Serdica

Mathematical Journal

Сердика

Математическо списание

Serdica Math. J. **21** (1995), 151-170

Serdica
Mathematical Journal

Institute of Mathematics
Bulgarian Academy of Sciences

STRONG-WEAK STACKELBERG PROBLEMS IN FINITE DIMENSIONAL SPACES

Abdelmalek Aboussoror and Pierre Loridan

Communicated by R. Lucchetti

Abstract. We are concerned with two-level optimization problems called strong-weak Stackelberg problems, generalizing the class of Stackelberg problems in the strong and weak sense. In order to handle the fact that the considered two-level optimization problems may fail to have a solution under mild assumptions, we consider a regularization involving ϵ-approximate optimal solutions in the lower level problems. We prove the existence of optimal solutions for such regularized problems and present some approximation results when the parameter ϵ goes to zero. Finally, as an example, we consider an optimization problem associated to a best bound given in [2] for a system of nondifferentiable convex inequalities.

1. Introduction and motivation. Let U and V be two finite dimensional Euclidean spaces, X (resp. Y) a nonempty subset of U (resp. of V). Let f_1 and f_2 be two functions:

$$f_1 : X \times T \times Y \to \mathbb{R}, \quad T \subset Y, \quad T \neq \emptyset, \quad f_2 : X \times Y \to \mathbb{R} \cup \{+\infty\}, \quad f_2 \not\equiv +\infty.$$

We consider the following two-level optimization problem:

$$(S) \qquad \begin{cases} \operatorname*{Min}_{x\in X} \inf_{y\in M_2(x)} \sup_{z\in M_2(x)} f_1(x,y,z) \\ \text{where } M_2(x) \text{ denotes the set of optimal solutions to the lower} \\ \text{ level problem } P(x) : \left\{ \operatorname*{Min}_{y\in Y} f_2(x,y) \right. \end{cases}$$

1991 *Mathematics Subject Classification:* 90D40, 90C31, 90C25
Key words: Marginal functions, two-level optimization, limits of sets, stability, convex analysis.

Let us notice that:

– when f_1 does not depend on y, we get the Stackelberg problem in the weak sense [4, 7, 12].

– when f_1 does not depend on z, we get the Stackelberg problem in the strong sense [4, 9].

Let:

$$u_1(x) = \inf_{y \in M_2(x)} \sup_{z \in M_2(x)} f_1(x, y, z) \quad \text{and} \quad v_1 = \inf_{x \in X} u_1(x).$$

The previous remark leads us to the following definitions:

Definition 1.1. *The problem (S) is called a strong-weak Stackelberg problem.*

Definition 1.2. *Any $\overline{x} \in X$ verifying $v_1 = u_1(\overline{x})$ is called a strong-weak Stackelberg solution to (S) and v_1 is termed the strong-weak Stackelberg value.*

Examples.

1) Let f_i, $i = 1, 2, \ldots, m$, be closed proper convex functions defined on $\mathbb{R}^N$, and:

$$C = \Big\{x \in \mathbb{R}^N / \ \ f_i(x) \leq 0, i = 1, 2, \ldots, m\Big\}, \qquad \theta(x) = (f_1^+(x), f_2^+(x), \ldots, f_m^+(x))$$

with the notation $a^+ = \max(0, a)$. Suppose that C is nonempty. Auslender and Crouzeix have been interested in [2] by the best constant $k_{2,\infty}$ verifying:

$$d_2(x, C) \leq k_{2,\infty} \|\theta(x)\|_\infty \quad \forall x \in \mathbb{R}^N, \qquad \text{that is to say} \quad k_{2,\infty} = \sup_{x \notin C} \frac{d_2(x, C)}{\|\theta(x)\|_\infty}$$

where $d_2(x, C) = \min_{y \in C} \|x - y\|_2$, $\|.\|_2$ denotes the Euclidean norm on $\mathbb{R}^N$ and $\|.\|_\infty$ the Tchebycheff norm on $\mathbb{R}^m$. Let f be the function defined by: $f(x) = \max_{i=1,2,\ldots,m} f_i(x)$. Under appropriate hypotheses (see section 3), $k_{2,\infty}$ is a positive finite real number and is given in [2] by:

$$k_{2,\infty} = \frac{1}{k_{2,\infty}^*} \qquad \text{where} \qquad k_{2,\infty}^* = \inf_{x \in \mathrm{bd}(C)} \inf_{y \in \partial f(x)} \sup_{z \in \partial f(x)} \langle z, \frac{y}{\|y\|_2} \rangle$$

with: $\mathrm{bd}(C)$ the boundary of C, $\partial f(x)$ the subdifferential of f at x and $\langle .,. \rangle$ the usual inner product in $\mathbb{R}^N$. The constant $k^*_{2,\infty}$ can be seen as a strong-weak Stackelberg value. In fact, let us define the following functions:

$$f_2(x,y) = f^*(y) - \langle x, y\rangle \qquad \text{and for any } x \in \mathbb{R}^N, \quad f_1(x,y,z) = \langle z, \frac{y}{\|y\|_2}\rangle, \quad y \neq 0$$

where f^* denotes the conjugate function of f. Let $X = \mathrm{bd}(C)$ and $M_2(x)$ be the set of optimal solutions to the optimization problem:

$$\operatorname*{Min}_{y\in\mathbb{R}^N} f_2(x,y)$$

Then, the constant $k^*_{2,\infty}$ can be rewritten in the form:

$$k^*_{2,\infty} = \inf_{x\in X} \inf_{y\in M_2(x)} \sup_{z\in M_2(x)} f_1(x,y,z)$$

by using the fact that $M_2(x) = \partial f(x), \ \forall x \in X$.

2) Stackelberg games: consider a two-player Stackelberg game in which the first player called the leader has the leadership in playing the game, with an objective function f_1 and $X \subset \mathbb{R}^n$, the set of his strategies. The leader has all information about the objective function f_2 and the constraints of the second player (follower). He chooses his optimal strategy, knowing that the follower reacts optimally. For an announced strategy, the follower selects one strategy and his choice cannot be affected by the leader. The aim of the two players is to minimize their objective functions. Let $Y \subset \mathbb{R}^m$, be the set of strategies and $M_2(x)$ the reaction set (for an announced strategy x by the leader) of the follower. There are two extreme cases for the leader [6]:

i) optimistic case: the leader has to minimize the function: $\inf_{y\in M_2(x)} f_1(x,y)$,

ii) pessimistic case: the leader has to minimize the function: $\sup_{z\in M_2(x)} f_1(x,z)$.

