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NEW UPPER BOUNDS FOR SOME SPHERICAL CODES

Peter Boyvalenkov, Peter Kazakov

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Abstract. The maximal cardinality of a code W on the unit sphere in n dimensions with $(x,y) \leq s$ whenever $x, y \in W,\ x \neq y$, is denoted by $A(n,s)$. We use two methods for obtaining new upper bounds on $A(n,s)$ for some values of n and s. We find new linear programming bounds by suitable polynomials of degrees which are higher than the degrees of the previously known good polynomials due to Levenshtein [11, 12]. Also we investigate the possibilities for attaining the Levenshtein bounds [11, 12]. In such cases we find the distance distributions of the corresponding feasible maximal spherical codes. Usually this leads to a contradiction showing that such codes do not exist.

1. Introduction. A finite non-empty subset of n-dimensional Euclidean sphere $\mathbf{S}^{n-1}$ is called a spherical code. A spherical code $W \subset \mathbf{S}^{n-1}$ has several characteristics, such as its cardinality, its maximal cosine $s = s(W) = \max\{(x,y) : x, y \in W,\ x \neq y\}$, its minimum distance $d = d(W) = \min\{d(x,y) : x, y \in W,\ x \neq y\}$, its degree $|A(W)| = |\{(x,y) : x, y \in W,\ x \neq y\}|$ and its strength $\tau = \tau(W) = \max\{\tau : W \text{ is a spherical } \tau\text{-design }\}$. For many links between these characteristics we refer to [10, 12, 13].

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If W is a spherical code and $x \in W$ then the distance distribution of W with respect to x [3, 9] is the system of nonnegative integer numbers $\{A_t(x) : -1 \leq t < 1\}$ where $A_t(x) = |\{y \in W : (x, y) = t\}|$.

The maximal cardinality $A(n, s)$ of a spherical code $W \subset \mathbf{S}^{n-1}$ with maximal cosine $s = s(W)$ is bounded above from [11, 12, 13]

$$(1)\ A(n,s) \leq \begin{cases} \binom{k+n-3}{k-1}\left[\frac{2k+n-3}{n-1} - \frac{P_{k-1}^{(n)}(s) - P_k^{(n)}(s)}{(1-s)P_k^{(n)}(s)}\right] & \text{for } \xi_{k-1} \leq s \leq \eta_k; \\ \binom{k+n-2}{k}\left[\frac{2k+n-1}{n-1} - \frac{(1+s)(P_k^{(n)}(s) - P_{k+1}^{(n)}(s))}{(1-s)(P_k^{(n)}(s) + P_{k+1}^{(n)}(s))}\right] & \text{for } \eta_k \leq s \leq \xi_k, \end{cases}$$

where ξ_k and η_k are the greatest zeros of Jacobi polynomials $P_k^{((n-1)/2,(n-1)/2)}(t)$ and $P_k^{((n-1)/2,(n-3)/2)}(t)$ respectively (see [1]); $P_k^{(n)}(t) = P_k^{((n-3)/2,(n-3)/2)}(t)$ are the Gegenbauer polynomials [1, 2] (normalised for $P_k^{(n)}(1) = 1$). A code $W \subset \mathbf{S}^{n-1}$ with $s(W) = s$ and $|W| = A(n, s)$ points is called maximal.

The bound (1) has been obtained by linear programming. Indeed, suitable polynomials for the following theorem were used.

Theorem 1. [10, 11] *Let $f(t)$ be a real polynomial such that:*
(C1) $f(t) \leq 0$ *for* $-1 \leq t \leq s$.
(C2) *The coefficients in the Gegenbauer expansion* $f(t) = \sum_{i=1}^{k} f_i P_i^{(n)}(t)$ *satisfy* $f_0 > 0$, $f_i \geq 0$ *for* $i = 1, 2, \ldots, k$.
Then $A(n, s) \leq f(1)/f_0$.

