for example, [13],[12]).

2.2. Topological spaces. A topological space X is said to be *totally disconnected* if, whenever x and y are distinct points, there is a clopen (i.e. both closed and open) set that contains one point but not the other. A completely regular Hausdorff space X is said to be *strongly zero-dimensional*, if whenever A and B are separated, in that there is a continuous function on X taking the value 0 on A and the value 1 on B, there is a clopen set containing A without meeting B (Engelking [9] gives a different definition but proves that it is equivalent to this one). A compact Hausdorff space is totally disconnected, if, and only if, it is strongly zero-dimensional.

A completely regular Hausdorff space X is said to be an *F-space* if each function defined, bounded and continuous on a cozero subset of X can be extended to a bounded continuous function on X. For equivalent definitions and a whole range of examples see Gillman and Jerison [14].

A topological space X is said to be *pseudocompact,* if each continuous function defined on X is bounded.

A set-valued function F from a topological space Y to a topological space X, that is a map from Y to the power set of X, is said to be *upper semi-continuous* if, for each η in Y and each open set G in X containing $F(\eta)$, the set of y in Y with $F(y) \subset G$ is open in Y. A space X is said to be *K-analytic* if is a Hausdorff space that is the image of complete separable metric space (that can always be taken to be

$\mathbb{N}^{\mathbb{N}}$) under a compact-valued upper semi-continuous set-valued map. For an account of such spaces see, for example, [25].

2.3. Banach Spaces. When K is a compact Hausdorff space we use $C(K)$ to denote the space of continuous real-valued functions on K with the supremum norm. The dual space of $C(K)$ is the space of differences of Radon measures on K, taken with the total variation norm. If Γ is a discrete set, $\ell^{\infty}(\Gamma)$ denotes the Banach space of bounded real-valued functions on Γ with the supremum norm, and we identify $\ell^{\infty}(\Gamma)$ in the natural way with $C(\beta\Gamma)$, $\beta\Gamma$ being the Stone-Čech compactification of Γ. Of course the usual space ℓ^{∞} is identified with $\ell^{\infty}(\mathbb{N}) = C(\beta\mathbb{N})$. We also use $\ell_c^{\infty}(\Gamma)$ to denote the Banach sub-space of $\ell^{\infty}(\Gamma)$ consisting of the function of $\ell^{\infty}(\Gamma)$ having countable support.

A Banach space E is said to be *weakly compactly generated*, if E is the norm closure of the linear span of a weakly compact subset. A Banach space E is said to be weakly K-analytic if $(E, weak)$, is a K-analytic space. A Banach space norm $\|\cdot\|$ is said to be *locally uniformly convex* if, for each point f of E and each sequence $\{g_n\}$ of points of E, the convergence of $\|g_n\|$ to $\|f\|$ together with the convergence of $\|f + g_n\|$ to $2\|f\|$ entail the convergence of g_n to f in norm. A Banach space norm $\|\cdot\|$ is said to be a *Kadec norm* if the restrictions of the norm topology and of the weak topology to the unit sphere $\{g : \|g\| = 1\}$ coincide.

It is easy to verify that any separable Banach space and any reflexive Banach space is weakly compactly generated. Talagrand [32] proves that any weakly compactly generated Banach space is, when naturally embedded in its second dual, a weak* $K_{\sigma\delta}$-set and so is weakly K-analytic; see [25], Section 2.11, for a simple proof. Now Vašák [37], building on Amir and Lindenstrauss's construction of long sequences of projections [1] and on Troyanski's renorming theorem [35], showed that a weakly K-analytic Banach space has an equivalent locally uniformly convex norm. Note that, when Γ is a set of arbitrary cardinality, Day's norm on $c_0(\Gamma)$ is locally uniformly convex, see [4], Chapter 4. It follows directly from the definitions that a locally uniformly convex norm is a Kadec norm. Edgar [7], Theorem 1.1, or [8] Corollary 2.4, shows that, when a Banach space admits a Kadec norm, the norm Borel sets coincide with the weak Borel sets (see also Section 3 below).

3. Borel Measures that are Radon Measures. In this section we will be concerned with circumstances when Borel measures can be shown to be Radon measures. We outline some of the steps that lead to some of the results mentioned in the introduction.

It is well-known that a totally bounded complete metric space is compact. See, for example, Engelking [9], Theorem 4.3.29. As Engelking remarks, this was proved by Fréchet in 1910, see [10]. Now suppose that μ is a Borel measure on a complete separable metric space X. For each $n \geq 1$, the separable metric space X can be covered by a sequence of closed balls of diameter less than $1/n$. Let $\varepsilon > 0$ be given. Since μ is a Borel measure, we can choose a set F_n that is the union of a finite subfamily of the countable family of closed balls of diameter less than $1/n$ covering X, and satisfying

$$\mu\,(X \backslash F_n) < \varepsilon 2^{-n-1}.$$

Then $K = \bigcap_{n=1}^{\infty} F_n$ is a totally bounded complete separable metric space and so is compact. Further

$$\mu\,(X \backslash K) \leq \sum_{n=1}^{\infty} \mu\,(X \backslash F_n) < \frac{1}{2}\varepsilon.$$

Since X is a metric space, μ is inner regular, in the strong sense, that for each Borel set B in X there is a closed set F contained in B with

$$\mu\,(F) > \mu\,(B) - \frac{1}{2}\varepsilon.$$

Now $F \cap K$ is a compact set contained in B with

$$\mu(F \cap K) > \mu\,(B) - \varepsilon.$$

Thus μ is a Radon measure. We have outlined a proof that

(a) *a Borel measure on a complete separable metric space is a Radon measure.*

For another proof of this result see Royden [31], Proposition 18, page 411.
The result of Marczewski and Sikorski [27] lies rather deeper. It asserts that

(b) *a Borel measure on a metric space X that has a dense subset having cardinal of measure zero, has a minimal closed support that is separable.*

We split the proof of (b) into the proofs of two assertions.

(c) *If a Borel measure on a metric space X has a minimal closed support, that support is separable.*

(d) *If a Borel measure is defined on a metric space X that has a dense subset having cardinal of measure zero, then it has a minimal closed support.*

Clearly (c) and (d) together imply (b), and (a) and (b) together imply Theorem 1 (MS) stated in Section 1.

We first prove (c). Let F be a minimal closed support for a Borel measure μ defined on a metric space X. Since F is a minimal closed support for μ, whenever G is an open set with $G \cap F \neq \emptyset$, we must have $\mu(G) > 0$, otherwise $F \backslash G$ would be a closed support for μ strictly contained in F. For each $n \geq 1$, choose a maximal set D_n of points in F with the distances between the points at least $2/n$. Then the closed balls with radii $2/n$ centred on the points of D_n cover F. Further, the open balls with radii $1/n$ centred at the points of D_n are disjoint, and each meets F at a point of D_n, and so has positive μ-measure. Thus D_n is countable, for each $n \geq 1$, and $\bigcup_{n=1}^{\infty} D_n$ is a countable dense set in F. Thus F is separable.

We now outline the proof of (d). Let μ be a Borel measure on a metric space X that has a dense subset having a cardinal of measure zero. Choose in X a transfinite sequence $\{x_\alpha : 0 \leq \alpha < \gamma\}$ that is dense in X, with γ an ordinal whose cardinal is of measure zero. We ignore the case when γ is finite. Since X is metric and γ is infinite we can choose a family $\{B_\alpha : 0 \leq \alpha < \gamma\}$ of open sets forming a base for the topology of X. Let A be the set of ordinals α, $0 \leq \alpha < \gamma$, for which

$$\mu(B_\alpha) = 0.$$

Let $\{N_\alpha : 0 \leq \alpha < \beta\}$, with $\beta \leq \gamma$, be a wellordering of the sets $\{B_\alpha : \alpha \in A\}$. Write

$$N = \bigcup \{N_\alpha : 0 \leq \alpha < \beta\}.$$

We note that if an open set G in X meets $X \backslash N$, then G contains a basic open set, B say, that meets $X \backslash N$ and so is not one of the sets $\{N_\alpha : 0 \leq \alpha < \beta\}$. This ensures that

$$\mu(G) \geq \mu(B) > 0.$$

We now use, and later prove the following simple lemma of Montgomery [28], Lemma 1.

(e) *Let $\{O_\alpha : 0 \leq \alpha < \beta\}$ be a transfinite sequence of open sets in a metric space. Write*

$$D_\alpha = O_\alpha \backslash \bigcup \{O_\gamma : 0 \leq \gamma < \alpha\}$$

for $0 \leq \alpha < \beta$ and let H_α be a relatively closed subset of D_α for $0 \leq \alpha < \beta$. Then

$$\bigcup \{H_\alpha : 0 \leq \alpha < \beta\}$$

is an $\mathcal{F}_\sigma$-set.

We apply this result to the family $O_\alpha = N_\alpha$, $0 \le \alpha < \beta$. For Θ any subset of the ordinal β we take

$$\begin{aligned} H_\alpha &= D_\alpha, \quad \text{if} \quad \alpha \in \Theta, \\ H_\alpha &= \emptyset, \quad \text{if} \quad \alpha \notin \Theta. \end{aligned}$$

Then, for each Θ the set

$$\bigcup \{H_\theta : \theta \in \Theta\}$$

is an $\mathcal{F}_\sigma$-set. We regard the ordinals δ with $0 \le \delta < \beta$ as a discrete set Δ. It is easy to verify that the formula

$$\nu(\Theta) = \mu\left(\bigcup\{H_\delta : \delta \in \Theta\}\right)$$

for all subsets Θ of Δ, defines a Borel measure on Δ. Further, for each δ in Δ,

$$\nu(\{\delta\}) = \mu(H_\delta) \le \mu(N_\delta) = 0.$$

The cardinal of Δ is at most that of γ and so is of measure zero. Hence

$$\mu(N) = \nu(\Delta) = 0.$$

Now $F = X \backslash N$ is a closed set in X with

$$\mu(X \backslash F) = 0, \quad \mu(F) = \mu(X).$$

Thus F is a closed support for μ. If F' were any proper closed subset of F, then $G = X \backslash F'$ would be an open set meeting $F = X \backslash N$ so that $\mu(G)$ would be positive and $\mu(F')$ would be less than $\mu(F)$. Hence F is a minimal closed support for μ.

To prove the result (e). For each $n \ge 1$ and $0 \le \alpha < \beta$ let $H_\alpha^{(n)}$ be the set of points of H_α whose distance from $X \backslash O_\alpha$ is at least $1/n$. Hence each set $H_\alpha^{(n)}$ is a relatively closed subset of D_α. Further, if $0 \le \gamma < \delta < \beta$, the distance between the points of $H_\gamma^{(n)}$ and $X \backslash O_\gamma$ is at least $1/n$. Since $H_\delta^{(n)} \subset X \backslash O_\gamma$ the distance between the $\mathcal{F}_\sigma$-sets $H_\gamma^{(n)}$ and $H_\delta^{(n)}$ is at least $1/n$. Thus the family

$$\left\{H_\alpha^{(n)} : 0 \le \alpha < \beta\right\}$$

is a discrete family of $\mathcal{F}_\sigma$-sets with $\mathcal{F}_\sigma$-union. Hence

$$\bigcup \{H_\alpha : 0 \le \alpha < \beta\} = \bigcup_{n=1}^{\infty} \bigcup \left\{H_\alpha^{(n)} : 0 \le \alpha < \beta\right\}$$

is an $\mathcal{F}_\sigma$-set in X.

Proof of Corollary 2 (MS). Let μ be a Borel measure on $(E, weak)$. When the Borel sets of $(E, weak)$ coincide with those on $(E, norm)$, the measure μ is a Borel measure on $(E, norm)$ and so is a Radon measure on $(E, norm)$ by the theorem. Since each norm compact set is weakly compact, it follows that μ is a Radon measure on $(E, weak)$.

In order to apply Corollary 2 (MS) to Banach spaces with locally uniformly convex norms or with Kadec norms we prove that the weak and the norm Borel sets of E coincide when E has a Kadec norm. We follow the proof of Edgar [7], Theorem 1.1, but we give more details and show that each norm open set is a countable union of differences of weakly closed sets. Another proof of the equality of the two Borel families, due to W. Schachermayer, is given by Edgar in [8].