As an intermediate case, the leader could make a choice by minimizing the function:

$$\alpha \inf_{y\in M_2(x)} f_1(x,y) + (1-\alpha) \sup_{z\in M_2(x)} f_1(x,z)$$

for some $\alpha \in]0,1[$. By setting $F_\alpha(x,y,z) = \alpha f_1(x,y) + (1-\alpha) f_1(x,z)$, the corresponding minimization problem amounts to:

$$\operatorname*{Min}_{x\in X} \inf_{y\in M_2(x)} \sup_{z\in M_2(x)} F_\alpha(x,y,z)$$

which is a strong-weak Stackelberg problem in the sense of Definition 1.1.

Remark 1.1. The problem (S) may fail to have a solution even if the decision variables x, y and z range over a compact set, whereas f_1 and f_2 are continuous, as it is seen in the following example (which is an adaptation of the one considered in [12]):

Let: $X = Y = [0,1]$, $\quad f_1(x,y,z) = -xyz$, $\quad f_2(x,y) = (x - \frac{1}{2})y$.

Then:

$$M_2(x) = \begin{cases} \{1\} & \text{for } x \in [0, \frac{1}{2}[\\ [0,1] & \text{for } x = \frac{1}{2} \\ \{0\} & \text{for } x \in]\frac{1}{2}, 1] \end{cases}$$

Let: $w_1(x,y) = \sup_{z \in M_2(x)} f_1(x,y,z)$, $\quad u_1(x) = \inf_{y \in M_2(x)} w_1(x,y)$, $\quad v_1 = \inf_{x \in X} u_1(x)$.

Then:

$$w_1(x,y) = \begin{cases} -xy & \text{for } x \in [0, \frac{1}{2}[\\ 0 & \text{for } x \in [\frac{1}{2}, 1] \end{cases} \quad \text{and} \quad u_1(x) = \begin{cases} -x & \text{for } x \in [0, \frac{1}{2}[\\ 0 & \text{for } x \in [\frac{1}{2}, 1] \end{cases}$$

The strong-weak Stackelberg value is $v_1 = -\frac{1}{2}$, but there is no $\overline{x}$ such that: $u_1(\overline{x}) = v_1$. Let us notice that the marginal function u_1 is not lower semicontinuous at $x = \frac{1}{2}$. In the sequel, we shall consider a regularization of (S) based on ϵ-approximate solutions to the lower level problems $P(x)$, $x \in X$ as in [13, 14]. More precisely, for $\epsilon > 0$, the regularized strong-weak Stackelberg problem will be defined by:

$$(S_\epsilon) \quad \begin{cases} \operatorname*{Min}_{x \in X} \inf_{y \in M_2(x)} \sup_{z \in M_2(x,\epsilon)} f_1(x,y,z) \\ \text{where } M_2(x,\epsilon) \text{ is the set of } \epsilon\text{-approximate solutions to the lower} \\ \text{level problem } P(x) : \left\{ \operatorname*{Min}_{y \in Y} f_2(x,y) \right. \end{cases}$$

Under appropriate assumptions, we shall prove the existence of solutions for the regularized problem (S_ϵ). Then, we shall give approximation results when ϵ goes to zero. Finally, we shall apply our results to the first example concerning the best bound given in [2].

2. Properties of the strong-weak regularized Stackelberg problem.

2.1. Convergence results for the lower level problems. In the sequel we shall use the following notations:

$$v_2(x) = \inf_{y \in Y} f_2(x,y) \quad \text{and for } \epsilon \geq 0, \quad M_2(x,\epsilon) = \left\{ y \in Y / \ f_2(x,y) \leq v_2(x) + \epsilon \right\},$$

with $M_2(x,0) = M_2(x) = \Big\{ y \in Y / \ f_2(x,y) = v_2(x) \Big\}$.

Let A_n , $n \in \mathbb{N}$, be a sequence of subsets of Y. We recall the following definitions:

$$\operatorname*{Lim\,inf}_{n \to +\infty} A_n = \Big\{ y \in Y / \ \exists y_n \to y, \ y_n \in A_n \Big\},$$

$$\operatorname*{Lim\,sup}_{n \to +\infty} A_n = \Big\{ y \in Y / \ \exists y_{n_k} \to y, \ y_{n_k} \in A_{n_k} \Big\}.$$

We report now some results given in [13, 14] for the lower level problems.

Proposition 2.1. *Suppose that the following assumptions are fulfilled:*

(2.1) f_2 *is lower semicontinuous on the topological space* $X \times Y$,

(2.2) *For any* $(x,y) \in X \times Y$, *for any sequence* $(x_n)_n$ *converging to* x *in* X, *there exists a sequence* $(y_n)_n$ *converging to* y *in* Y, *such that:*

$$\limsup_{n \to +\infty} f_2(x_n, y_n) \leq f_2(x,y)$$

(2.3) Y *is a convex compact subset of* V,

(2.4) $\forall x \in X$, *the function:* $y \to f_2(x,y)$ *is strictly quasiconvex on* Y [16], *that is to say:* $\forall (y,y') \in Y \times Y$, *such that:* $f_2(x,y) \neq f_2(x,y')$, $\forall \theta \in]0,1[$, *we have:*

$$f_2(x, \theta y + (1-\theta)y') < max\Big\{ f_2(x,y), f_2(x,y') \Big\}.$$

Then, for any $\epsilon > 0$, *for any* $x \in X$ *and any sequence* $(x_n)_n$ *converging to* x *in* X, *we have:*

$$M_2(x,\epsilon) \subset \operatorname*{Lim\,inf}_{n \to +\infty} M_2(x_n, \epsilon)$$

that is to say the multifunction $M_2(.,\epsilon)$ *is lower semicontinuous on* X.

Proof. The previous result is only but a particular case of the one stated in [14, Proposition 2.1] for not necessarily first countable spaces. □

Proposition 2.2. *Let us suppose that the assumption* (2.1) *and the following assumption are fulfilled:*

(2.5) *For any* $(x,y) \in X \times Y$, *for any sequence* $(x_n)_n$ *converging to* x *in* X, *there exists a sequence* $(y_n)_n$ *in* Y *such that:*

$$\limsup_{n \to +\infty} f_2(x_n, y_n) \leq f_2(x,y).$$

Then, for any $x \in X$ and any sequence $(x_n)_n$ converging to x in X, we have:

1) $\limsup\limits_{n\to+\infty} v_2(x_n) \le v_2(x)$;

2) $\operatorname*{Lim\,sup}\limits_{n\to+\infty} M_2(x_n,\epsilon_n) \subset M_2(x)$, *for any sequence* $\epsilon_n \to 0$, $\epsilon_n > 0$ $\forall n$;

3) $\operatorname*{Lim\,sup}\limits_{n\to+\infty} M_2(x_n,\epsilon) \subset M_2(x,\epsilon)$, $\forall \epsilon \ge 0$.