In this paper we improve in some cases the bound (1) using two different methods. The first one is based on the results in [5, 6], and the second uses an approach from [4, 7].

Firstly, in some cases we are able to find polynomials of degrees which are higher than the degrees of the corresponding Levenshtein polynomials. In fact, we search for good polynomials, having one double zero more [6, Theorem 2.2]. Moreover, our polynomials have several consecutive zero Gegenbauer coefficients [6, Theorem 3.1]. The best polynomials we obtain in this way are extremal in some sense [5, 6]. More precisely, they are best [6, Theorem 5.2] among the polynomials of the same or lower degree which satisfy the conditions of Theorem 1.

The above method does not work for all values of n and s. Then we investigate the possibilities for attaining the bound (1). The distance distribution of all maximal codes attaining (1) can be computed by a Vandermonde system [10, Theorem 7.4], [4, Theorem 2.1]. Thus we would obtain a contradiction unless the solutions are nonnegative integers. This gives nonexistence of corresponding maximal codes.

2. Good polynomials of higher degrees. Levenshtein [11, 12, 13] obtains the bound (1) by suitable polynomials having form $A^2(t)(t-s)$ or $A^2(t)(t+1)(t-s)$. Following [5, 6], we apply Theorem 1 for polynomials $B^2(t)G(t)(t-s)$ where

$$G(t) = \begin{cases} (t-p)^2 + q(s-t)(1+t) & \text{if } \deg(f) \text{ is odd;} \\ (t+1)(t-p)^2 + q(s-t)(t-u)^2 & \text{if } \deg(f) \text{ is even,} \end{cases}$$

$0 < q < 1$, $p, u \in [-1, s]$. Furthermore, $B(t)$ has one zero (in $[-1, s]$) more than the corresponding $A(t)$.

We express all coefficients of $B(t)$ by p and q (or p, q and u respectively). To do this, we use equalities $f_i = 0$ [6, Theorems 3.1, 3.2] and equations we have obtained by equating to zero the partial derivatives of $F = f(1)/f_0$ as a function of p, q, u. The remaining two or three parameters we determine by a Monte Carlo method minimizing the ratio $f(1)/f_0$ under the condition (C2) from Theorem 1.

As remarked, our polynomials have one double zero more than the corresponding Levenshtein polynomials. Their degree must be higher by three or four at least [6, Section 2.1]. Usually, two or three consecutive Gegenbauer coefficients of the best polynomials we find are equal to zero.

Example 1. The good polynomials of degree 9 must have the following form:

$$f(t) = (t^3 + at^2 + bt + c)^2((t-p)^2 + q(s-t)(1+t))(t-s) = \sum_{i=1}^{9} f_i P_i^{(n)}(t),$$

where $0 < q < 1$, $-1 \le p \le s$. The requirements $f_8 = f_7 = f_6 = 0$ give $a = -\alpha/2$, $b = -(\beta + 2a\alpha + a^2 + 36/(n+14))/2$ and $c = -(28(2a+\alpha)/(n+12) + \gamma + 2a\beta + \alpha(2a + b^2) + 2ab)/2$ respectively (where $(t-p)^2 + q(s-t)(1+t) = \alpha t^2 + \beta t + \gamma$). When $s > \eta_3$ we allow $f_6 > 0$ and compute c by the linear equation $2f_0 - (1+a+b+c)f'_{0c} = 0$ (where $f'_{0c} = 0$ is the partial derivative of the function $f_0 = f_0(c, p, q)$ with respect to c). This approach gives new bounds for $\xi_3 = 1/\sqrt{n+2} < s < \xi_4 = \sqrt{3/(n+4)}$ in dimensions $n \le 11$. In some cases the improvements are not in the integral part. In two tables below we give some new bounds on $A(n, s)$ in this range. These are the typical cases.