We suppose that $\|\cdot\|$ is a Kadec norm on the Banach space E. It will be convenient to use

$$\begin{aligned} I\,(y;r) &= \{x : \|x-y\| < r\}\,, \\ B\,(y;r) &= \{x : \|x-y\| \le r\}\,, \\ S\,(y;r) &= \{x : \|x-y\| = r\}\,, \end{aligned}$$

to denote the open ball, the closed ball and the sphere with centre y and radius $r > 0$ in E.

Our first objective is to prove that, if y is any point of E with $\|y\| = 1$ and $\varepsilon > 0$, then y does not belong to the weak closure

$$\mathrm{cl}_w \{B\,(0;1) \setminus B\,(y;\varepsilon)\}\,.$$

Now

$$S\,(0;1) \bigcap I\left(y;\frac{1}{2}\varepsilon\right)$$

is an open subset of $S(0;1)$ in the norm topology on $S(0;1)$ and so also in the weak topology on $S(0;1)$. Hence we can choose a weakly open neighbourhood V of 0 in E so that

$$S(0;1) \cap (V+y) \subset S(0;1) \cap I(y;\frac{1}{2}\varepsilon).$$

Since the weakly open sets that are convex and symmetrical in 0 form a base for the weak neighbourhoods of 0, we may take V to be such a convex symmetrical weakly open neighbourhood of 0. Now we can choose a $\delta > 0$ with $\delta < \frac{1}{2}\varepsilon$ and

$$B(0;\delta) \subset \frac{1}{2}V.$$

Since $\|y\| = 1$, the point y is not in the weakly closed set $B(0;1-\delta)$. Hence

$$W = \left(y + \frac{1}{2}V\right) \backslash B(0;1-\delta)$$

is weakly open set containing y. We show that all points x of

$$W \cap B(0;1)$$

lie in $B(y;\varepsilon)$. For x in $W \cap B(0;1)$ we have

$$1 - \delta < \|x\| \leq 1 \quad \text{and} \quad x - y \in \frac{1}{2}V.$$

Write

$$x' = x/\|x\|.$$

Then

$$\begin{aligned}\|x - x'\| &= \left\|x - \frac{x}{\|x\|}\right\| \\ &= \|x\|^{-1}\|(\|x\| - 1)x\| \\ &= |\|x\| - 1| \\ &< \delta.\end{aligned}$$

Thus

$$x' = (x' - x) + (x - y) + y \in B(0;\delta) + \frac{1}{2}V + y \subset \frac{1}{2}V + \frac{1}{2}V + y = V + y.$$

Hence

$$x' \in (V + y) \cap S(0;1) \subset I\left(y;\frac{1}{2}\varepsilon\right).$$

Since $x - x' \in B(0;\delta) < B\left(0;\frac{1}{2}\varepsilon\right)$, this yields $x \in B(y;\varepsilon)$. Thus

$$x \in B(y;\varepsilon).$$

Hence W is a weakly open set containing y and not meeting

$$B(0;1) \backslash B(y;\varepsilon).$$

Thus, when $\|y\| = 1$ and $\varepsilon > 0$, the point y is not in

$$\mathrm{cl}_w \{B(0;1) \backslash B(y;\varepsilon)\}.$$

Consider any weak open set U in E. Then U is also norm open. It follows that every weak Borel set in E is a norm Borel set in E.

On the other hand let U be a norm open subset of E. We prove that U is a countable union of differences of weakly closed subsets of E. Once proved this will show that each norm Borel set in E is also a weak Borel set. The case when $U = E$ is trivial. So we may suppose that U is a proper subset of E. Further, after a translation, we may suppose that $0 \notin U$. Consider the set

$$U' = \cup \{B(0;r) \cap \mathrm{int}_w (U \cup E \backslash B(0;r)) : r > 0, r \text{ rational}\},$$

where we use int_w to denote the weak interior of a set. Clearly U' is a countable union of differences of weakly closed sets. Further $U' \subset U$. We verify that in fact $U' = U$. Consider any point y in U. Then we can choose $\varepsilon > 0$ so that

$$B(y;\varepsilon) \subset U.$$

Since $\|y\| \neq 0$, the result of the main paragraph implies that

$$y \notin \mathrm{cl}_w \left\{B(0;\|y\|) \backslash B\left(y;\frac{1}{2}\varepsilon\right)\right\}$$

and so

$$y \in A = \mathrm{int}_w \left\{B\left(y;\frac{1}{2}\varepsilon\right) \cup \{E \backslash B(0;\|y\|)\}\right\}.$$

Since A is weakly open, for some rational $r > \|y\|$ we have

$$y \in (r/\|y\|) A,$$

and

$$(r/\|y\|) B\left(y;\frac{1}{2}\varepsilon\right) \subset B(y;\varepsilon) \subset U.$$

Thus

$$y \in (r/\|y\|) A = \mathrm{int}_w \left\{(r/\|y\|) B\left(y;\frac{1}{2}\varepsilon\right) \cup \{E \backslash B(0;r)\}\right\}.$$

Since $\|y\| < r$ we have $y \in B(0;r)$ and so $y \in U'$. Hence $U = U'$, as required.

We now turn our attention to Theorem 3 (PG). Let E be a Banach space and let μ be a Radon measure on $(E, weak)$. Then we can choose an increasing sequence $K_0 = \emptyset, K_1, K_2, \ldots$ of weakly compact sets with

$$\lim_{n\to\infty} \mu(K_n) = \mu(E).$$

For each $i \geq 1$, define set function μ_i on the weakly Borel sets B, by the formula

$$\mu_i(B) = \mu(B \cap K_i \backslash K_{i-1}).$$

Then μ_i is a Radon measure on $(E, weak)$ supported by the weakly compact set K_i for each $i \geq 1$.

We next prove the following weak version of Theorem 3 (PG).

(f) *If μ is a Radon measure on $(E, weak)$ that is supported by a weakly compact set, then μ extends from the weak Borel sets to form a Radon measure on $(E, norm)$.*

Consider a Radon measure μ on $(E, weak)$ that is supported by a weakly compact set K. Let F be the norm closure in E of the linear span of K. Then F is a Banach subspace of E that inherits both its norm and its weak topologies from E. Since F is weakly compactly generated, it has an equivalent Kadec norm and, consequently, the weak and norm Borel sets coincide on F. Thus μ is already defined on the norm Borel sets of F and can be extended to a norm Borel measure on E by taking

$$\hat{\mu}(B) = \mu(B \cap F),$$

for each norm Borel subset B of E.

Being a Radon measure on $(F, weak)$, μ has a minimal weakly closed support, say L. Then L is necessarily contained in the weakly compact set K, and so is weakly compact. Now, we have $\mu(L \cap G) > 0$ whenever G is a weakly open set that meets L. Now L is fragmented by the norm, see, for example, Namioka [30] and [29]. This means that for each $\delta > 0$, each non-empty subset of L has a non-empty relatively weakly open subset of diameter less than δ. In particular, we can choose a weakly open set G with $L \cap G$ a non-empty set of diameter less than δ. Let H be the weak closure of $L \cap G$. Then the diameter of H, being equal to that of $L \cap G$, is less than δ. Further, H, being a subset of K, is weakly compact. Now

$$\mu(H) \geq \mu(L \cap G) > 0.$$

Having established the existence of at least one weakly compact subset of H with positive μ-measure and diameter less than δ, we may consider a maximal disjoint family $\mathcal{H}$ of such weakly compact sets H contained in L with positive μ-measure and diameter less than δ. Write

$$J = \bigcup \{H : H \in \mathcal{H}\}\,.$$

Since μ is a finite measure, $\mathcal{H}$ is necessarily countable and so J is a weak Borel set. We prove that

$$\mu\,(J) = \mu\,(L)\,.$$

Clearly $\mu\,(J) \leq \mu\,(L)$. If we had $\mu(J) < \mu(L)$ then $L \backslash J$ would be a weak Borel set of positive μ-measure, and so would contain a weakly compact set, say M, of positive μ-measure. Applying to the restriction $\mu_{|M}$ of μ to M, the argument we have applied to μ on K, we could find within M, which is within $L \backslash J$, a weakly compact set of positive μ-measure and diameter less than δ. This would contradict the maximality of the family $\mathcal{H}$. Thus

$$\mu\,(J) = \mu\,(L)$$

as required.

Now, for each integer $n \geq 1$, we can choose a set J_n that is a countable union of weakly compact sets each of diameter less than $1/n$ with

$$J_n \subset L \quad \text{and} \quad \mu\,(J_n) = \mu\,(K)\,.$$

Now

$$C = \bigcap_{n=1}^{\infty} J_n$$

is weak Borel set with a countable dense set and with $\mu\,(C) = \mu\,(K)$. Thus the norm closure of C is a complete separable metric space supporting the norm Borel measure $\hat{\mu}$. By the result (a) obtained above, $\hat{\mu}$ is a Radon measure on $(E, norm)$.

Applying this result to the measures μ_i introduced above we verify that

$$\hat{\mu} = \sum_{i=1}^{\infty} \hat{\mu}_i$$

is the extension of μ to form a Radon measure on $(E, norm)$.

We now consider Theorem 4 (C), which is a simplified version of Choquet's capacitability theorem. Let X be a K-analytic space. Let μ be an inner regular Borel measure on X and let $\varepsilon > 0$ be given. Then X is a Hausdorff space and

$$X = K\left(M\right),$$

with K a compact valued upper semi-continuous set-valued map from a complete separable metric space M. For each $n \geq 1$, we can choose a sequence $F\left(n, m\right)$, $m = 1, 2, \ldots$ of closed subsets of M of diameter less than $1/n$, covering M. Now

$$K\left(\bigcup_{k=1}^{m} F\left(n, k\right)\right), \ m = 1, 2, \ldots$$

is an increasing sequence of K-analytic sets with union X. Since K-analytic sets are μ-measurable, see, for example [25], Theorem 2.5.2 and Corollary 2.9.3, we can choose $m\left(n\right)$ so large that, on writing

$$F_n = \bigcup_{k=1}^{m(n)} F\left(n, k\right),$$

we have

$$\mu\left(X \backslash K\left(F_n\right)\right) < \varepsilon 2^{-n}.$$

Since each set F_n is a finite union of closed sets of diameter less than $1/n$, the set

$$F = \bigcap_{n=1}^{\infty} F_n$$

is a totally bounded complete metric space and so is compact. So $K\left(F\right)$, being the upper semi-continuous image in X of a compact subset of M, is compact in X. Now

$$\mu\left(X \backslash K\left(F\right)\right) \leq \sum_{n=1}^{\infty} \mu\left(X \backslash K\left(F_n\right)\right) < \varepsilon.$$

This shows that μ is a Radon measure on X.

Now suppose that E is a Banach space that is K-analytic in its weak topology, and that μ is an inner regular Borel measure on $(E, weak)$. Theorem 4 (C) tells us that μ is a Radon measure on $(E, weak)$. Further, E has a Kadec norm so that the weak and norm Borel sets on E coincide. By Theorem 3 (PG), μ is a Radon measure on $(E, norm)$. This yields Corollary 5 (C).

To amplify our remarks concerning $c_0(\Gamma)$ we note that $c_0(\Gamma)$ is the space of all bounded real-valued functions f on Γ such that for each $\varepsilon > 0$, $\{\gamma : |f(\gamma)|\}$ is finite, with the supremum norm

$$\|f\| = \sup\{f(\gamma) : \gamma \in \Gamma\}.$$

Let χ_γ be the function of $c_0(\Gamma)$ defined by

$$\chi_\gamma(\gamma) = 1,$$

$$\chi_\gamma(\delta) = 0, \quad \text{if} \quad \delta \in \Gamma,\ \delta \neq \gamma.$$

It is easy to verify that the set $\{\chi_\gamma : \gamma \in \Gamma\}$ is weakly discrete, norm discrete and norm closed. However, as we remarked $c_0(\Gamma)$ is weakly K-analytic and so weakly Lindelöf and so can contain no weakly closed uncountable weakly discrete space.