P r o o f. Apply Propositions 4.1 and 4.2 and Remark 4.2 in [13] by letting $f_{2,n} = f_2$, for any $n \in \mathbb{N}$. □

Proposition 2.3. *If the assumptions of Proposition* 2.2 *are satisfied and if* Y *is compact, then, for any* $x \in X$ *and any sequence* $(x_n)_n$ *converging to* x *in* X*, we have:*

$$\lim_{n\to+\infty} v_2(x_n) = v_2(x).$$

P r o o f. Apply Proposition 4.3 in [13] or see Propositions 3.1.1 and 4.1.1 in [10]. □

Remark 2.1. The assumption (2.1) and the compactness of Y imply that $M_2(x,\epsilon)$ is a nonempty compact set, $\forall \epsilon \ge 0$, $\forall x \in X$. In particular, $v_2(x)$ is a finite real number.

2.2. Convergence results for the upper level problem. In the sequel, we shall let: $T = \bigcup\limits_{x\in X} M_2(x) = M_2(X)$.

For $\epsilon > 0$, we introduce the following notations:

$$w_1(x,y,\epsilon) = \sup_{z\in M_2(x,\epsilon)} f_1(x,y,z) \quad u_1(x,\epsilon) = \inf_{y\in M_2(x)} w_1(x,y,\epsilon) \quad v_1(\epsilon) = \inf_{x\in X} u_1(x,\epsilon).$$

Proposition 2.4. *Let $\epsilon > 0$. Suppose that the assumptions of Proposition* 2.1 *and the following assumption are satisfied:*

(2.6) f_1 *is lower semicontinuous on the topological space* $X \times T \times Y$.

Then, the marginal function $w_1(.,.,\epsilon)$ *is lower semicontinuous on the topological space* $X \times T$.

P r o o f. See the results given in [11, page 160]. □

Proposition 2.5. *Let $\epsilon > 0$. Suppose that the assumptions of Proposition* 2.4 *are satisfied. Then, the marginal function* $u_1(.,\epsilon)$ *is lower semicontinuous on* X.

Proof. See [10, Proposition 4.2.1]. □

Proposition 2.6. *Let $\epsilon > 0$. Under the assumptions of Proposition 2.5 and if moreover X is compact, then the problem (S_ϵ) has at least one solution, that is to say: there exists $\overline{x} \in X$, such that: $u_1(\overline{x}, \epsilon) = v_1(\epsilon)$.*

Proof. Since the marginal function $u_1(., \epsilon)$ is lower semicontinuous (Proposition 2.5) on the compact space X, the existence of solutions to (S_ϵ) is obvious. □

Proposition 2.7. *Let $\epsilon > 0$. Suppose that the assumptions of Proposition 2.6 and the following assumption are satisfied:*

(2.7) *$\forall (x, y) \in X \times T$, the function: $z \to f_1(x, y, z)$ is upper semicontinuous on Y.*

Then there exists $(\overline{x}(\epsilon), \overline{y}(\epsilon), \overline{z}(\epsilon)) \in X \times T \times Y$ verifying:

$$v_1(\epsilon) = f_1(\overline{x}(\epsilon), \overline{y}(\epsilon), \overline{z}(\epsilon))$$

and $(\overline{y}(\epsilon), \overline{z}(\epsilon)) \in M_2(\overline{x}(\epsilon)) \times M_2(\overline{x}(\epsilon), \epsilon)$.

Proof. From Proposition 2.6, there exists $\overline{x}(\epsilon)$ such that: $u_1(\overline{x}(\epsilon), \epsilon) = v_1(\epsilon)$. By using Proposition 2.4, we deduce that the function $w_1(\overline{x}(\epsilon), ., \epsilon)$ is lower semicontinuous on T. From Remark 2.1, $M_2(\overline{x}(\epsilon))$ is a compact set. Then, there exists $\overline{y}(\epsilon) \in M_2(\overline{x}(\epsilon))$ such that:

$$w_1(\overline{x}(\epsilon), \overline{y}(\epsilon), \epsilon) = u_1(\overline{x}(\epsilon), \epsilon) = v_1(\epsilon).$$

Finally, from the upper semicontinuity of the function $f_1(\overline{x}(\epsilon), \overline{y}(\epsilon), .)$ (assumption (2.7)) and the compactness of $M_2(\overline{x}(\epsilon), \epsilon)$, there exists $\overline{z}(\epsilon) \in M_2(\overline{x}(\epsilon), \epsilon)$ verifying:

$$f_1(\overline{x}(\epsilon), \overline{y}(\epsilon), \overline{z}(\epsilon)) = w_1(\overline{x}(\epsilon), \overline{y}(\epsilon), \epsilon) = v_1(\epsilon). \quad \square$$

Proposition 2.8. *Let $x \in X$. Suppose that the following assumption is satisfied:*

(2.8) *$\forall (x, z) \in X \times Y$, the function: $y \to f_1(x, y, z)$ is lower semicontinuous on the topological space T.*

Then, for any sequence $\epsilon_n \to 0$, $\epsilon_n > 0$ $\forall n$, for any $y \in T$ and any sequence $(y_n)_n$ converging to y in T, we have:

$$w_1(x, y) \leq \liminf_{n \to +\infty} w_1(x, y_n, \epsilon_n).$$

P r o o f. Let $z \in M_2(x)$ and $\epsilon_n \to 0$, $\epsilon_n > 0$ $\forall n$. Then:

$$f_1(x, y_n, z) \leq \sup_{t \in M_2(x, \epsilon_n)} f_1(x, y_n, t) = w_1(x, y_n, \epsilon_n).$$

The assumption (2.8) implies that:

$$f_1(x, y, z) \leq \liminf_{n \to +\infty} f_1(x, y_n, z) \leq \liminf_{n \to +\infty} w_1(x, y_n, \epsilon_n).$$

Since $f_1(x, y, z) \leq \liminf_{n \to +\infty} w_1(x, y_n, \epsilon_n)$ for any $z \in M_2(x)$, we get:

$$w_1(x, y) \leq \liminf_{n \to +\infty} w_1(x, y_n, \epsilon_n). \quad \square$$