Table 1. New upper bounds for $A(n, 0.45)$, $5 \leq n \leq 10$.

n	Bound (1)	New bound	p	q	$f_6 = 0$
5	38.5009	38.0423	0.04674609	0.7687089	Yes
6	63.7149	62.004	0.0543148	0.757356	Yes
7	103.2661	99.4735	0.06189406	0.7458104	Yes
8	163.5957	160.1935	0.06995187	0.7334939	Yes
9	250.6434	249.737	-0.06348625	0.86231578	No
10	383.2484	382.6357	-0.07007453	0.8659478	No

Table 2. New upper bounds for $A(n, 0.48)$, $4 \leq n \leq 8$.

n	Bound (1)	New bound	p	q	$f_6 = 0$
4	24.3733	24.15	0.0756345	0.7586056	Yes
5	43.6207	42.447	0.0835181	0.7464715	Yes
6	75.7119	72.6399	0.0914318	0.73411734	Yes
7	124.944	124.0729	0.0139593	0.80664359	No
8	202.3733	201.6748	-0.0405325	0.8560384	No

In the next table we give bounds for $s = \sqrt{2/7} = 0.534522$ by polynomials of degree 9, 10 and 11.

Table 3. New upper bounds for $A(n, \sqrt{2/7})$, $3 \leq n \leq 10$ (R is the number of the double zeros of the corresponding extremal polynomial)

n	Bound (1)	New bound	deg(f)	R	$f_i = 0$
3	14.348	14.233	9	3	$f_7 = f_6 = 0$
4	29.561	28.77	10	3	$f_8 = f_7 = 0 = f_6$
5	56.066	55.27	10	3	$f_8 = f_7 = 0$
6	102.429	101.16	10	3	$f_9 = f_8 = f_7 = 0$
7	181.586	179.562	11	4	$f_9 = f_8 = 0$
8	307.929	303.51	11	4	$f_{10} = f_9 = f_8 = 0$
9	514.853	497.22	11	4	$f_{10} = f_9 = f_8 = 0$

3. Nonexistence of some maximal spherical codes. We need definition for spherical designs. The most convenient for using here is the following:

Definition. *A spherical code $W \subset \mathbf{S}^{n-1}$ is called a spherical τ-design if and only if*

$$\sum_{x \in W} f((x, y)) = f_0 |W| \tag{2}$$

holds for all real polynomials $f(t)$ of degree at most τ and any point $y \in S^{n-1}$. Here f_0 is as in Theorem 1. The maximal number τ for which W is a spherical τ-design is called strength of W.

Let $W \subset \mathbf{S}^{n-1}$ be a maximal code and $|W| = f(1)/f_0$ for the corresponding Levenshtein polynomial $f(t)$. Then we have the following necessary conditions (obtained by the complementary slackness in the linear programming) for attaining the bound (1) [10, 11, 12]. Firstly, $A_t(x) \neq 0$ is possible only if $f(t) = 0$. Moreover, since $f_i > 0$ for $i = 1, 2, \ldots, \deg(f)$ [11], W must be a spherical design with strength at least $\deg(f)$.

Theorem 2. ([10, Theorem 7.4], [4, Theorem 2.1]) *The distance distribution of a maximal spherical code $W \subset \mathbf{S}^{n-1}$ attaining* (1) *depends only on its cardinality and the zeros of the corresponding Levenshtein polynomial. It can be found by the following Vandermonde system:*

$$A_{\alpha_1}(x)\alpha_1^i + A_{\alpha_2}(x)\alpha_2^i + \cdots + A_{\alpha_r}(x)\alpha_r^i = f_0^{(i)}|W| - 1, \; i = 0, 1, \ldots, r-1, \tag{3}$$

where $A(W) = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$, $\alpha_1, \alpha_2, \ldots, \alpha_r$ are all different zeros of $f(t)$, $f_0^{(2j+1)} = 0$, $f_0^{(2j)} = \dfrac{(2j-1)!!}{n(n+2)\cdots(n+2j-2)}$ ($f_0^{(0)} = 1$).