4. The topology of pointwise convergence. In 1939 Dieudonné [5] showed that the ordinal interval $[0, \omega_1)$ with its order topology provided an example of a normal space that admits no complete uniform structure. In particular, he showed that if A and B are disjoint closed subsets of $[0, \omega_1)$, then at least one of the sets A and B is bounded in the order topology. Starting from this observation, Halmos [15], Exercise 10 on page 231 (see also his references on page 292 and his acknowledgments on page vii, where Halmos attributes this exercise to Dieudonné), observed that the family $\mathcal{H}$ of all closed unbounded subsets of $[0, \omega_1)$ is closed under countable intersections, and deduced that, if B is any Borel set in $[0, \omega_1)$, then either B contains some set H of $\mathcal{H}$ or $[0, \omega_1)\backslash B$ contains such a set (but of course not both). It is easy to check that the measure ν defined for each Borel set B by

$$\nu(B) = 1, \text{ if } H \subset B \text{ for some } H \in \mathcal{H},$$

$$\nu(B) = 0, \text{ if } H \subset [0, \omega_1)\backslash B \text{ for some } H \in \mathcal{H}.$$

As an introduction to the use of Talagrand's methods to prove Theorems 6 to 10, we prove Theorem 11, stated in the introduction, by showing how the construction of the above measure can be lifted from $[0, \omega_1)$ to $\ell_c^\infty(\Gamma)$, when Γ is an uncountable discrete set.

Proof of Theorem 11. With a minor abuse of notation, let Γ also denote the least ordinal with cardinal equal to that of the discrete set Γ. Let F be the subset of $\ell_c^\infty(\Gamma)$ consisting of all bounded functions f on Γ satisfying the conditions:

(a) $f(\gamma) = 0$ or 1 for $0 \le \gamma < \Gamma$; and

(b) if $f(\gamma) = 1$, then $f(\delta) = 1$ when $0 \le \delta < \gamma < \Gamma$.

Here the condition (b) can be rewritten in the form

(b′) $(f(\delta), f(\gamma)) = (0,0), (1,0)$ or $(1,1)$ when $0 \le \delta < \gamma < \Gamma$.

It is now clear that F is a closed subset of the space X taken to be $\ell_c^\infty(\Gamma)$ with the topology of pointwise convergence on Γ. Further, these conditions, together with the condition of countable support, ensure that we have

$$f(\gamma) = 0 \text{ for } \omega_1 \le \gamma < \Gamma,$$

for all f in F.

It will be convenient to index the functions f in F by the ordinals γ less than ω_1. For each γ with $0 \le \gamma < \omega_1$ we use $f^{(\gamma)}$ to denote the function f in F uniquely defined by

$$\begin{aligned} f(\delta) &= 1 \text{ for } 0 \le \delta < \gamma, \\ f(\delta) &= 0 \text{ for } \gamma \le \delta < \Gamma. \end{aligned}$$

Note that, for $0 \le \gamma < \omega_1$, the function $f^{(\gamma)}$ defined in this way is a bounded function with countable support, satisfying the conditions (a) and (b), and so belongs to F. On the other hand, if f is any function in F, the condition of countable support ensures that $f(\delta) = 0$ for some δ with $0 \le \delta < \omega_1$, the conditions (a) and (b) then ensure that $f = f^{(\gamma)}$ with γ the least ordinal δ with $f(\gamma) = 0$. Thus we have a one-to-one mapping $\gamma \leftrightarrow f^{(\gamma)}$ between the countable ordinals and the functions of F.

It will be convenient to say that a subset H of F has no countable bound if, for each γ with $0 \le \gamma < \omega_1$, there is a function $f^{(\delta)}$ in H with $\gamma < \delta$. Following the construction method outlined above, we study the family $\mathcal{H}$ of all subsets H of F that are closed (i.e. pointwise closed) and without any countable bound.

We note that *if $\gamma(1), \gamma(2), \ldots$ is any strictly increasing sequence of ordinals converging to a countable ordinal γ, then $f^{(\gamma)}$ belongs to the closure of the set*

$$\{f^{\gamma(n))} : n \ge 1\}.$$

To verify this, first note that the condition that $\gamma(1), \gamma(2), \ldots$ is a strictly increasing sequence ensures that γ is a limit ordinal. Thus $f^{(\gamma)}$ has

$$\begin{aligned} f(\delta) &= 1 \text{ for } 0 \le \delta < \gamma, \\ f(\delta) &= 0 \text{ for } \gamma \le \delta < \Gamma. \end{aligned}$$

Now any open neighbourhood of $f^{(\gamma)}$ contains a basic open neighbourhood of $f^{(\gamma)}$. The condition that a function f of F lies in such a basic open neighbourhood is of the form

$$\begin{aligned} f(\rho) &= 1 \text{ for a finite number of ordinals } \rho \text{ all less than } \gamma, \\ f(\rho) &= 0 \text{ for a finite number of ordinals } \rho \text{ all at least } \gamma. \end{aligned}$$

This condition is satisfied by all the functions $f^{(\gamma(n))}$ with n so large that $\gamma(n)$ exceeds all the $\rho < \gamma$ with $f(\rho)$ restricted to take the value 1. Thus $f^{(\gamma)}$ belongs to the closure of the set $\{f^{(\gamma(n))} : n \geq 1\}$, as required.

We now show that the family $\mathcal{H}$ is closed under the operation of countable intersection. Let $H_1, H_2, \ldots$ be any countable sequence of sets in $\mathcal{H}$. Let β be a given countable ordinal. Since none of the sets has a countable bound we can find a strictly increasing sequence $\delta(1), \delta(2), \ldots$ of ordinals all exceeding β and all countable so that

$$f^{(\delta(\ell))} \in H_m \text{ with } \ell = 2^n(2m-1).$$

The supremum λ of the sequence $\delta(1), \delta(2), \ldots$ is the limit of each of the sequences

$$\delta(2^n(2m-1)),\ n = 0, 1, \ldots$$

for each $m \geq 1$. Hence the function $f^{(\lambda)}$, with $\lambda > \beta$, is a common point of all the sets $H_m, m \geq 1$. Thus

$$\bigcap_{m=1}^{\infty} H_m \in \mathcal{H}.$$

We next observe that, if M is any closed subset of F, either M has no countable bound and $M \in \mathcal{H}$, or M has a countable bound, say β, and $F \backslash M$ contains the set

$$\{f^{(\gamma)} : \beta \leq \gamma < \omega_1\}$$

which belongs to $\mathcal{H}$.

Take $\mathcal{M}$ to be the family of all subsets M of F such that either M contains a set in $\mathcal{H}$ or $F \backslash M$ contains a set in $\mathcal{H}$. By the remark of the last paragraph, $\mathcal{M}$ contains all closed sets in F. Directly from the definition, $\mathcal{M}$ is closed under complementation. If $M_1, M_2, \ldots$ is any sequence of sets of $\mathcal{M}$, either one at least of the sets contains a set of $\mathcal{H}$ in which case $\bigcup_{n=1}^{\infty} M_n$ contains such a set, or each complementary set $F \backslash M_n, n \geq 1$, contains a set of $\mathcal{H}$ in which case

$$\bigcap_{n=1}^{\infty} (F \backslash M_n) = F \backslash \bigcup_{n=1}^{\infty} M_n$$

contains a set of $\mathcal{H}$. Thus $\mathcal{M}$ is closed under countable unions. Hence $\mathcal{M}$ contains all Borel subsets of F.

We now define a set function μ on the Borel sets B of

$$X = (\ell_c^\infty(\Gamma), pointwise)$$

by taking

$$\begin{aligned} \mu(B) &= 1, \text{ if } F \cap B \text{ contains a set of } \mathcal{H}, \\ \mu(B) &= 0, \text{ otherwise.} \end{aligned}$$

Clearly μ takes only the values 0 and 1, takes the value 0 on each one-point set, and takes the value 1 on F and so also on X. We verify that μ is additive over countable disjoint unions of Borel sets. Let $B_1, B_2, \ldots$ be any sequence of disjoint Borel sets in X. Perhaps $\mu(B_i) = 0$ for $i \geq 1$. In this case $F \cap B_i$ contains no set of $\mathcal{H}$, so $F \backslash B_i$ and contains a set of $\mathcal{H}$ for $i \geq 1$, and $F \backslash \bigcup B_i$ contains a set of $\mathcal{H}$, so that $F \cap \bigcup B_i$ contains no set of $\mathcal{H}$ and $\mu(\bigcup B_i) = 0$. On the other hand if $\mu(B_i) = 1$ for some $i \geq 1$, say for i^*, then $F \cap B_{i^*}$ contains a set of $\mathcal{H}$ but the sets $F \cap B_i$, $i \geq i^*$, being disjoint from $F \cap B_{i^*}$, can contain no set of $\mathcal{H}$. Thus

$$\mu(B_{i^*}) = 1, \; \mu(B_i) = 0 \text{ for } i \neq i^*,$$

and

$$\mu(\bigcup_{i=1}^{\infty} B_i) = 1 = \sum_{i=1}^{\infty} \mu(B_i).$$

This shows that μ is a Borel measure on X. However μ can have no minimal closed support and so is not a Radon measure.

Much of this proof of Theorem 11 would work for the space $(\ell_c^\infty(\Gamma), weak)$. However, the set F turns out to be discrete in the weak topology, causing a fundamental breakdown.

We remark that if $[0, \omega_1]$ is the space of ordinals with the order topology and $C^1([0, \omega_1])$ is the Banach space of bounded Baire first class functions (i.e. pointwise limits of sequences of continuous functions) on $[0, \omega_1]$ with the supremum norm, then essentially the same proof yields a Borel measure on $(C^1([0, \omega_1]), pointwise)$ that is not a Radon measure.

5. A Reduction. The conditions (a) and (b) in Theorem 8, which is our main theorem, come from the work of de Maria and Rodriguez-Salinas [3]. They are relatively easy to check, but are not in a form really suited to our purpose. For this reason we prove a lemma showing that these conditions ensure the existence of a family $\mathcal{C}$ of clopen sets. Note that a set C of $\mathcal{C}$ can be regarded as bounded by the ordinal γ if $C \subset D^\gamma$. The existence of such a family $\mathcal{C}$ of clopen sets in $\ell_c^\infty(\Gamma)$, when Γ is an uncountable discrete set, is the basis of Talagrand's proof of his Theorem 6, stated in the introduction.

Lemma 12. *Let K be a compact Hausdorff space with a non-empty family $\mathcal{D}$ of non-empty proper clopen subsets with the properties (a) and (b) of Theorem 8. Then, for some ordinal Γ, of uncountable cofinality, there is a maximal strictly increasing sequence*

$$\left\{D^{(\gamma)} : 0 \leq \gamma < \Gamma\right\}$$

of sets chosen from $\mathcal{D}$. Further, the family $\mathcal{C}$ of all clopen subsets C of K, satisfying

$$C \subset D^{(\gamma)} \textit{ for some } \gamma \textit{ with } 0 \leq \gamma < \Gamma,$$

has the following properties.

(*c*) *If C_1 and C_2 belong to $\mathcal{C}$, then so do $C_1 \cup C_2$, $C_2 \cap C_2$ and $C_2 \backslash C_1$.*

(*d*) *If $C_1, C_2, \ldots$ and $C_1', C_2', \ldots$ are increasing sequences of sets of $\mathcal{C}$ with*

$$C_n \cap C_n' = \emptyset, \textit{ for } n \geq 1,$$

then there are disjoint sets C_0 and C_0' in $\mathcal{C}$ with

$$C_n \subset C_0 \quad \textit{and} \quad C_n' \subset C_0', \textit{ for } n \geq 1.$$

In particular C_0, can be chosen when $C_0' = C_1' = C_2' = \ldots = \emptyset$.