Remark 2.2. Let $x \in X$ and $\epsilon_n \to 0$, $\epsilon_n > 0$ $\forall n$. Since we have: $u_1(x) \leq u_1(x, \epsilon_n)$, $\forall n$, then:

$$u_1(x) \leq \liminf_{n \to +\infty} u_1(x, \epsilon_n).$$

Proposition 2.9. *Let $x \in X$ and $\epsilon_n \to 0$, $\epsilon_n > 0$ $\forall n$. Suppose that the assumption* (2.7) *and the assumptions of Proposition* 2.2 *are satisfied and that Y is compact. Then:*

$$\limsup_{n \to +\infty} u_1(x, \epsilon_n) \leq u_1(x).$$

P r o o f. Let $y \in M_2(x)$. Then:

$$(*) \qquad \limsup_{n \to +\infty} w_1(x, y, \epsilon_n) \leq w_1(x, y).$$

Indeed, if it is not true, there exists $\alpha \in \mathbb{R}$ such that:

$$w_1(x, y) < \alpha < \limsup_{n \to +\infty} w_1(x, y, \epsilon_n) = \lim_{\substack{n \to +\infty \\ n \in N'}} w_1(x, y, \epsilon_n)$$

where $N' \subset \mathbb{N}$. Then, there exists $n_0 \in N'$ such that for all $n \geq n_0$, $n \in N'$: $w_1(x, y, \epsilon_n) > \alpha$. So, for all $n \geq n_0$, $n \in N'$, there exists $z_n \in M_2(x, \epsilon_n)$ such that: $f_1(x, y, z_n) > \alpha$. From the compactness of Y, there exists a subsequence z_n, $n \in N'' \subset N'$ converging to $\overline{z} \in M_2(x)$ (from Proposition 2.2). Then, by using (2.7), we have:

$$\alpha \leq \limsup_{\substack{n \to +\infty \\ n \in N''}} f_1(x, y, z_n) \leq f_1(x, y, \overline{z}) \leq w_1(x, y)$$

and the contradiction. Since: $u_1(x, \epsilon_n) \leq w_1(x, y, \epsilon_n)$, $\forall n \in \mathbb{N}$, we get:

$$\limsup_{n \to +\infty} u_1(x, \epsilon_n) \leq \limsup_{n \to +\infty} w_1(x, y, \epsilon_n) \leq w_1(x, y)$$

by using inequality $(*)$. Since $\limsup\limits_{n \to +\infty} u_1(x, \epsilon_n) \leq w_1(x, y)$ for any $y \in M_2(x)$, we obtain:

$$\limsup_{n \to +\infty} u_1(x, \epsilon_n) \leq u_1(x). \quad \square$$

Corollary 2.1. *Under the assumptions of Proposition* 2.9*, for any $x \in X$, we have:*

$$\lim_{\epsilon \to 0^+} u_1(x, \epsilon) = u_1(x)$$

P r o o f. It is sufficient to apply Remark 2.2 and Proposition 2.9. $\square$

Proposition 2.10. *If the assumptions of Proposition* 2.9 *are satisfied, then for any $\epsilon_n \to 0$, $\epsilon_n > 0$ $\forall n$, we have:*

$$\lim_{n \to +\infty} v_1(\epsilon_n) = v_1.$$

P r o o f. For any $x \in X$ and any n, we have : $v_1(\epsilon_n) \leq u_1(x, \epsilon_n)$. Then, by using Proposition 2.9, we get:

$$\limsup_{n \to +\infty} v_1(\epsilon_n) \leq \limsup_{n \to +\infty} u_1(x, \epsilon_n) \leq u_1(x).$$

Since $\limsup\limits_{n \to +\infty} v_1(\epsilon_n) \leq u_1(x)$ for any $x \in X$, we obtain: $\limsup\limits_{n \to +\infty} v_1(\epsilon_n) \leq v_1$. Noticing that $v_1 \leq v_1(\epsilon)$, for any $\epsilon \geq 0$, we deduce that: $v_1 \leq \liminf\limits_{n \to +\infty} v_1(\epsilon_n)$. Then:

$$\lim_{n \to +\infty} v_1(\epsilon_n) = v_1. \quad \square$$

Remark 2.3. From the previous result, we also have: $\lim\limits_{\epsilon \to 0^+} v_1(\epsilon) = v_1$.

Proposition 2.11. *Suppose that the assumption* (2.8) *is satisfied. Then: $\forall \epsilon > 0$, $\forall x \in X$, the function: $y \to w_1(x, y, \epsilon)$ is lower semicontinuous on the topological space T.*

P r o o f. Let $\epsilon > 0$ and $x \in X$. Let $y \in T$ such that $w_1(x, y, \epsilon) \neq -\infty$, and let $(y_n)_n$ be a sequence converging to y in T. Let $a \in \mathbb{R}$ such that $a < w_1(x, y, \epsilon) =$

$\sup_{z \in M_2(x,\epsilon)} f_1(x,y,z)$. Then, there exists $z_a \in M_2(x,\epsilon)$ verifying: $f_1(x,y,z_a) > a$. So, by using (2.8):

$$a < f_1(x,y,z_a) \le \liminf_{n\to+\infty} f_1(x,y_n,z_a)$$

$$\le \liminf_{n\to+\infty}\{\sup_{z\in M_2(x,\epsilon)} f_1(x,y_n,z)\} = \liminf_{n\to+\infty} w_1(x,y_n,\epsilon)$$

Then:

$$w_1(x,y,\epsilon) \le \liminf_{n\to+\infty} w_1(x,y_n,\epsilon),$$

since we have $a < \liminf_{n\to+\infty} w_1(x,y_n,\epsilon)$ for all $a \in \mathbb{R}$ such that: $a < w_1(x,y,\epsilon)$. $\square$

Definition 2.1. *Any $(x,y,z) \in X \times Y \times Y$, verifying $f_1(x,y,z) = v_1$ and $(y,z) \in M_2(x) \times M_2(x)$ is called an hybrid exact lower Stackelberg equilibrium of (S).*

This definition generalizes the one introduced in [1] for weak Stackelberg problems.