Proof. We set consecutively $f(t) = 1, t, \ldots, t^{r-1}$ in (2) ($r \leq \deg(f)$, $y \in W$). The system (3) has a Vandermonde matrix. Hence its solution is unique and gives the distance distribution of W. □

In the sequel we compute the distance distributions of putative maximal spherical codes which would attain the bound (1) for degrees 3 and 4. Since the distributions do not depend on the point $x \in W$, we shall write A_t instead of $A_t(x)$. If $W \subset \mathbf{S}^{n-1}$ has $|W| = M$ points and maximal cosine $s = s(W)$ we refer to as an (n, M, s) code.

The third Levenshtein polynomial is

$$f(t) = (t+a)^2(t-s), \; a = \frac{1+s}{1+ns}.$$

It gives bound (1) when $0 \leq s \leq \dfrac{\sqrt{n+3}-1}{n+2}$. In this range we have (see (1), [12, p.66]):

$$A(n,s) \leq \frac{f(1)}{f_0} = \frac{n(1-s)(2+s+ns)}{1-ns^2}. \tag{4}$$

Let us suppose that an (n, M, s) code with $s \in (0, \dfrac{\sqrt{n+3}-1}{n+2}]$ attains the bound (4). Then by Theorem 2 the maximal code W is a spherical 3-design, $A(W) = \{s, -a\}$ and its distance distribution satisfies the following system

$$A_s + A_{-a} = \frac{n(1-s)(2+s+ns)}{1-ns^2} - 1, \; sA_s - aA_{-a} = -1.$$

Therefore we have

$$A_s(x) = \frac{(n-1)(2+s+ns)}{(1-ns^2)(1+ns^2+2s)}, \ A_{-a}(x) = \frac{(1+ns)^3(1-s)}{(1-ns^2)(1+ns^2+2s)}.$$

Example 2. a) For $n = 5$, $s = 1/5$ we obtain the distance distribution of a known 3-design with 16 points (a $(5, 16, 1/5)$ code) [10, 13, 4]. Namely, we have $A_{1/5} = 10$, $A_{-3/5} = 5$. In fact, the first author proves [8] that this code is unique up to isometry.

b) For $n = 4$, $s = 1/4$ we have $A(4, 1/4) \leq 13$. If W attains this bound then $A_{1/4} = 52/7$. Therefore $(4, 13, 1/4)$ codes do not exist, i.e. $A(4, 1/4) \leq 12$.

We set $s = 1/m \in (0, \frac{\sqrt{n+3}-1}{n+2}]$. So we have $m \geq 1+\sqrt{n+3}$. For a putative maximal $(n, M, 1/m)$ code we obtain

$$|W| = M = \frac{n(m-1)(2m+n+1)}{m^2-n},$$

$$A_{\frac{1}{m}} = B_1 = \frac{m^3(n-1)(2m+n+1)}{(m^2-n)(m^2+2m+n)}, \quad A_{-\frac{m+1}{m+n}} = B_2 = \frac{(m+n)^3(m-1)}{(m^2-n)(m^2+2m+n)}.$$

Therefore the following assertion is true:

Theorem 3. *If for some $k \geq 1 + \sqrt{n+3}$ the numbers B_1 and B_2 are not integer then $(n, M, 1/m)$ maximal codes do not exist.*

Example 3. Maximal spherical $(4, 13, 1/4)$, $(7, 28, 1/5)$, $(9, 45, 1/5)$, $(10, 56, 1/5)$, $(6, 19, 1/6)$, $(8, 30, 1/6)$, $(16, 116, 1/6)$, $(18, 165, 1/6)$, $(21, 238, 1/6)$ codes do not exist. We set $m = 4, 5, 6$ and choose the suitable cases.