(*e*) *No countable union of sets from $\mathcal{C}$ coincides with*

$$\bigcup \{C : C \in \mathcal{C}\}.$$

(*f*) *If μ is any Radon measure on K, there is a C in $\mathcal{C}$ with the property that*

$$\mu(I) = \mu(J)$$

whenever I and J are clopen sets in $\mathcal{C}$ with

$$C \cap I = C \cap J.$$

Proof. Since $\mathcal{D}$ is a non-empty family we can choose a set $D^{(0)}$ in $\mathcal{D}$. By transfinite induction we can choose a maximal strictly increasing sequence (finite, infinite or transfinite)

$$\{D^{(\gamma)} : 0 \leq \gamma < \Gamma\}$$

of sets of $\mathcal{D}$. By the maximality, the set

$$D^{(\Gamma)} = \bigcup \left\{ D^{(\gamma)} : 0 \leq \gamma < \Gamma \right\}$$

is properly contained in no element of $\mathcal{D}$. If Γ were of countable cofinality, there would be an increasing sequence $\alpha(n)$, $n \geq 1$, with $0 \leq \alpha(n) < \Gamma$, and

$$\bigcup \left\{ D^{(\alpha(n))} : n \geq 1 \right\} = D^{(\Gamma)}.$$

By condition (a) this set is properly contained in a set of $\mathcal{D}$, providing a contradiction. Thus Γ is of uncountable cofinality.

We now define $\mathcal{C}$ to be the family of all clopen subsets C of K, for which there is a γ, with $0 \leq \gamma < \Gamma$ and

$$C \subset D^{(\gamma)}.$$

This ensures that each C in $\mathcal{C}$ is a proper subset of K.

If C_1 and C_2 are sets of $\mathcal{C}$, we can choose γ_1 and γ_2 with $0 \leq \gamma_1 < \Gamma$ and $0 \leq \gamma_2 < \Gamma$, so that

$$C_1 \subset D^{(\gamma_1)} \text{ and } C_2 \subset D^{(\gamma_2)}.$$

Taking $\gamma_3 = \max\{\gamma_1, \gamma_2\}$ we have $0 \leq \gamma_3 < \Gamma$ and $C_1 \cup C_2$, $C_1 \cap C_2$ and $C_2 \backslash C_1$ are clopen sets of K contained in $D^{(\gamma_3)}$ and so belong to $\mathcal{C}$. Thus condition (c) is satisfied.

Now suppose that $C_1, C_2, \ldots$ and $C'_1, C'_2, \ldots$ are increasing sequences of sets of $\mathcal{C}$ with

$$C_n \cap C'_n = \emptyset, \text{ for } n \geq 1.$$

Since $C_n \cup C'_n \in \mathcal{C}$, we can choose $\alpha(n)$ with $0 \leq \alpha(n) < \Gamma$ and

$$C_n \cup C'_n \subset D^{(\alpha(n))}, \quad \text{for} \quad n \geq 1.$$

Since Γ is of uncountable cofinality, we can choose γ with $0 \leq \alpha(n) \leq \gamma < \Gamma$, for $n \geq 1$. By condition (b) there are disjoint clopen sets S_0 and S'_0 in K with

$$C_n \subset S_0 \text{ and } C'_n \subset S'_0, \text{ for } n \geq 1.$$

Now

$$C_0 = S_0 \cap D^{(\gamma)} \text{ and } C_0' = S_0' \cap D^{(\gamma)}$$

are disjoint clopen sets in $\mathcal{C}$ with

$$C_n \subset C_0 \text{ and } C_n' \subset C_0', \text{ for } n \geq 1.$$

Thus condition (d) is satisfied.

If $C_1, C_2, \ldots$ is any sequence of sets of $\mathcal{C}$, the argument of the last paragraph shows that there is a γ with $0 \leq \gamma < \Gamma$ and

$$C_n \subset D^{(\gamma)}, \text{ for } n \geq 1.$$

Thus

$$\bigcup \{C_n : n \geq 1\} \subset D^{(\gamma)}$$

and $D^{(\gamma)}$ is a proper subset of $D^{(\gamma+1)}$ which is contained in

$$\bigcup \{C : C \in \mathcal{C}\}.$$

Hence condition (e) is satisfied.

Now let μ be a Radon measure on K. Then μ is τ-additive. Since the family

$$\left\{D^{(\gamma)} : 0 \leq \gamma < \Gamma\right\}$$

is a nested family of open subset of K,

$$\sup\left\{\mu\left(D^{(\gamma)}\right) : 0 \leq \gamma < \Gamma\right\} = \mu\left(\bigcup\left\{D^{(\gamma)} : 0 \leq \gamma < \Gamma\right\}\right).$$

Hence we can choose a sequence $\alpha(n)$, $n \geq 1$, of ordinals with $0 \leq \alpha(n) < \Gamma$, and

$$\lim_{n\to\infty} \mu\left(D^{(\alpha(n))}\right) = \mu\left(D^{(\Gamma)}\right)$$

with $D^{(\Gamma)}$ the open set

$$D^{(\Gamma)} = \bigcup\left\{D^{(\gamma)} : 0 \leq \gamma < \Gamma\right\}.$$

Since Γ is of uncountable cofinality, we can choose γ with $\alpha(n) < \gamma < \Gamma$ for $n \geq 1$. Now

$$\mu\left(D^{(\Gamma)}\right) = \lim_{n\to\infty} \mu\left(D^{(\alpha(n))}\right) \leq \mu\left(D^{(\gamma)}\right) \leq \mu\left(D^{(\Gamma)}\right).$$

Thus

$$\mu\left(D^{(\gamma)}\right) = \mu\left(D^{(\Gamma)}\right).$$

Write $C = D^{(\gamma)}$. Now when I and J are sets in $\mathcal{C}$ with

$$C \cap I = C \cap J$$

we have

$$I \cup J \subset D^{(\Gamma)}$$

and

$$I \triangle J = (I \backslash J) \cup (J \backslash I) \subset D^{(\Gamma)} \backslash D^{(\gamma)},$$

so that

$$\mu(I \triangle J) \leq \mu\left(D^{(\Gamma)} \backslash D^{(\gamma)}\right) = 0,$$

and

$$\mu(I) = \mu(J).$$

Thus condition (f) holds, as required.

6. The Main Theorem. In this section we assume that K is a compact Hausdorff space with a non-empty family $\mathcal{D}$ of non-empty proper clopen subsets with the properties (a) and (b) of Theorem 8. We assume, as we may, that the sequence

$$\left\{D^{(\gamma)} : 0 \leq \gamma < \Gamma\right\}$$

and the family $\mathcal{C}$ are those provided by Lemma 12. We prove a sequence of lemmas that enable us to prove Theorem 8. We shall make use of methods used by Talagrand and by de Maria and Rodriguez-Salinas in their considerations of their special spaces.

We shall mainly work within the set

$$I = \{\chi_C : C \in \mathcal{C}\}$$

of characteristic functions of sets in $\mathcal{C}$. Since each set in $\mathcal{C}$ is clopen in K, the set I is a subset of the space $C(K)$ of continuous functions on K.

The reader who looks forward to the proof of Theorem 8 and the statement of Lemma 18 will see that once the family $\mathcal{H}$ of subsets of I, satisfying the conditions 1-4 of Lemma 18, has been constructed the proof of Theorem 8 follows the simple pattern of the proof of Theorem 11. The main difficulty being the construction of the family $\mathcal{H}$.

We need to introduce and to study some families of sets in $C(K)$. For each C in $\mathcal{C}$ we introduce the sets

$$\begin{aligned} F_C &= \{f \in I : f(x) = 1 \text{ for all } x \in C\}, \\ G_C &= \{f \in I : f(x) = 0 \text{ for all } x \in C\}. \end{aligned}$$

We use E_1 to denote the unit ball of $C(K)$ with its weak topology, and E^* and E_1^* to denote the dual Banach space of $C(K)$ and its unit ball, both taken with the weak* topology.

We next introduce some families of open subsets of I taken with its weak topology; we can think of these open sets as 'thick' or 'coarse'. For each integer $p \geq 1$, we introduce the family $\mathcal{V}_p$ of all sets of the form

$$V(p, \mathbf{y}^*, \mathbf{z}) = \{f \in I : |y_i^*(f) - z_i| < 1/p,\ 1 \leq i \leq p\},$$

where

$$\begin{aligned} \mathbf{y}^* &= (y_1^*, y_2^*, \ldots, y_p^*) \in (E_1^*)^p, \\ \mathbf{z} &= (z_1, z_2, \ldots, z_p) \in (\mathbb{R}_1)^p, \end{aligned}$$

with $\mathbb{R}_1 = [-1, 1]$. If N is any neighbourhood in I of a point g in I, it is possible to find a set $V(p, \mathbf{y}^*, \mathbf{z})$ containing g and contained in N, by taking p to be sufficiently large, taking $y_1^*, y_2^*, \ldots, y_p^*$ to be suitable points of E_1^* and then taking $z_i = y_i^*(g) = \langle g, y_i^* \rangle$, for $1 \leq i \leq p$. Note that the conditions $\|g\| \leq 1, \|y_i^*\| \leq 1$ for $1 \leq i \leq p$, ensure that the requirement that $z_i \in \mathbb{R}_1$ is satisfied for $1 \leq i \leq p$. Thus the family

$$\mathcal{V} = \bigcup_{p=1}^{\infty} \mathcal{V}_p$$

is a base for the open sets of I.

We take $\mathcal{Q}_p$ to be the family of all (arbitrary) unions of sets taken from $\mathcal{V}_p$, and write

$$\mathcal{Q} = \bigcup_{p=1}^{\infty} \mathcal{Q}_p.$$

Thus the family $\mathcal{Q}$ is a family of (coarse) open subsets of I, and each open subset G of I has the form

$$G = \bigcup_{p=1}^{\infty} \bigcup \{V \in \mathcal{V}_p : V \subset G\}$$

and so is a countable union of sets of $\mathcal{Q}$.

We can now introduce the families of sets that we really have to study. Write

$$\mathcal{L}_1 = \{Q \in \mathcal{Q} : Q \cap F_C \neq \emptyset \text{ for all } C \text{ in } \mathcal{C}\},$$

$$\mathcal{L}_2 = \{I \backslash Q : Q \in \mathcal{Q} \text{ and } Q \notin \mathcal{L}_1\},$$

$$\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2.$$

Here the sets in $\mathcal{L}_1$ are 'large' since they are 'course' open sets that meet all sets F_C with $C \in \mathcal{C}$. The sets in $\mathcal{L}_2$ are 'large', since they are complements of sets that are not 'large'. The family $\mathcal{H}$ will later be taken to be the set of all countable intersections of members of $\mathcal{L}$.

We now prove

Lemma 13. *The families $\mathcal{Q}$ and $\mathcal{L}$ satisfy the following conditions.*
(α) *Each open set in I is a countable union of sets from $\mathcal{Q}$.*
(β) *For each Q in $\mathcal{Q}$, either $Q \in \mathcal{L}$ or $I \backslash Q \in \mathcal{L}$.*
(γ) *The intersection of all the sets of $\mathcal{L}$ is empty.*

Proof. We have already noted the condition (α). The result (β) follows from the definition of $\mathcal{L}$; the question of whether $Q \in \mathcal{L}$ or $I \backslash Q \in \mathcal{L}$ depending on whether or not $Q \in \mathcal{L}_1$.

To prove (γ) we consider the sets

$$M_x = \{g \in I : g(x) > 0\},$$

where $x \in C \in \mathcal{C}$. For each such x, M_x is an open set in I that is of the form

$$\{g \in I : |g(x) - 1| < 1\},$$

and so belongs to $\mathcal{V}_1 \subset \mathcal{Q}_1$. Further, for any $C' \in \mathcal{C}$ and $x \in C \in \mathcal{C}$, we have

$$\chi_{C \cup C'} \in M_x \cap F_{C'},$$

so that $M_x \cap F_{C'} \neq \emptyset$. Thus $M_x \in \mathcal{L}_1$. However, the only point of I that could belong to

$$\bigcap \{M_x : x \in C \in \mathcal{C}\}$$

is the characteristic function of

$$\bigcup \{C \in \mathcal{C}\} = D^{(\Gamma)}.$$

Since, by the construction of the sequence

$$\left\{D^{(\gamma)} : 0 \le \gamma < \Gamma\right\},$$

we have

$$D^{(\Gamma)} \notin \mathcal{C},$$

the characteristic function does not belong to I. Thus

$$\bigcap \{M_x : x \in C \in \mathcal{C}\} = \emptyset$$

and so

$$\bigcap \{L : L \in \mathcal{L}_1\} = \emptyset,$$

as required.

Our next objective is to establish the result.