For $x \in X$ and $\epsilon > 0$, let:

$$M_1(x,\epsilon) = \Big\{y \in M_2(x)/\ u_1(x,\epsilon) = w_1(x,y,\epsilon)\Big\}$$

and:

$$M_1(x) = \Big\{y \in M_2(x)/\ u_1(x) = w_1(x,y)\Big\}$$

be respectively the sets of optimal solutions to the minimization problems:

$$S_\epsilon(x):\ \operatorname*{Min}_{y\in M_2(x)} \sup_{z\in M_2(x,\epsilon)} f_1(x,y,z) \qquad \text{and} \qquad S(x):\ \operatorname*{Min}_{y\in M_2(x)} \sup_{z\in M_2(x)} f_1(x,y,z)$$

Let us denote

$$M_{1,\epsilon} = \Big\{x \in X/\ u_1(x,\epsilon) = v_1(\epsilon)\Big\}$$

the set of optimal solutions to (S_ϵ), and let M^0 be the projection onto X of the set of hybrid exact lower Stackelberg equilibria of (S):

$$M^0 = \Big\{x \in X/\ \exists y \in M_2(x), \exists z \in M_2(x) \text{ verifying: } f_1(x,y,z) = v_1\Big\}.$$

Let us point out that when f_1 does not depend on y, M^0 amounts to the projection onto X of the set of exact lower Stackelberg equilibrium pairs [1]. Then, we have the following stability results:

Proposition 2.12. *Suppose that the assumptions* (2.7), (2.8) *and the assumptions of Proposition* 2.2 *are satisfied and that* Y *is compact. Then, for any* $x \in X$ *and any sequence* $\epsilon_n \to 0$, $\epsilon_n > 0\ \forall n$, *we have:*

$$\operatorname*{Lim\,sup}_{n\to+\infty} M_1(x,\epsilon_n) \subset M_1(x)$$

and $M_1(x)$ *is a nonempty set.*

Proof. For any n, the marginal function $w_1(x,.,\epsilon_n)$ is lower semicontinuous on T (Proposition 2.11) and the set $M_2(x)$ is compact, then: $M_1(x,\epsilon_n) \neq \emptyset$ (that is the problem $S_{\epsilon_n}(x)$ has at least one solution). Since $M_1(x,\epsilon_n) \neq \emptyset$, $\forall n$, it follows from the compactness of Y that: $\operatorname*{Lim\,sup}_{n\to+\infty} M_1(x,\epsilon_n) \neq \emptyset$. Let $\overline{y} \in \operatorname*{Lim\,sup}_{n\to+\infty} M_1(x,\epsilon_n)$. Then, there exists a subsequence y_n, $n \in N' \subset \mathbb{N}$, converging to $\overline{y}$ in Y and $y_n \in M_1(x,\epsilon_n)$, $\forall n \in N'$, that is to say:

$$u_1(x,\epsilon_n) = w_1(x,y_n,\epsilon_n) \quad \text{and} \quad y_n \in M_2(x) \quad \forall n \in N'.$$

Then, $\overline{y} \in M_2(x)$. By using Proposition 2.8 and Corollary 2.1, we obtain:

$$w_1(x,\overline{y}) \leq \liminf_{\substack{n\to+\infty\\ n\in N'}} w_1(x,y_n,\epsilon_n) = \lim_{\substack{n\to+\infty\\ n\in N'}} u_1(x,\epsilon_n) = u_1(x) \leq w_1(x,\overline{y}).$$

Then: $w_1(x,\overline{y}) = u_1(x)$ and $\overline{y} \in M_1(x)$. $\square$

Proposition 2.13. *Suppose that* Y *is compact and that the assumptions of Proposition* 2.2 *are fulfilled. If moreover:*

(2.9) f_1 *is continuous on the topological space* $X \times T \times Y$, *then, for any* $\epsilon_n \to 0$, $\epsilon_n > 0\ \forall n$, *we have:*

$$\operatorname*{Lim\,sup}_{n\to+\infty} M_{1,\epsilon_n} \subset M^0.$$

Proof. If $\operatorname*{Lim\,sup}_{n\to+\infty} M_{1,\epsilon_n} = \emptyset$, then there is nothing to prove. Otherwise, let $\overline{x}$ be an element of $\operatorname*{Lim\,sup}_{n\to+\infty} M_{1,\epsilon_n}$. By definition, there exists a subsequence x_n, $n \in N' \subset \mathbb{N}$, converging to $\overline{x}$ in X, verifying $x_n \in M_{1,\epsilon_n}$, $\forall n \in N'$, that is: $u_1(x_n,\epsilon_n) = v_1(\epsilon_n)$, $\forall n \in N'$. From Proposition 2.11, $\forall n \in N'$, the function $w_1(x_n,.,\epsilon_n)$ is lower semicontinuous on T, and since the set $M_2(x_n)$ is compact, there exists $y_n \in M_2(x_n)$ such that:

$$u_1(x_n,\epsilon_n) = w_1(x_n,y_n,\epsilon_n) = v_1(\epsilon_n).$$

For each $n \in N'$, we deduce from (2.9) and the compactness of $M_2(x_n, \epsilon_n)$ that there exists $z_n \in M_2(x_n, \epsilon_n)$ such that:

$$w_1(x_n, y_n, \epsilon_n) = \sup_{z \in M_2(x_n, \epsilon_n)} f_1(x_n, y_n, z) = f_1(x_n, y_n, z_n).$$

From the compactness of Y, there exists a subsequence (y_n, z_n), $n \in N'' \subset N'$, converging to $(\overline{y}, \overline{z}) \in M_2(\overline{x}) \times M_2(\overline{x})$ (by using Proposition 2.2). Then:

$$f_1(\overline{x}, \overline{y}, \overline{z}) = \lim_{\substack{n \to +\infty \\ n \in N''}} f_1(x_n, y_n, z_n) = \lim_{\substack{n \to +\infty \\ n \in N''}} v_1(\epsilon_n) = v_1$$

by using (2.9) and Proposition 2.10. Then: $\overline{x} \in M^0$. □

Remark 2.4. The previous proposition stress the importance of the set M^0 in the approximation of (S) by a sequence of regularized problems (S_{ϵ_n}).

For $x \in X$, $\epsilon > 0$ and $\eta > 0$, we let:

$$M_1(x, \epsilon, \eta) = \left\{ y \in M_2(x) / \ w_1(x, y, \epsilon) \leq u_1(x, \epsilon) + \eta \right\}$$

the set of η-approximate solutions to $S_\epsilon(x)$. Then, as a complementary result concerning the intermediate problem $S(x)$ previously defined, we have:

Proposition 2.14. *Let $\eta > 0$ and $x \in X$. Suppose that the assumptions of Proposition 2.12 are fulfilled. Then, for any sequence $\epsilon_n \to 0$, $\epsilon_n > 0\ \forall n$, we have:*

$$M_1(x) \subset \operatorname*{Lim\,inf}_{n \to +\infty} M_1(x, \epsilon_n, \eta)$$

Proof. By using Proposition 2.12, we deduce that $M_1(x)$ is nonempty set. Let $\overline{y} \in M_1(x)$. From the inequality: $w_1(x, \overline{y}, \epsilon_n) \geq w_1(x, \overline{y})$, $\forall n$, and the proof of Proposition 2.9, we deduce:

$$\lim_{n \to +\infty} w_1(x, \overline{y}, \epsilon_n) = w_1(x, \overline{y})$$

and Corollary 2.1 implies that:

$$\lim_{n \to +\infty} u_1(x, \epsilon_n) = u_1(x) = w_1(x, \overline{y}).$$

Then, there exists $(n_0, n_1) \in \mathbb{N} \times \mathbb{N}$, such that:

$$| \ w_1(x, \overline{y}, \epsilon_n) - w_1(x, \overline{y}) \ | < \frac{\eta}{2} \qquad \forall n \geq n_0$$

and

$$| w_1(x, \overline{y}) - u_1(x, \epsilon_n) | < \frac{\eta}{2} \qquad \forall n \geq n_1$$

So, for all $n \geq \max\{n_0, n_1\}$, we have:

$$w_1(x, \overline{y}, \epsilon_n) \leq u_1(x, \epsilon_n) + \eta$$

that is $\overline{y} \in M_1(x, \epsilon_n, \eta)$. Then:

$$M_1(x) \subset \operatorname*{Lim\,inf}_{n \to +\infty} M_1(x, \epsilon_n, \eta). \quad \square$$

For related results in a topological setting, the reader is referred to [11].