To obtain infinitely many nonexistence results we set suitable values of m in Theorem 3. Let us take $m = n \geq 1 + \sqrt{n+3}$ for $n \geq 4$. Then by (4) we obtain $A(n, 1/n) \leq 3n + 1$ for $n \geq 4$. However, $B_2 = 8n/(n + 3)$ is integer only for $n = 5, 9$, and 21. Since $A(5, 1/5) = 16$, we have:

Theorem 4. *$A(n, 1/n) \leq 3n$ for $n \geq 4$ except for $n = 5$, where $A(5, 1/5) = 16$, and for $n = 9$ and 21 where $A(9, 1/9) \leq 28$ and $A(21, 1/21) \leq 64$ by (4).*

We consider now the bound (1) for degree 4. It has been obtained by

$$f(t) = (t + a)^2(t + 1)(t - s), \ a = 1/s(n + 2).$$

For $\frac{\sqrt{n+3}-1}{n+2} \leq s \leq \frac{1}{\sqrt{n+2}}$ we have ((1), [12, p.66])

$$A(n, s) \leq \frac{f(1)}{f_0} = \frac{2n(1-s)(ns+2s+1)}{1+2s-(2+n)s^2}. \tag{5}$$

Let us suppose that a maximal (n, M, s) code W attains (5). Then W is a spherical 4-design. We resolve the corresponding system (3). In fact, we need only the expression for $A_{-1} \in \{0, 1\}$:

$$A_{-1} = \frac{2s|W| + n[s^2(n+2) - s(n+1) - 1]}{n(1+s)[s(n+2) - 1]} \in \{0, 1\}.$$

Example 4. For $n = 7$, $s = 1/3$ we obtain the inner product distribution of a $(7, 56, 1/3)$ maximal code: $A_{-1} = 1$, $A_{1/3} = A_{-1/3} = 27$. In fact, this is the unique tight 7-design on $\mathbf{S}^6$ [10, 3]. For $n = 8$, $s = 1/4$ we obtain nonexistence of maximal $(8, 48, 1/4)$ codes, i.e. $A(8, 1/4) \leq 47$.

Again we set $s = 1/m \in [\frac{\sqrt{n+3} - 1}{n+2}, \frac{1}{\sqrt{n+2}}]$ (i.e. $1+\sqrt{n+3} \geq m \geq \sqrt{n+2}$). Then we find

$$A_{-1} = \frac{(m-1)(n+m+2)(n+2+2m-m^2)}{(m+1)(n-m+2)(m^2+2m-n-2)} \in \{0, 1\}.$$

Considering the two possibilities, we obtain the following necessary condition:

Theorem 5. *If a maximal $(n, M, 1/m)$ code attains the bound* (5) *then* $n + 2 + 2m - m^2 = 0$ *($A_{-1} = 0$ respectively) or $m^2 = n + 2$ ($A_{-1} = 1$).*

Example 5. Maximal spherical $(4, 16, 1/3)$, $(5, 25, 1/3)$, $(8, 48, 1/4)$, $(10, 80, 1/4)$, $(11, 102, 1/4)$, $(17, 204, 1/5)$, $(18, 240, 1/5)$, $(21, 392, 1/5)$, $(22, 464, 1/5)$ codes do not exist. We set $k = 3, 4, 5$ and choose the suitable codes. For the unique $(6, 27, 1/4)$ code [4, Theorem 4.1] we find its distance distribution $A_{1/4} = 16$, $A_{-1/2} = 10$.

Example 6. Using a similar argument we show the nonexistence of $(9, 108, 1/3)$, $(10, 146, 1/3)$, $(13, 338, 1/3)$, $(15, 576, 1/3)$, $(16, 752, 1/3)$, $(17, 986, 1/3)$, $(16, 312, 1/4)$, $(21, 756, 1/4)$ ect. codes. Such codes would attain the bound (1) for degree 5. The new bound $A(10, 1/3) \leq 145$ improves by one the last entry in [9, Table 9.2].

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Institute of Mathematics
Bulgarian Academy of Sciences
8, Akad. G. Bonchev str.
1113 Sofia, Bulgaria
e-mail: sectmoi@bgearn.bitnet

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