(δ) *The intersection of any countable sequence of sets from $\mathcal{L}$ is non-empty.*

We first prove

Lemma 14. *Let $L_1, L_2, \ldots$ be any countable sequence of sets chosen from $\mathcal{L}_2$. Then there is a set C in $\mathcal{C}$ with*

$$F_C \subset \bigcap_{n=1}^{\infty} L_n.$$

Proof. By the definition of $\mathcal{L}_2$, for each $n \ge 1$, we have

$$L_n = I \backslash Q_n$$

with $Q_n \in \mathcal{Q} \backslash \mathcal{L}_1$. By the definition of $\mathcal{L}_1$, we can choose $C\,(n)$ in $\mathcal{C}$ with

$$Q_n \cap F_{C(n)} = \emptyset, \quad n \ge 1.$$

Thus

$$F_{C(n)} \subset L_n, \quad n \ge 1.$$

By condition (d) we can choose C in $\mathcal{C}$ with

$$\bigcup_{n=1}^{\infty} C\,(n) \subset C.$$

This ensures that

$$F_C \subset \bigcap_{n=1}^{\infty} F_{C(n)} \subset \bigcap_{n=1}^{\infty} L_n,$$

as required.

Our next lemma is also simple.

Lemma 15. *Suppose that L in $\mathcal{L}_1$ is of the form*

$$L = \bigcup_{n=1}^{\infty} Q_n$$

with each set Q_n in $\mathcal{Q}$. Then for at least one $n \geq 1$ we have

$$Q_n \in \mathcal{L}_1.$$

P r o o f. For each γ with $0 \leq \gamma < \Gamma$, we have

$$L \cap F_{D(\gamma)} \neq \emptyset,$$

so that

$$Q_n \cap F_{D(\gamma)} \neq \emptyset,$$

for at least one $n = n(\gamma)$. Let $\Gamma(n)$ be the set of all γ with $0 \leq \gamma < \Gamma$ and

$$Q_n \cap F_{D(\gamma)} \neq \emptyset.$$

Then

$$\bigcup_{n=1}^{\infty} \Gamma(n) = \{\gamma : 0 \leq \gamma < \Gamma\}.$$

Since Γ is of uncountable cofinality, at least one of the sets $\Gamma(n)$, say $\Gamma(n^*)$, is cofinal in Γ. Now, for each C in $\mathcal{C}$, there is a γ^* in $\Gamma(n^*)$ with $C \subset D^{(\gamma^*)}$, so that

$$\emptyset \neq Q_{n*} \cap F_{D(\gamma*)} \subset Q_{n^*} \cap F_c.$$

Thus $Q_{n^*} \in \mathcal{L}_1$, as required.

We recall that the sets F_C and G_C for C in $\mathcal{C}$ are defined by

$$F_C = \{f \in I : f(x) = 1 \text{ for all } x \in C\},$$

$$G_C = \{f \in I : f(x) = 0 \text{ for all } x \in C\}.$$

The next lemma is the most difficult step in the proof. It is the key to the proof that the intersection of any countable sequence of sets from $\mathcal{L}$ is non-empty. It depends essentially on the introduction of the 'course' open sets and uses, for the first time, the condition (f) of Lemma 12, in order to make the choice of the sets U_0 and U_1 possible. Notice that any set of the form

$$F_{U_1} \cap G_{U_0},$$

with U_0, U_1 dispoint sets in C, is necessarily 'large' in a new sense, since the condition $f \in F_{U_1} \cap G_{U_0}$ restricts the values of f only on the set $U_0 \cap U_1$.

Lemma 16. *Let L be a set in $\mathcal{L}_1$. Then there is a set A in $\mathcal{C}$ such that for any sets R, S in $\mathcal{C}$ with*

$$A \subset R \subset S,$$

there are sets U_0 and U_1 in $\mathcal{C}$ with

$$R \subset U_1, \quad S \backslash R \subset U_0,$$

$$U_0 \cap U_1 = \emptyset,$$

and

$$F_{U_1} \cap G_{U_0} \subset L.$$

P r o o f. Since $L \in \mathcal{L}_1$ we have $L = Q$ where

$$Q \in \mathcal{Q}_p,$$

for some $p \geq 1$, and Q is of the form

$$Q = \bigcup \{V(p, \mathbf{y}^*, \mathbf{z}) : (\mathbf{y}^*, \mathbf{z}) \in Z\}$$

with

$$Z \subset (E_1^*)^p \times (\mathbb{R}_1)^p,$$

and, for each set C of $\mathcal{C}$,

$$Q \cap F_C \neq \emptyset.$$

For each $(\mathbf{y}^*, \mathbf{z})$ in Z, we can write

$$V(p, \mathbf{y}^*, \mathbf{z}) = \bigcup_{q>p} W(p, q, \mathbf{y}^*, \mathbf{z}),$$

with

$$W(p,q,\mathbf{y}^*,\mathbf{z}) = \left\{ f \in I : |y_i^*(f) - z_i| < p^{-1} - q^{-1},\, 1 \le i \le p \right\}.$$

Thus

$$Q = \bigcup_{q>p} Q_q$$

with

$$Q_q = \bigcup \{ W(p,q,\mathbf{y}^*,\mathbf{z}) : (\mathbf{y}^*,\mathbf{z}) \in Z \}.$$

Clearly each set Q_q, $q > p$, belongs to $\mathcal{Q}$. Hence, by Lemma 15, we can choose a fixed $q > p$ so that

$$Q_q \in \mathcal{L}_1.$$

Now for each γ with $0 \le \gamma < \Gamma$,

$$Q_q \cap F_{D(\gamma)} \neq \emptyset,$$

and so we can choose $\left(\mathbf{y}_\gamma^*, \mathbf{z}_\gamma\right)$ in Z with

$$W\left(p,q,\mathbf{y}_\gamma^*,\mathbf{z}_\gamma\right) \cap F_{D(\gamma)} \neq \emptyset.$$

Since E_1^* is the unit ball of the dual space of $C(K)$ taken with its weak topology and $\mathbb{R}_1$ is the unit interval $[-1,1]$, the set $(E_1^*)^p \times (\mathbb{R}_1)^p$ is compact. Hence the transfinite sequence

$$\left(\mathbf{y}_\gamma^*, \mathbf{z}_\gamma\right),\ 0 \le \gamma < \Gamma,$$

has a cluster point, (η^*, ζ) say, in $(E_1^*)^p \times (\mathbb{R}_1)^p$, that is a point (η^*, ζ) with the property that for any open neighbourhood N of (η^*, ζ) in $(E_1^*)^p \times (\mathbb{R}_1)^p$ the set of ordinals γ with

$$\left(\mathbf{y}_\gamma^*, \mathbf{z}_\gamma\right) \in N$$

is cofinal in Γ.

Now $\eta_i^* = \left(\eta_1^*, \eta_2^*, \ldots, n_p^*\right)$ where each element η_i^*, $1 \le i \le p$, of E_1^* takes the form

$$\langle f, \eta_i^* \rangle = \int f d\mu_i$$

where μ_i is the difference of two Radon measures on K. By the conditions (f) and (c) of Lemma 12 we can choose a set A in $\mathcal{C}$ so that

$$\int f d\mu_i = \int g d\mu_i, \quad 1 \le i \le p,$$

for all pairs f, g of continuous characteristic functions that coincide on A. Thus

$$\langle f, \eta_i^* \rangle = \langle g, \eta_i^* \rangle, \; 1 \le i \le p,$$

whenever f and g functions of I that coincide on A.

The set A having been chosen in this way, we show that it satisfies the requirement of the lemma. We suppose that R and S are any sets in $\mathcal{C}$ with

$$A \subset R \subset S.$$

Then $S \backslash R \in \mathcal{C}$ and so $\chi_{S\backslash R} \in I$. We consider the neighbourhood N of $(\boldsymbol{\eta}^*, \boldsymbol{\zeta})$ in $(E_1^*)^p \times (\mathbb{R}_1)^p$ defined to be the set of points $(\mathbf{y}^*, \mathbf{z})$ with

$$\left| y_i^*(\chi_{S\backslash R}) - \eta_i^*(\chi_{S\backslash R}) \right| < (2q)^{-1}, \quad 1 \le i \le p,$$

$$|z_i - \zeta_i| < (2q)^{-1}, \quad 1 \le i \le p.$$

Since $(\boldsymbol{\eta}^*, \boldsymbol{\zeta})$ is a cluster point of the sequence

$$\left(\mathbf{y}_\gamma^*, \mathbf{z}_\gamma \right), \quad 0 \le \gamma < \Gamma,$$

we can choose γ with $0 \le \gamma < \Gamma$,

$$S \subset D^{(\gamma)},$$

and

$$\left(\mathbf{y}_\gamma^*, \mathbf{z}_\gamma \right) \in N.$$

By the choice of $\left(\mathbf{y}_\gamma^*, \mathbf{z}_\gamma \right)$, we have

$$W\left(p, q, \mathbf{y}_\gamma^*, \mathbf{z}_\gamma \right) \cap F_{D^{(\gamma)}} \ne \emptyset,$$

and so we can choose $h = \chi_T$ for some set T in $\mathcal{C}$ with

$$S \subset D^{(\gamma)} \subset T,$$

and

$$\chi_T \in W\left(p, q, \mathbf{y}_\gamma^*, \mathbf{z}_\gamma \right).$$

By the definition of $W\left(p, q, \mathbf{y}_\gamma^*, \mathbf{z}_\gamma \right)$, we have

$$\left| y_{\gamma,i}^* (\chi_T) - z_{\gamma,i} \right| \le p^{-1} - q^{-1}, \; 1 \le i \le p.$$

Now

$$(S \backslash R) \cap A = \emptyset,$$

so that $\chi_{S\backslash R}$ vanishes on A and

$$\eta_i^* \left(\chi_{S\backslash R} \right) = 0, \quad 1 \le i \le p.$$

Since $\left(\mathbf{y}_\gamma^*, \mathbf{z}_\gamma \right) \in N$, we have

$$\left| y_{\gamma,i}^* \left(\chi_{S\backslash R} \right) - \eta_i^* \left(\chi_{S\backslash R} \right) \right| < (2q)^{-1}, \quad 1 \le i \le p,$$

$$\left| z_{\gamma,i} - \zeta_i \right| < (2q)^{-1}, \ 1 \le i \le p.$$

Thus

$$\left| y_{\gamma,i}^* \left(\chi_{S\backslash R} \right) \right| < (2q)^{-1}, \ 1 \le i \le p.$$

Since

$$\left| y_{\gamma,i}^* \left(\chi_T \right) - z_{\gamma,i} \right| < p^{-1} - q^{-1}, \ 1 \le i \le p,$$

these inequalities imply that

$$\chi_T - \chi_{S\backslash R} \in V \left(p, \mathbf{y}_\gamma^*, \mathbf{z}_\gamma \right).$$

Using the conditions (f) and (c) again we can choose U in $\mathcal{C}$ with $T \subset U$ so that

$$y_{\gamma,i}^* \left(\chi_C \right) = y_{\gamma,i}^* \left(\chi_{C'} \right), \ 1 \le i \le p,$$

whenever C, C' in $\mathcal{C}$ satisfy

$$U \cap C = U \cap C'.$$

It follows that if Θ is any set in $\mathcal{C}$ and χ_Θ coincides with $\chi_T - \chi_{S\backslash R}$ on U, then

$$\chi_\Theta \in V(p, y_\gamma^*, z_\gamma).$$

Recall the inclusions

$$A \subset B \subset R \subset S \subset T \subset U$$

and note their alphabetical nature. Take

$$U_1 = R \cup (T \backslash S),$$
$$U_0 = (S \backslash R) \cup (U \backslash T).$$

Then U_0, U_1 belong to $\mathcal{C}$ and

$$U_0 \cap U_1 = \emptyset, \quad U_0 \cup U_1 = U,$$

$$R \subset U_1 \quad \text{and} \quad S \backslash R \subset U_0.$$

Consider any function f in $f_{U_1} \cap G_{U_0}$. Then $f = \chi_\Theta$ with Θ a set of $\mathcal{C}$ with

$$U_1 \subset \Theta, \quad U_0 \cap \Theta = \emptyset.$$

Write

$$\Phi = \Theta \backslash U.$$

Then $\Phi \in \mathcal{C}$ and it is easy to check that

$$\chi_\Theta = \chi_T - \chi_{S\backslash R} + \chi_\Phi.$$

Thus the functions χ_Θ and $\chi_T - \chi_{S\backslash R}$ coincide on U, and so

$$\chi_\Theta \in V(p, y^*_\gamma, z_\gamma) \subset L.$$

Hence

$$F_{U_1} \cap G_{U_0} \subset L,$$

as required.