3. Example: an optimization problem associated to a system of non-differentiable convex inequalities.

3.1. Introduction. First, we recall the example considered in section 1.

Let $f_i : \mathbb{R}^N \to \overline{\mathbb{R}}$, $i = 1, 2, \ldots, m$, be closed proper convex functions and:

$$C = \Big\{x \in \mathbb{R}^N / \ f_i(x) \leq 0, i = 1, 2, \ldots, m\Big\}, \quad \theta(x) = (f_1^+(x), f_2^+(x), \ldots, f_m^+(x))$$

with the notation $a^+ = \max(0, a)$. Suppose that C is nonempty. One of the problems considered in [2] is to find the best constant $k_{2,\infty}$ verifying:

$$d_2(x, C) \leq k_{2,\infty} \|\theta(x)\|_\infty \quad \forall x \in \mathbb{R}^N, \qquad \text{that is to say} \qquad k_{2,\infty} = \sup_{x \notin C} \frac{d_2(x, C)}{\|\theta(x)\|_\infty}$$

where $d_2(x, C) = \min_{y \in C} \|x - y\|_2$, $\|.\|_2$ the Euclidean norm on $\mathbb{R}^N$ and $\|.\|_\infty$ the Tcheby-cheff norm on $\mathbb{R}^m$. Let $k_{2,\infty}^* = \frac{1}{k_{2,\infty}}$ (if $k_{2,\infty} \neq 0$). Then, we have the following result:

Proposition 3.1 [2]. *Let f be the function defined by: $f(x) = \max_{i=1,2,\ldots,m} f_i(x)$. If the following assumptions are satisfied:*

(3.1) *C is a compact set,*

(3.2) *$C \subset \operatorname{int}(\operatorname{dom} f_i), i = 1, 2, \ldots, m$ (where $\operatorname{int}(\operatorname{dom} f_i)$ denotes the interior of the effective domain of f_i),*

(3.3) *Slater's condition: $\exists \overline{x}, \exists \eta > 0$ such that: $f_i(\overline{x}) < -\eta, \quad i = 1, 2, \ldots, m$, then, $k_{2,\infty}$ is a positive finite real number, and:*

$$k_{2,\infty}^* = \inf_{x \in \operatorname{bd}(C)} \inf_{y \in \partial f(x)} \sup_{z \in \partial f(x)} \langle z, \frac{y}{\|y\|_2} \rangle$$

(where bd(C) *denotes the boundary of* C *and* $\partial f(x)$ *the subdifferential of* f *at* x*)*

Remark 3.1. Noticing that the function f is closed, proper and convex and that $C \subset \text{int}(\text{dom} f)$, we deduce that f is a continuous function on C [18].

3.2. Preliminary results. Before going further, let us establish the following results relevant to convex analysis:

Theorem 3.1. *Let* $\epsilon \geq 0$. *Let* g *be a closed proper convex function on* $\mathbb{R}^N$, *and* G *be a nonempty compact subset of int(domg). Then, the set* $\bigcup_{x \in G} \partial_\epsilon g(x)$ *is compact.*

Proof. The multifunction $\partial_\epsilon g$ is upper semicontinuous and compact valued. Since G is compact, then $\partial_\epsilon g(G)$ is also compact (see [3], Theorem 3, page 116). □

Remark 3.2. For $\epsilon = 0$, we get a result given in [18, Theorem 24.7].

Corollary 3.1. *Under the assumptions* (3.1) *to* (3.3), *there exists* $\epsilon_0 > 0$ *such that:*

$$0 \notin Y^0 = \bigcup_{x \in \text{bd}(C)} \partial_{\epsilon_0} f(x)$$

and Y^0 *is a compact set in* $\mathbb{R}^N$.

Proof. From Slater's condition (3.3) and by using the fact that: $f(x) = 0$, $\forall x \in \text{bd}(C)$, we deduce that:

$$0 \notin \partial_\eta f(x) \quad \forall x \in \text{bd}(C).$$

Then:

$$0 \notin \bigcup_{x \in \text{bd}(C)} \partial_\eta f(x).$$

Let $\epsilon_0 = \eta$. Then:

$$0 \notin Y^0 = \bigcup_{x \in \text{bd}(C)} \partial_{\epsilon_0} f(x)$$

and Y^0 is a compact set by using Theorem 3.1. □

Now, we consider the following strong-weak Stackelberg problem associated to the constant $k^*_{2,\infty}$: find $\overline{x} \in \text{bd}(C)$ solving the minimization problem:

$$(S): \underset{x \in \text{bd}(C)}{\text{Min}} \inf_{y \in \partial f(x)} \sup_{z \in \partial f(x)} \langle z, \frac{y}{\|y\|_2} \rangle$$

Let:

$$f_1(x,y,z) = \langle z, \frac{y}{\|y\|_2}\rangle, \quad f_2(x,y) = f^*(y) - \langle x, y\rangle$$

with $f^*(y) = \sup_{x\in\mathbb{R}^N}\{\langle x,y\rangle - f(x)\}$ the conjugate function of f, and

$$X = \mathrm{bd}(C), \qquad Y = \mathrm{co}(Y^0), \qquad T = \bigcup_{x\in X} \partial f(x)$$

where $\mathrm{co}(Y^0)$ denotes the convex hull of Y^0.

The problem (S) can be rewritten as follows:

$$(S) \qquad \begin{cases} \operatorname*{Min}_{x\in X} \inf_{y\in M_2(x)} \sup_{z\in M_2(x)} f_1(x,y,z) \\ \text{where } M_2(x) = \partial f(x) \text{ denotes the set of optimal solutions to the lower} \\ \text{ level problem } P(x) : \left\{ \operatorname*{Min}_{y\in Y} f_2(x,y) \right. \end{cases}$$

We notice that $k^*_{2,\infty}$ corresponds to the strong-weak Stackelberg value considered in section 2.