Our next lemma uses condition (d) of Lemma 12 to prove that the intersection of the 'large' sets

$$F_{U_1(n)} \cap G_{U_0(n)}, \ n \geq 1,$$

contained in the sets

$$L_n^{(1)}, \quad n \geq 1,$$

contains the 'large' set

$$F_{U_1} \cap G_{U_0}$$

contained in $\bigcap_{n=1}^\infty L_n^{(2)}$.

Lemma 17. *Let H be the intersection of a countable sequence of sets of $\mathcal{L}$. Then for some disjoint sets U_1, U_0 in $\mathcal{C}$, we have*

$$\emptyset \neq F_{U_1} \cap G_{U_0} \subset H.$$

Proof. We write

$$H = \left(\bigcap_{n=1}^\infty L_n^{(1)} \right) \cap \left(\bigcap_{n=1}^\infty L_n^{(2)} \right),$$

with $L_n^{(1)}$, $n \geq 1$, a sequence of sets of $\mathcal{L}_1$ and $L_n^{(2)}$, $n \geq 1$, a sequence of sets of $\mathcal{L}_2$. By Lemma 14 we can choose a set $A(0)$ in $\mathcal{C}$ with

$$F_{A(0)} \subset \bigcap_{n=1}^{\infty} L_n^{(2)}.$$

By Lemma 16, for each $n \geq 1$, there is a set $A\,(n)$ in $\mathcal{C}$, such that the conclusion of Lemma 16 holds when $A = A\,(n)$ and $L = L_n^{(1)}$. Using the conditions (d) and (c) of Lemma 12 we can choose B in $\mathcal{C}$, independent of n, with

$$A\,(n) \subset B, \text{ for } n \geq 0.$$

This ensures, in the first place, that

$$F_B \subset F_{A(0)} \subset \bigcap_{n=1}^{\infty} L_n^{(2)}.$$

Secondly, for each $n \geq 1$, for any sets R, S in $\mathcal{C}$ with

$$B \subset R \subset S,$$

we have $A(n) \subset R \subset S$, and there are sets $U_1\,(n)$ and $U_0\,(n)$ in $\mathcal{C}$ with

$$R \subset U_1\,(n)\,, \quad S \backslash R \subset U_0\,(n)\,,$$

$$U_0\,(n) \cap U_1\,(n) = \emptyset,$$

and

$$F_{U_1(n)} \cap G_{U_0(n)} \subset L_n^{(1)}.$$

By a suitable inductive choice of $R\,(n)$ and $S\,(n)$, we shall ensure the existence of such sets

$$U_0\,(n)\,, \quad U_1(n), \quad n = 1, 2, \ldots \text{ with}$$

$$U_0\,(1) \subset U_0\,(2) \subset \ldots\,,$$

$$U_1\,(1) \subset U_1\,(2) \subset \ldots\,.$$

We start the inductive process by taking $R(1) = S(1) = B$, and choosing $U_0(1)$, $U_1(1)$ in $\mathcal{C}$ with

$$R(1) \subset U_1(1), \quad U_0(1) \cap U_1(1) = \emptyset,$$

and

$$F_{U_1(1)} \cap G_{U_0(1)} \subset L_n^{(1)}.$$

When $n \geq 1$, and $R\,(n)$, $S\,(n)$, $U_0\,(n)$ and $U_1\,(n)$ have been chosen in $\mathcal{C}$ with

$$B \subset R\,(n) \subset S\,(n)\,,$$

$$R\,(n) \subset U_1\,(n)\,, \quad S\,(n) \setminus R\,(n) \subset U_0\,(n)\,,$$

$$U_0\,(n) \cap U_1\,(n) = \emptyset,$$

we take $R\,(n+1) = U_1\,(n)$, and $S\,(n+1) = U_0\,(n) \cup U_1\,(n)$, so that

$$R(n+1) \text{ and } S(n+1)$$

are sets of $\mathcal{C}$ with

$$B \subset U_1\,(n) = R\,(n+1)\,,$$

$$R\,(n+1) \subset U_0\,(n) \cup U_1\,(n) = S\,(n+1)\,.$$

This ensures that

$$U_0\,(n) = S\,(n+1) \setminus R\,(n+1)\,.$$

By our condition, we can then choose $U_0\,(n+1)$ and $U_1\,(n+1)$ in $\mathcal{C}$ with

$$U_1\,(n) = R\,(n+1) \subset U_1\,(n+1)\,,$$

$$U_0\,(n) = S\,(n+1) \setminus R\,(n+1) \subset U_0\,(n+1)\,,$$

$$U_0\,(n+1) \cap U_1\,(n+1) = \emptyset,$$

and

$$F_{U_1(n+1)} \cap G_{U_0(n+1)} \subset L_{n+1}^{(1)}.$$

In this way we construct increasing sequences of sets of $\mathcal{C}$,

$$B \subset U_1\,(1) \subset U_1\,(2) \subset \ldots \subset U_1\,(n) \subset \ldots,$$

$$U_0\,(1) \subset U_0\,(2) \subset \ldots \subset U_1\,(n) \subset \ldots,$$

with

$$U_0\,(n) \cap U_1\,(n) = \emptyset,$$

$$F_{U_1(n)} \cap G_{U_0(n)} \subset L_n^{(1)},$$

for $n \geq 1$. By condition (d) we can choose sets U_0 and U_1 in $\mathcal{C}$ with

$$U_0\,(n) \subset U_0, \quad n \geq 1,$$

$$U_1\,(n) \subset U_1, \quad n \geq 1,$$

$$U_0 \cap U_1 = \emptyset.$$

Since U_0, U_1 are disjoint sets of $\mathcal{C}$

$$F_{U_1} \cap G_{U_0} \neq \emptyset.$$

Now

$$F_{U_1} \cap G_{U_0} \subset F_{A(0)} \subset \bigcap_{n=1}^{\infty} L_n^{(2)},$$

$$F_{U_1} \cap G_{U_0} \subset F_{U_1(n)} \cap G_{U_0(n)} \subset L_n^{(1)}, \quad n \geq 1,$$

so that

$$\emptyset \neq F_{U_1} \cap G_{U_0} \subset \left(\bigcap_{n=1}^{\infty} L_n^{(1)} \right) \cap \left(\bigcap_{n=1}^{\infty} L_n^{(2)} \right) = H$$

as required.

We have now verified that the families $\mathcal{Q}$ and $\mathcal{L}$ satisfy the conditions (α), (β), (γ) of Lemma 13 and the condition (δ) stated just before Lemma 14.

Lema 18. *The family $\mathcal{H}$ of all countable intersections of sets from $\mathcal{L}$ has the following properties.*

1. *The family $\mathcal{H}$ of subsets of I is closed under countable intersections.*

2. *The intersection of all the sets in $\mathcal{H}$ is empty.*

3. *Each set H in $\mathcal{H}$ is uncountable.*

4. *For each Borel subset B of I, either B or $I \backslash B$ contains a set H of $\mathcal{H}$.*

P r o o f. The property (1) is immediate from the definition of $\mathcal{H}$. The property (2) follows immediately from condition (γ) of Lemma 13.

If $H \in \mathcal{H}$, then Lemma 17 ensures the existence of disjoint sets U_1, U_0 in $\mathcal{C}$ with

$$F_{U_1} \cap G_{U_0} \subset H.$$

We can choose γ with $0 \leq \gamma < \Gamma$ and

$$U_0 \cup U_1 \subset D^{(\gamma)}.$$

Now there are uncountably many disjoint clopen sets in $\mathcal{C}$ of the form

$$D^{(\delta+1)} \backslash D^{(\delta)}, \quad \gamma < \delta < \Gamma,$$

all being non-empty. So the characteristic functions of the sets

$$U_1 \cup \left(D^{(\delta+1)} \backslash D^{(\delta)} \right), \quad \gamma < \delta < \Gamma$$

form an uncountable system of distinct points in

$$F_{U_1} \cap G_{U_0}.$$

Hence H is uncountable and $\mathcal{H}$ has the property (3).

Now consider the family $\mathcal{A}$ of all subsets A of I, with the property:

either A contains a set of $\mathcal{H}$,

or $I \backslash A$ contains a set of $\mathcal{H}$.

By condition (β) of Lemma 13, the family $\mathcal{Q}$ is contained in $\mathcal{A}$. By the defining property $\mathcal{A}$ is closed under the operation of complementation with respect to I. Now suppose that $A_1, A_2, \ldots$ is any sequence of sets in $\mathcal{A}$. If any one of the sets A_i, $i \geq 1$, contains a set of H, then so does their union. If none of the sets A_i, $i \geq 1$, contain a set of $\mathcal{H}$, then for each $i \geq 1$, the set $I \backslash A_i$ contains a set, H_i say, of $\mathcal{H}$ and

$$I \backslash \bigcup_{i=1}^{\infty} A_i \supset \bigcap_{i=1}^{\infty} H_i \in \mathcal{H}.$$

Thus, in each case,

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}.$$

Thus $\mathcal{A}$ is closed under countable unions. Since each open subset of I is a countable union of sets of $\mathcal{Q}$, each open subset of I belongs to $\mathcal{A}$. Hence $\mathcal{A}$ contains all the Borel subsets of I and $\mathcal{H}$ has the property (4), as required.

Proof of Theorem 8. Let K be a compact Hausdorff space satisfying the conditions of Theorem 8. Then Lemmas 12 to 18 ensure the existence of a family $\mathcal{H}$ of subsets of $C(K)$ with the properties (1) to (4) of Lemma 18.

We define a set function ν on the Borel sets $\mathcal{B}$ of $(C(K), weak)$ by taking

$$\nu(B) = 0 \text{ if } B \in \mathcal{B} \text{ and } B \cap I \text{ contains no set } H \text{ in } \mathcal{H},$$

$$\nu(B) = 1 \text{ if } B \in \mathcal{B} \text{ and } B \cap I \text{ contains some set } H \text{ in } \mathcal{H}.$$

Then ν is a set-function defined on $\mathcal{B}$ taking only the values 0 or 1. By the property (3), no countable set can contain a set of $\mathcal{H}$, and so all countable sets are assigned the value 0. By property (4), since $I \backslash I = \emptyset$ contains no set of $\mathcal{H}$, I itself must contain a set of $\mathcal{H}$ and so $\nu(I) = 1$.

Now consider any disjoint sequence $B_1, B_2, \ldots$ of sets of $\mathcal{B}$. If for some i we have $\nu(B_i) = 1$, then $B_i \cap I$ contains some set H of $\mathcal{H}$. Hence

$$B_0 = \bigcup_{k=1}^{\infty} B_k$$

contains H and $\nu(B_0) = 1$. However, for $j \neq i$,

$$B_j \cap I \subset I \backslash B_i$$

and $B_j \cap I$ can contain no set of H. Thus $\nu(B_j) = 0$ for $j \neq i$. Hence

$$1 = \nu(B_0) = \nu(B_i) = \sum_{k=1}^{\infty} \nu(B_k).$$

On the other hand, if for no $i \geq 1$, do we have $\nu(B_i) = 1$, then each $I \backslash B_i$ contains a set, H_i say, of $\mathcal{H}$, so that

$$\bigcap_{i=1}^{\infty} I \backslash B_i \supset \bigcap_{i=1}^{\infty} H_i \in \mathcal{H}$$

and

$$\nu\left(I \backslash \bigcup_{i=1}^{\infty} B_i\right) = 1.$$

In this case

$$\nu\left(\bigcup_{i=1}^{\infty} B_i\right) = 0 = \sum_{i=1}^{\infty} \nu(B_i).$$

Thus ν is countably additive on $\mathcal{B}$ and ν is a countably additive Borel measure on $(C(K), weak)$ taking only the values 0 and 1 and taking the value 0 on each point of I and the value 1 on I.

To prove that ν is non-τ-additive we consider two cases. First suppose that each point f of $C(K)$ has a weak neighbourhood, G_f say, of ν-measure zero. Then for all countable sequences $f_1, f_2, \ldots$ in $C(K)$

$$\nu\left(\bigcup_{i=1}^{\infty} G_{f_i}\right) = 0,$$

but

$$\nu\left(\bigcup\{G_f : f \in C(K)\}\right) = \nu(C(K)) = 1.$$

Thus, in this case ν is non-τ-additive. Secondly suppose that there is a point f of $C(K)$ that has no weak neighbourhood of measure zero. Since $(C(K), \mathit{weak})$ is Hausdorff, for each point $g \neq f$ of I we can choose disjoint weak neighbourhoods N_g of f and G_g of g. Then

$$\nu(N_g \cap I) = 1$$

so that

$$\nu(G_g \cap I) = 0,$$

for all $g \neq f$. Now, for any sequence $g_1, g_2, \ldots$ of functions of $C(K)$ distinct from f we have

$$\nu\left(\bigcup_{i=1}^{\infty} G_{g_i} \cap I\right) = 0.$$

However

$$\nu\left(\bigcup\{G_g \cap I : g \neq f,\ g \in I\}\right) = \nu(I \backslash \{f\}) = 1.$$

Thus we again conclude that ν is non-τ-additive. Thus ν is not a Radon measure.

Of course, this last argument is well-known and works in any Hausdorff space; it is only included for completeness.

7. Talagrand's Theorem. In this section we deduce Talagrand's theorem as a consequence of Theorem 8. However, we cannot apply Theorem 8 directly to the space $\ell_c^\infty(\Gamma)$ of bounded real-valued functions of countable support on the uncountable discrete space Γ. We take $X = X(\Gamma) = \Gamma \cup \{\infty\}$, with $\infty \notin \Gamma$, topologized by taking the points of Γ to be open sets in X and taking the neighbourhoods of ∞ to be the sets of the form

$$\Theta \cup \{\infty\}$$

with $\Theta \subset \Gamma$ and $\Gamma \backslash \Theta$ countable. The space X obtained in this way is completely regular and we take $K = K(\Gamma)$ to be the Stone-Čech compactification of $X(\Gamma)$.

We first use Theorem 8 to prove a result that Talagrand gives in [33] together with an outline proof.

Proposition 19. (T) *Let* Γ *be an uncountable discrete space and let* $K(\Gamma)$ *be the space defined above. Then* $(C(K(\Gamma)), weak)$ *admits a Borel measure that is not a Radon measure, and takes only the values* 0 *or* 1.

P r o o f. We study the family $\mathcal{D}$ of all closures $\overline{\Delta}$ in $K = \beta X$ of the non-empty countable subsets Δ of Γ. Since Δ is clopen in X the extension to K of the characteristic function of Δ, as a subset of X, is a continuous function taking only the values 0 and 1. Thus the closures $\overline{\Delta}$ and $\overline{X\backslash\Delta}$ of Δ and $X\backslash\Delta$ in K are complementary clopen sets in K, and $\infty \in \overline{X\backslash\Delta}$. On the other hand, if F is any clopen set in K that does not contain the point ∞, then $F \cap X$ is a clopen set in X that does not contain ∞ and so is a countable subset, Δ say, of Γ. As above the closures $\overline{\Delta}$ and $\overline{X\backslash\Delta}$ are complementary clopen sets in K, the first contained in F and the second contained in $K\backslash F$. Thus $F = \overline{\Delta}$ and so $F \in \mathcal{D}$. This shows that $\mathcal{D}$ is just the family of all non-empty clopen subset of K that do not contain ∞.

We now know that $\mathcal{D}$ is a non-empty family of non-empty proper clopen subsets of K, as required in the hypothesis of Theorem 8.

Suppose that $D_1, D_2, \ldots$ is an increasing sequence of member of $\mathcal{D}$. Then the sets $D_1 \cap \Gamma$, $D_2 \cap \Gamma$, $\ldots$ form an increasing sequence of countable subsets of Γ. Take Δ to be any countable set in Γ properly containing the union

$$\bigcup \{D_i \cap \Gamma : i \geq 1\};$$

this being possible as Γ is uncountable. Now

$$\bigcup \{D_i : i \geq 1\}$$

is properly contained in $\overline{\Delta}$, which belongs to $\mathcal{D}$. Thus the condition (a) of Theorem 8 is satisfied.

Now suppose that

$$D_1, D_2, \ldots \quad \text{and} \quad D'_1, D'_2, \ldots$$

are two increasing sequences of clopen sets in K all contained in a fixed set D of $\mathcal{D}$, and with

$$D_n \cap D'_n = \emptyset, \quad \text{for} \quad n \geq 1.$$

Then

$$D_1 \cap \Gamma, D_2 \cap \Gamma, \ldots, \quad \text{and} \quad D_1' \cap \Gamma, D_2' \cap \Gamma, \ldots,$$

are two increasing sequences of countable subsets of Γ, with

$$(D_n \cap \Gamma) \cap (D_n' \cap \Gamma) = \emptyset, \quad \text{for} \quad n \geq 1.$$

Thus

$$\Delta = \bigcup \{D_i \cap \Gamma : i \geq 1\}, \quad \text{and} \quad \Delta' = \bigcup \{D_i' \cap \Gamma : i \geq 1\}$$

are disjoint countable subsets of Γ and

$$\overline{\Delta} \quad \text{and} \quad \overline{\Delta}'$$

are disjoint clopen sets in K, both in $\mathcal{D}$ with

$$D_n \subset \overline{\Delta}, \; D_n' \subset \overline{\Delta}', \quad \text{for} \quad n \geq 1.$$

This shows that the condition (b) of Theorem 8 is satisfied.

Now Theorem 8 shows that $(C(K), weak)$ admits a Borel measure that is not a Radon measure. By the proof of Theorem 8, the measure can be taken to have only the values 0 or 1.

Our next lemma concerns linear subspaces of codimension one of a Banach space; is relevant to our work since, as we shall see, $\ell_c^\infty(\Gamma)$ is a subspace of $C(K(\Gamma))$ of codimension one.

Lemma 20. *Let L be a linear subspace of codimension* 1 *in a Banach space E. If E admits a weak zero-one Borel measure that is not a Radon measure, then so does L.*

Proof. We may suppose that ν is a weak zero-one Borel measure on E that is not a Radon measure, and also that L is the linear subspace defined by

$$\langle x, x^* \rangle = 0$$

for some $x^* \neq 0$ in E^*. If $a \leq b$, let $E([a, b])$ denote the set

$$\{x : \langle x, x^* \rangle \in [a, b]\}.$$

Since ν takes only the values 0 and 1, we can choose an integer $m \geq 0$ so that

$$\nu(E([-m, m])) = 1.$$

Now, if $[a, b]$ with $a < b$ is any real interval with

$$\nu(E([a, b])) = 1,$$

then either

$$\nu\left(E\left(\left[\frac{1}{2}a + \frac{1}{2}b, \frac{1}{2}a + \frac{1}{2}b\right]\right)\right) = 1$$

or just one of the sets

$$E\left(\left[a, \frac{1}{2}a + \frac{1}{2}b\right]\right) \quad \text{or} \quad E\left(\left[\frac{1}{2}a + \frac{1}{2}b, b\right]\right)$$

has measure 1. Hence we can choose a nested sequence of intervals

$$[a_1, b_1], [a_2, b_2], \ldots,$$

all of ν-measure 1, so that either the sequence terminates with a degenerate interval, $[c, c]$ say, or it is infinite with a single point of intersection, c say. Since ν is σ-additive, we must have

$$\nu(E([c, c])) = 1$$

in each case. In particular, $E([c, c]) \neq \emptyset$, and we can choose x_0 in this set. Let μ be the measure on L defined from ν by translating x_0 to the origin, or more formally by taking

$$\mu(B) = \nu(B + x_0)$$

for all weak Borel subsets of L. Then ν is a measure on L satisfying our requirements.

Proof of Theorem 6 (T). By Proposition 19 (T), there is a Borel measure ν on $(C(K), weak)$, taking only the values 0 or 1, that is not a Radon measure on $(C(K), weak)$.

Recall that $K = K(\Gamma)$ is the Stone-Čech compactification of the space

$$\Gamma \cup \{\infty\}$$

topologized by taking the points of Γ to be open and the neighbourhoods of ∞ to be the sets of the form

$$\Theta \cup \{\infty\}$$

with $\Theta \subset \Gamma$ and $\Gamma \backslash \Theta$ countable. If y is any function in $C(K)$ with $y(\infty) = 0$, let y_Γ denote the restriction of y to Γ. By the continuity of y at ∞, for each $\varepsilon > 0$, the set of γ in Γ with

$$|y_\Gamma(\gamma)| \geq \varepsilon$$

is a countable subset of Γ. Thus y_Γ has countable support, and being bounded, belongs to $\ell_c^\infty(\Gamma)$. Indeed we can identify $\ell_c^\infty(\Gamma)$ with the closed linear subspace

$$\{y_\Gamma : y(\infty) = 0,\ y \in C(K)\}$$

of $C(K)$.

It follows from Lemma 20 that $\ell_c^\infty(\Gamma)$ admits a weak zero-one Borel measure that is not a Radon measure.

8. Theorem 9 and the Theorem of de Maria and Rodriguez-Salinas. We first prove Theorem 9.

Proof of Theorem 9. Let K be an infinite compact Hausdorff space that is a totally disconnected F-space with the property that each non-empty zero set in K contains an infinite open subset. Let $\mathcal{D}$ be the family of non-empty clopen proper subsets of K. Since K is infinite compact and totally disconnected the family $\mathcal{D}$ is not empty. We verify that $\mathcal{D}$ satisfies the following hypothesis of Theorem 8.

(a) The union of any increasing sequence of members of $\mathcal{D}$ is properly contained in a member of $\mathcal{D}$.

(b) If $S_1, S_2, \ldots$ and $T_1, T_2, \ldots$ are two increasing sequences of clopen sets in K, all contained in a fixed set of $\mathcal{D}$, with

$$S_n \cap T_n = \emptyset,\ \text{for } n \geq 1,$$

then there disjoint clopen sets S_0 and T_0 with

$$S_n \subset S_0 \quad \text{and} \quad T_n \subset T_0, \quad \text{for} \quad n \geq 1.$$

Consider any increasing sequence $D_1, D_2, \ldots$ of sets of $\mathcal{D}$. Then

$$K \backslash D_i, \quad i = 1, 2, \ldots,$$

is a decreasing sequence of non-empty compact sets, and

$$\bigcap_{i=1}^{\infty} K \backslash D_i$$

is a non-empty compact set. Since each set D_i, $i \geq 1$, is a cozero set, $\bigcup_{i=1}^{\infty} D_i$ is also a cozero set and

$$\bigcap_{i=1}^{\infty} K \backslash D_i$$

is a non-empty zero set. By our hypotheses this set contains an infinite open set G. Take any point g in G. Since K is compact and totally disconnected we can choose a clopen set H containing g and properly contained in G. Now $D_0 = K\backslash H$ is a non-empty clopen proper subset of K properly containing

$$\bigcup_{i=1}^{\infty} D_i,$$

as required.

Let $S_1, S_2, \ldots$ and $T_1, T_2, \ldots$ be two increasing sequences of clopen sets in K with

$$S_n \cap T_n = \emptyset, \quad \text{for} \quad n \geq 1.$$

Then

$$\bigcup_{n=1}^{\infty} S_n \quad \text{and} \quad \bigcup_{n=1}^{\infty} T_n$$

are disjoint cozero sets in K. Since K is an F-space these two sets have disjoint closures. Since K is compact and totally disconnected there is a clopen set S_0 with

$$\bigcup_{n=1}^{\infty} S_n \subset S_0 \quad \text{and} \quad S_0 \cap \bigcup_{n=1}^{\infty} T_n = \emptyset.$$

Now S_0 and $T_0 = K\backslash S_0$ satisfy our requirements. (Note that this result of this paragraph is proved in this way in the proof of Proposition 2.23 of R. C. Walker [38]).

Now K satisfies the hypotheses of Theorem 8 and so Theorem 9 follows from Theorem 8.