For $\epsilon > 0$, we shall consider the following ϵ-regularized problem of (S) defined by:

$$(S_\epsilon) \qquad \begin{cases} \operatorname*{Min}_{x\in X} \inf_{y\in M_2(x)} \sup_{z\in M_2(x,\epsilon)} f_1(x,y,z) \\ \text{where } M_2(x,\epsilon) = \partial_\epsilon f(x) \text{ denotes the set of } \epsilon\text{-approximate solutions} \\ \text{ to the lower level problem } P(x) : \left\{ \operatorname*{Min}_{y\in Y} f_2(x,y) \right. \end{cases}$$

The following remarks collect some useful and obvious properties.

Remark 3.3.

1) Y is a convex compact set and T is a compact set (Theorem 3.1).

2) The problem $P(x)$ and the problem: $\operatorname*{Min}_{y\in Y} f_2(x,y)$, are equivalent (that is the two problems have the same set of solutions).

3) f_2 is lower semicontinuous on $X\times Y$ and the function: $y \to f_2(x,y)$ is convex on Y, for any $x\in X$.

Remark 3.4.

1) X is a compact set.

2) $\forall x \in X$, $\forall \epsilon > 0$ sufficiently small, we have: $0 \notin M_2(x, \epsilon)$ and the function f_1 is continuous on $X \times Y^0 \times \mathbb{R}^N$ (see Corollary 3.1), and in particular on $X \times T \times Y$.

3.3. Convergence results for the lower level problems. In the following, we shall always assume that the assumptions (3.1) to (3.3) are fulfilled.

Proposition 3.2. *Let $\epsilon > 0$. Then, for any $x \in X$ and for any sequence $(x_n)_n$ converging to x in X, we have:*

$$M_2(x, \epsilon) \subset \operatorname*{Lim\,inf}_{n \to +\infty} M_2(x_n, \epsilon)$$

i.e

$$\partial_\epsilon f(x) \subset \operatorname*{Lim\,inf}_{n \to +\infty} \partial_\epsilon f(x_n).$$

P r o o f. We deduce the result by using Remarks 3.1 and 3.3 and Proposition 2.1 (take $y_n = y$ $\forall n$ in (2.2)). $\square$

Proposition 3.3. *For any $x \in X$ and any sequence $(x_n)_n$ converging to x in X, we have:*

1) $\operatorname*{Lim\,sup}_{n \to +\infty} M_2(x_n, \epsilon_n) \subset M_2(x)$, *for any sequence $\epsilon_n \to 0$, $\epsilon_n > 0$ $\forall n$, that is to say:*

$$\operatorname*{Lim\,sup}_{n \to +\infty} \partial_{\epsilon_n} f(x_n) \subset \partial f(x).$$

2) $\operatorname*{Lim\,sup}_{n \to +\infty} M_2(x_n, \epsilon) \subset M_2(x, \epsilon)$, *for any $\epsilon \geq 0$, that is to say:*

$$\operatorname*{Lim\,sup}_{n \to +\infty} \partial_\epsilon f(x_n) \subset \partial_\epsilon f(x).$$

P r o o f. See Proposition 2.2. (take $y_n = y$ $\forall n$ in (2.5)). $\square$

For a further analysis concerning approximation results for the subdifferential of a convex function see [17].

3.4. Convergence results for the upper level problem. As in subsection 3.3, in order to state the following results, we always assume that the assumptions (3.1) to (3.3) are satisfied. We notice that the assumptions (2.1) to (2.9) are satisfied as a consequence of the assumptions (3.1) to (3.3). As in section 2, for $\epsilon > 0$, we shall use the following notations:

$$w_1(x,y,\epsilon) = \sup_{z \in M_2(x,\epsilon)} f_1(x,y,z), \quad u_1(x,\epsilon) = \inf_{y \in M_2(x)} w_1(x,y,\epsilon), \quad v_1(\epsilon) = \inf_{x \in X} u_1(x,\epsilon).$$

Proposition 3.4. *Let $\epsilon > 0$. The problem (S_ϵ) has at least one solution.*

Proof. Apply Proposition 2.6. $\square$

Proposition 3.5. *Let $\epsilon > 0$. There exists $(\overline{x}(\epsilon), \overline{y}(\epsilon), \overline{z}(\epsilon)) \in X \times T \times Y$, verifying:*

$$f_1(\overline{x}(\epsilon), \overline{y}(\epsilon), \overline{z}(\epsilon)) = v_1(\epsilon)$$

and $(\overline{x}(\epsilon), \overline{y}(\epsilon), \overline{z}(\epsilon)) \in X \times M_2(\overline{x}(\epsilon)) \times M_2(\overline{x}(\epsilon), \epsilon)$.

Proof. Apply Proposition 2.7. $\square$

Proposition 3.6. *The following result holds:*

$$\lim_{\epsilon \to 0^+} v_1(\epsilon) = k^*_{2,\infty}.$$

Proof. Apply Proposition 2.10. $\square$

For $x \in X$ and $\epsilon > 0$, let:

$$\begin{aligned} M_1(x,\epsilon) &= \Big\{\overline{y} \in M_2(x)/\ u_1(x,\epsilon) = w_1(x,\overline{y},\epsilon)\Big\} \\ &= \Big\{\overline{y} \in \partial f(x)/ \sup_{z \in \partial_\epsilon f(x)} \langle z, \frac{\overline{y}}{\|\overline{y}\|_2}\rangle = \inf_{y \in \partial f(x)} \sup_{z \in \partial_\epsilon f(x)} \langle z, \frac{y}{\|y\|_2}\rangle\Big\}, \\ M_1(x) &= \Big\{\overline{y} \in M_2(x)/\ u_1(x) = w_1(x,\overline{y})\Big\} \\ &= \Big\{\overline{y} \in \partial f(x)/ \sup_{z \in \partial f(x)} \langle z, \frac{\overline{y}}{\|\overline{y}\|_2}\rangle = \inf_{y \in \partial f(x)} \sup_{z \in \partial f(x)} \langle z, \frac{y}{\|y\|_2}\rangle\Big\} \end{aligned}$$

be respectively the sets of optimal solutions to the minimization problems:

$$S_\epsilon(x): \operatorname*{Min}_{y \in \partial f(x)} \sup_{z \in \partial_\epsilon f(x)} \langle z, \frac{y}{\|y\|_2}\rangle \qquad \text{and} \qquad S(x): \operatorname*{Min}_{y \in \partial f(x)} \sup_{z \in \partial f(x)} \langle z, \frac{y}{\|y\|_2}\rangle$$