We now consider the case when $K = \beta\mathbb{N}\backslash\mathbb{N}$. For a proof that K is a compact F-space and that each non-empty $\mathcal{G}_\delta$-subset of K contains an infinite open set see, for example, van Mill [36] Theorem 1.2.5. Since $\mathbb{N}$ is strongly zero-dimensional $\beta\mathbb{N}$ is also strongly zero-dimensional, see, for example, Engelking [9] Theorem 6.2.12. This implies that $\beta\mathbb{N}\backslash\mathbb{N}$ is totally disconnected. Thus the conditions of Theorem 9 are satisfied and the result of de Maria and Rodriguez-Salinas follows.

9. Locally Compact Hausdorff Spaces that are not Pseudocompact. In this section we establish Theorem 10. We start with a construction in any locally compact Hausdorff space that is not pseudocompact. We prove three lemmas.

Lemma 21. *Let X be a locally compact Hausdorff space that is not pseudocompact. Then X is completely regular and it is possible to choose a discrete family $\{K_i : i \geq 1\}$ of compact sets, each having non-empty interior.*

Proof. Since X is not pseudocompact we can choose a function k that is continuous on X but unbounded on X. We suppose that $k(x) \geq 0$ for all x in X. We choose y_1 arbitrarily in X and then choose $y_2, y_3, \dots$ inductively in X so that

$$k(y_{i+1}) > k(y_i) + 1, \quad \text{for} \quad i \geq 1.$$

For each $i \geq 1$ write

$$D_i = \left\{ x : |k(x) - k(y_i)| < \frac{1}{3} \right\}.$$

Then $D_1, D_2, \dots$ are disjoint open sets in X with

$$y_i \in D_i, \quad i \geq 1.$$

Since X is locally compact, for each $i \geq 1$, we can choose a compact set K_i with

$$y_i \in \operatorname{int} K_i \quad \text{and} \quad K_i \subset D_i.$$

Then $K_1, K_2, \dots$ is a disjoint sequence of compact sets with non-empty interiors.

It remains to prove that the family $\{K_i : i \geq 1\}$ is discrete in X. If η is any point of X with

$$k(\eta) < k(y_1) + \frac{1}{2},$$

the set

$$\{x : |k(x) - k(\eta)| < \frac{1}{6}\}$$

is an open neighbourhood of η that may meet the set K_1 but can meet no set k_i with $i \geq 2$. Otherwise if

$$k(\eta) \geq k(y_1) + \frac{1}{2}$$

we can choose $i \geq 2$ so that

$$k(y_{i-1}) + \frac{1}{2} \leq k(\eta) \leq k(y_i) + \frac{1}{2} < k(y_{i=1}) - \frac{1}{2}.$$

In this case

$$\{x : |k(x) - k(\eta)| < \frac{1}{6}\}$$

is an open neighbourhood of η that may meet the set K_i but can meet no set K_j with $j \neq i$. Thus the family $\{K_i : i \geq 1\}$ is discrete in X.

Lemma 22. *Let X be a completely regular Hausdorff space that contains a discrete family $\{K_i : i \geq 1\}$ of compact sets, each with a non-empty interior. Then it is possible to choose a sequence $y_1, y_2, \ldots$ of points and a sequence $h_1, h_2, \ldots$ of continuous functions satisfying the following conditions.*

(a) $y_i \in \mathrm{int}\ K_i$, $i \geq 1$.

(b) $0 \leq h_i(x) \leq 1$ *for* $x \in X$, $i \geq 1$.

(c) $h_i(y_i) = 1$, *for* $i \geq 1$.

(d) $h_i(x) = 0$, *for* $x \notin K_i$, $i \geq 1$.

(e) The set $Y = \{y_i : i \geq 1\}$ *is a discrete subset of* X *that is* C^* *embedded in* X.

(f) The closure $\overline{Y}$ *of* Y *in* βX *is homeomorphic to* $\beta\mathbb{N}$ *and in the homeomorphism* Y *corresponds to* $\mathbb{N}$ *and* $\overline{Y}\backslash Y$ *corresponds to* $\beta\mathbb{N}\backslash\mathbb{N}$.

Proof. Since each set K_i has a non-empty interior we can choose the points y_i to satisfy condition (a). Since X is completely regular we can choose the continuous function h_i to satisfy

$$0 \leq h_i(x) \leq 1, \quad x \in X,$$

$$h_i(y_i) = 1,$$

$$h_i(x) = 0, \quad \text{for } x \notin \text{int } K_i.$$

Thus conditions (b) to (d) are satisfied.

Since the family $\{K_i : i \geq 1\}$ is discrete in X the set of points $Y = \{y_i : i \geq 1\}$ is discrete in X. Since Y is closed in X it is C^*-embedded in βX (that is, all continuous real-valued functions on Y extend to continuous real-valued functions on βX. By a property of the Stone-Čech compactification, see, for example, Gillman and Jerison [14] Section 6.9, the closure $\overline{Y}$ of Y in βX is homeomorphic to βY and in the homeomorphism Y is fixed and $\overline{Y}\backslash Y$ corresponds to $\beta Y\backslash Y$. Since Y is countably infinite and discrete, the closure $\overline{Y}$ of Y is homeomorphic to $\beta\mathbb{N}$ and in the homeomorphism Y corresponds to $\mathbb{N}$ and $\overline{Y}\backslash Y$ corresponds to $\beta\mathbb{N}\backslash\mathbb{N}$. This completes the proof of the lemma.

Lemma 23. *Under the conditions of Lemma 21, there is an isometric linear injective map of* $C(\beta\mathbb{N}\backslash\mathbb{N})$ *into* $C(\beta X\backslash X)$.

P r o o f. We suppose that X is a locally compact Hausdorff space that is not pseudocompact. We further suppose that the sequence $y_1, y_2, \ldots$ of points, the sequence $K_1, K_2, \ldots$ of compact sets, the sequence $h_1, h_2, \ldots$ of continuous functions on X, the set $Y = \{y_i : i \geq 1\}$ and the closure $\overline{Y}$ of Y in βX, satisfy the conditions (a) to (f) of Lemmas 22.

By condition (f) we may, and we do, identify $\mathbb{N}$ with Y, $\beta\mathbb{N}$ with $\overline{Y}$ and consequently $\beta\mathbb{N}\backslash\mathbb{N}$ with $\overline{Y}\backslash Y$. Each function f of $C(\beta\mathbb{N}\backslash\mathbb{N})$ identified with $C\left(\overline{Y}\backslash Y\right)$ has a continuous extension, $\hat{f}$ say, to $\overline{Y}$. Note that the choice of $\hat{f}$ is not unique, but, if $\hat{g}$ is any other continuous extension of g to $\overline{Y}$, one necessarily has

$$\hat{g}(y_i) - \hat{f}(y_i) \to 0 \quad \text{as} \quad i \to \infty.$$

We insist, as we may, that the extension $\hat{f}$ satisfies

$$\left\|\hat{f}\right\| = \|f\|,$$

and also that when f and g are proportion to each other on $\overline{Y}\backslash Y$ then $\hat{f}$ and $\hat{g}$ are proportional to each other on $\overline{Y}$.

We now extend $\hat{f}$ to X by taking $h = \eta(f)$ where

$$h = \sum_{i=1}^{\infty} \hat{f}(y_i)\, h_i.$$

Since the closed supports of the functions $h_i, i \geq 1$, form a discrete family in X, the function h is continuous on X. Note also that

$$\begin{aligned} |h(x)| &= \left|\sum_{i=1}^{\infty} \hat{f}(y_i) h_i(x)\right| \\ &= \max\left\{\left|\hat{f}(y_i) h_i(x)\right| : i \geq 1\right\} \\ &\leq \left\|\hat{f}\right\| \max\{|h_i(x)| : i \geq 1\} \\ &\leq \|f\|, \end{aligned}$$

for all x in X. Hence $\|h\| \leq \|f\|$.

Now h, being continuous on X, has a unique continuous extension $\hat{h} = \hat{\eta}(f)$ to βX. Let ψ denote the map from $C(\beta\mathbb{N}\backslash\mathbb{N})$ to $C(\beta X\backslash X)$ defined by taking $\psi(f)$ to be the restriction to $\beta X\backslash X$ of $\hat{h} = \hat{\eta}(f)$, for each f in $C(\beta\mathbb{N}\backslash\mathbb{N})$. Since $\left\|\hat{h}\right\| = \|h\|$, we have $\|\psi(f)\| = \left\|\hat{h}\right\| = \|h\| \leq \|f\|$ for all f in $C(\beta\mathbb{N}\backslash\mathbb{N})$. Since h coincides with $\hat{f}$

on Y, the extension $\hat{h}$ of h to βX coincides with $\hat{f}$ on $\overline{Y}$. Hence $\left\|\hat{h}\right\| = \|f\|$ and ψ is norm-preserving.

It is now necessary to prove that ψ is linear. We first note that our construction ensures that $\psi(\lambda f) = \lambda\psi(f)$ for any real λ. Let f_1, f_2 be two functions of $C(\beta\mathbb{N}\backslash\mathbb{N})$ and consider the third function

$$f_3 = f_1 + f_2.$$

Although we will not, in general, have

$$\hat{f}_3 = \hat{f}_1 + \hat{f}_2$$

on Y, the function $\hat{f}_1 + \hat{f}_2$ is a continuous extension to $\overline{Y}$ of the function $f_3 = f_1 + f_2$ on $\overline{Y}\backslash Y$, and so

$$\lim_{i\to\infty}\left(\hat{f}_3(y_i) - \hat{f}_1(y_i) - \hat{f}_2(y_i)\right) = 0.$$

Consider any point ξ of $\beta X\backslash X$ and any $\varepsilon > 0$. Then we can choose $n \geq 1$ so that

$$\left|\hat{f}_3(y_i) - \hat{f}_1(y_i) - \hat{f}_2(y_i)\right| < \varepsilon \quad \text{for} \quad i \geq n.$$

Now

$$\bigcup\{K_i : i < n\}$$

is a compact set in X, ensuring that

$$G = \beta X\backslash\bigcup\{K_i : i < n\}$$

is an open set containing ξ. Now for all x in G

$$\left|\hat{h}_3(x) - \hat{h}_1(x) - \hat{h}_2(x)\right| \leq \max\left\{\left|\hat{f}_3(y_i) - \hat{f}_1(y_i) - \hat{f}_2(y_i)\right| \; |h_i(x)| : i \geq n\right\} \leq \varepsilon.$$

Hence

$$\hat{h}_3(\xi) = \hat{h}_1(\xi) + \hat{h}_2(\xi).$$

Thus ψ is a linear isomorphic injection of $C(\beta\mathbb{N}\backslash\mathbb{N})$ into $C(\beta X\backslash X)$, as required.

Proof of Theorem 10. Let X be a locally compact Hausdorff space that is not pseudocompact. By Lemmas 21, 22 and 23, there is a linear injective isometric map ψ of $C(\beta\mathbb{N}\backslash\mathbb{N})$ into $C(\beta X\backslash X)$. Thus ψ embeds $C(\beta\mathbb{N}\backslash\mathbb{N})$ isometrically as a norm closed linear subspace, say L, of $C(\beta X\backslash X)$. The restriction of the weak topology of $C(\beta X\backslash X)$ to L coincides with the topology on L inherited from the weak topology on $C(\beta\mathbb{N}\backslash\mathbb{N})$. By the theorem of de Maria and Rodriguez-Salinas, L with the weak topology of $C(\beta X\backslash X)$ carries a non-τ-additive countably additive Borel measure μ taking only the values 0 and 1 and taking the value 0 on each point. Since L is weakly closed in $C(\beta X\backslash X)$ the space $C(\beta X\backslash X)$ with its weak topology also carries the similar measure ν defined by

$$\nu(B) = \mu(B \cap L)$$

for all weak Borel sets B in $C(\beta X\backslash X)$.

Corollary 24. *Let X be a completely regular Hausdorff space that contains a discrete family $\{K_i : i \geq 1\}$ of compact sets, each with a non-empty interior. Then $C(\beta\mathbb{N}\backslash\mathbb{N})$ embeds isometrically as a closed linear subspace of $C(\beta X\backslash X)$ and so $C(\beta X\backslash X)$ with its weak topology carries a non-τ-additive countably additive Borel measure taking only the values 0 and 1 and taking the value 0 on each point.*

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