$$\begin{aligned} M_{1,\epsilon} &= \Big\{\overline{x} \in X/\ u_1(\overline{x},\epsilon) = v_1(\epsilon)\Big\} \\ &= \Big\{\overline{x} \in \mathrm{bd}(C)/ \inf_{y \in \partial f(\overline{x})} \sup_{z \in \partial_\epsilon f(\overline{x})} \langle z, \frac{y}{\|y\|_2}\rangle = \inf_{x \in X} \inf_{y \in \partial f(x)} \sup_{z \in \partial_\epsilon f(x)} \langle z, \frac{y}{\|y\|_2}\rangle\Big\} \end{aligned}$$

the set of optimal solutions to (S_ϵ) and:

$$\begin{aligned} M^0 &= \Big\{x \in X/\ \exists y \in M_2(x), \exists z \in M_2(x) : f_1(x,y,z) = k^*_{2,\infty}\Big\} \\ &= \Big\{x \in \mathrm{bd}(C)/\ \exists y \in \partial f(x), \exists z \in \partial f(x) : \langle z, \tfrac{y}{\|y\|_2}\rangle = k^*_{2,\infty}\Big\}. \end{aligned}$$

Then, as in section 2, we have:

Proposition 3.7. *For any $x \in X$ and any sequence $\epsilon_n \to 0$, $\epsilon_n > 0\ \forall n$, we have:*

$$\operatorname*{Lim\,sup}_{n\to+\infty} M_1(x,\epsilon_n) \subset M_1(x)$$

and $M_1(x)$ is a nonempty set.

Proof. Apply Proposition 2.12. □

Proposition 3.8. *For any $\epsilon_n \to 0$, $\epsilon_n > 0\ \forall n$, we have:*

$$\operatorname*{Lim\,sup}_{n\to+\infty} M_{1,\epsilon_n} \subset M^0$$

and M^0 is a nonempty set.

Proof. By using Proposition 2.13, we get:

$$\operatorname*{Lim\,sup}_{n\to+\infty} M_{1,\epsilon_n} \subset M^0.$$

Since $M_{1,\epsilon_n} \neq \emptyset$, $\forall n$ (Proposition 2.6), it follows from the compactness of X that: $\operatorname*{Lim\,sup}_{n\to+\infty} M_{1,\epsilon_n} \neq \emptyset$. Then: $M^0 \neq \emptyset$. □

Remark 3.5.

1) From Proposition 3.8, there exist $\overline{x} \in \mathrm{bd}(C)$, $y^* \in \partial f(\overline{x})$ and $z^* \in \partial f(\overline{x})$ such that:

$$\langle z^*, \frac{y^*}{\|y^*\|_2}\rangle = k^*_{2,\infty} \quad \text{and} \quad d_2(x,C) \le \|\theta(x)\|_\infty \frac{1}{\langle z^*, \frac{y^*}{\|y^*\|_2}\rangle} \quad \forall x \in \mathbb{R}^N.$$

2) By taking into account the eventual vacuity of the set of optimal solutions to the problem (S) associated to $k^*_{2,\infty}$, we have introduced a sequence of regularized problems approximating (S) in the sense of Proposition 3.6 and Proposition 3.8.

Remark 3.6. Further results will appear in the thesis prepared by A. Aboussoror.

Acknowledgement. The authors wish to thank a referee for some remarks which have improved the presentation of the paper.

REFERENCES

[1] S. Addoune, P. Loridan, J. Morgan. Sous-équilibres de Stackelberg. Communication présentée aux journées d'Optimisation, Perpignan, 1992.

[2] A. Auslender, J. P. Crouzeix. Global regularity theorems. *Math. Oper. Res.*, **13** (1988), 243-253.

[3] C. Berge. Espaces topologiques, Dunod, Paris, 1966.

[4] M. Breton, A. Alj, A. Haurie. Sequential Stackelberg equilibria in two-person games. *J. Optim. Theory Appl.*, **59** (1988), 71-97.

[5] I. Ekeland, R. Temam. Analyse convexe et problèmes variationnels, Dunod, Paris, 1974.

[6] Y. Ishizuka, E. Aiyoshi. Double penalty method for bilevel optimization problems. *Ann. Oper. Res.*, **34** (1992), 73-88.

[7] G. Leitmann. On generalized Stackelberg strategies. *J. Optim. Theory Appl.*, **26** (1978), 637-643.

[8] M. B. Lignola, J. Morgan. Convergence of marginal functions with dependent constraints. *Optimization*, **23** (1992), 189-213.

[9] M. B. Lignola, J. Morgan. Regularized bilevel programming problem. Preprint No 22, 1993. Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università degli Studi di Napoli "Federico 2".

[10] M. B. Lignola, J. Morgan. Semi-continuities of marginal functions in a sequential setting. *Optimization*, **24** (1994), 241-252.

[11] M. B. Lignola, J. Morgan. Topological existence and stability for Stackelberg problems. *J. Optim. Theory Appl.*, **84** (1995), 145-169.

[12] P. Loridan, J. Morgan. Approximate solutions for two-level optimization problems. In: Trends in Mathematical Optimization. (eds. K. H. Hoffman, J. B. Hiriart-Urruty, C. Lemarechal and J. Zowe), International Series of Num. Math. **84**, Birkhäuser Verlag, Basel, 1988, 181-196.

[13] P. Loridan, J. Morgan. New results on approximate solutions in two-level optimization. *Optimization*, **20** (1989), 819-836.

[14] P. Loridan, J. Morgan. On strict ϵ-solutions for a two level optimization problem. Operations Research Proceedings of the International Conference on Operations Research 90 in Vienna, (eds. W. Buhler, G. Feichtinger, F. Hartl, F. J. Radermacher and P. Stahly), Springer Verlag, Berlin, 1992, 165-172.

[15] R. Lucchetti, F. Mignanego, G. Pieri. Existence theorems of equilibrium points in Stackelberg games with constraints. *Optimization*, **18** (1987), 857-866.

[16] O. L. Mangasarian. Non linear programming, Mc Graw-Hill, New-York, 1969.

[17] E. Meccariello. Approximation results for the subdifferential of a convex function via the ϵ-subdifferential. Preprint No 18 (1990). Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università degli Studi di Napoli "Federico 2".

[18] R. T. Rockafellar. Convex analysis, Princeton University Press, Princeton, N. J (1970).

Abdelmalek Aboussoror, Pierre Loridan
Université de Paris 1
C. E. R. M. S. E. M.
90, rue de Tolbiac
75634 Paris Cedex 13, France
Fax 40 77 19 80

Received November 11, 1994
Revised February 9, 1995