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NONLINEAR VARIATIONAL INEQUALITIES DEPENDING ON A PARAMETER

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Abstract. This paper develops the results announced in the Note [14]. Using an eigenvalue problem governed by a variational inequality, we try to unify the theory concerning the post-critical equilibrium state of a thin elastic plate subjected to unilateral conditions.

1. Introduction. Let Ω be an elastic homogeneous and isotropic thin plate identified with a bounded open connected subset of $\mathbb{R}^2$ referred to a coordinate system $0x_1\ x_2$. We assume that Ω is clamped on a part Γ_1 (with measure $\mu(\Gamma_1)$ strictly positive) of its boundary $\partial\Omega$ and is simply supported on $\Gamma_2 = \partial\Omega\backslash\Gamma_1$. We also suppose that $\partial\Omega$ is regular[1].

Let X be the subspace of

$$H^2(\Omega) := \{u \in L^2(\Omega) |\ u_{,i}, u_{,ij} \in L^2(\Omega); i, j = 1, 2\},$$

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[1] A one-dimensional manifold of class $\mathcal{C}^1$ such that the domain Ω is located in one side of $\partial\Omega$ is enough

defined by

$$X := \{u \in H^2(\Omega) |\ u = 0 \text{ on } \partial\Omega, \quad \frac{\partial u}{\partial n} = 0 \quad a.e. \text{ on } \Gamma_1\}$$

where as usually $\frac{\partial u}{\partial n}$ denotes the outward normal to Ω. In the case of an elastic plate subjected to unilateral conditions and to a transverse load, for a fixed real parameter λ measuring the magnitude of the lateral loading, the transversal displacement of the plate u is governed by the following variational equality (principle of virtual work) [27]:

$$(\mathcal{E}) \quad \int_\Omega M_{ij}(u)\cdot(v-u)_{,ij}dx + \int_\Omega \sigma_{ij}(u)\cdot u_{,i}\cdot(v-u)_{,j}dx - \int_\Omega F\cdot(v-u)dx = 0, \ \forall v \in X,$$

where $M := \{M_{ij}\}$ is the *bending moment tensor*, i.e.,

$$M_{ij}(u) = m_{ijkl} \cdot u_{,ij}$$

where the coefficients m_{ijkl} are assumed to satisfy the following assumptions:

$* \ m_{ijkl} = m_{jikl} = m_{lkij}$

$* \ m_{ijkl}\cdot u_{ij}\cdot u_{kj} \geq c\cdot u_{ij}\cdot u_{kl}$, for all $(2\times 2)-$symmetric matrices u $(c > 0)$,

$* \ m_{ijkl} \in L^\infty(\Omega)$.

We have

$$\int_\Omega M_{ij}v_{,ij}dx = \frac{Eh^3}{12(1-\nu^2)}\int_\Omega (1-v)u_{,ij}v_{,ij} + v\Delta u\Delta v dx$$

where E is the Young's modulus, ν the Poisson ratio and h the thickness of the plate.

If $\xi = (\xi_1, \xi_2)$ denotes the in-plane displacements we then have:

$$\sigma_{ij} = C_{ijkl}\left(\epsilon_{kl}(\xi) + \frac{1}{2}u_{,k}u_{,l}\right).$$

Here, $\{\sigma_{ij}\}, \{\epsilon_{ij}\}$ and $\{C_{ijkl}\}$ are respectively, the stress, strain and elasticity tensors in the plane of the plate. The components of $\{C_{ijkl}\}$ are assumed to be elements of $L^\infty(\Omega)$ and to satisfy the usual symmetry and ellipticity properties. However, a classical computation used in the Von Karman theory of plates permits to eliminate the in-plane displacements in order to describe the stress tensor as follows:

$$\sigma_{ij}(w) = \lambda \cdot \sigma^\circ_{ij} + \hat{\sigma}_{ij}(w),$$

where σ° does not depend on w and $\hat{\sigma}(w)$ is a quadratic function of w. The tensor $\lambda \cdot \sigma^\circ$ describes the constraints in the plane of the plate.

We assume that F splits into $f \in L^2(\Omega)$, which represents the external loading applied on the plate and Φ which describes the imposed unilateral conditions by obeying to a phenomenological superpotential law of the form

$$\Phi \in \partial\Psi(u)$$

where $\Psi : X \to \mathbb{R} \cup +\{\infty\}$ is a proper (not identically equal to $+\infty$) convex lower semicontinuous function. By taking into account this unilateral effect, Problem $(\mathcal{E})$ reduces to the following variational inequality:

$$(\mathcal{V}) \qquad \begin{gathered} \textstyle\int_\Omega M_{ij}(u)(v-u)_{,ij}dx + \int_\Omega \sigma_{ij}(u) \cdot u_{,i} \cdot (v-u)_{,j}dx - \\ - \textstyle\int_\Omega f(v-u)dx + \Psi(v) - \Psi(u) \geq 0, \\ \forall v \in X. \end{gathered}$$

Let us define for all $u, v \in X$

$$\begin{aligned} \langle Au, v\rangle &:= \int_\Omega M_{ij}(u) \cdot (v-u)_{,ij}dx, \\ \langle Cu, v\rangle &:= \int_\Omega \hat{\sigma}_{ij}(u) \cdot u_{,i} \cdot v_{,j}dx, \\ \langle Lu, v\rangle &:= -\int_\Omega \sigma^\circ_{ij}(u) \cdot u_{,i} \cdot v_{,j}dx, \\ \langle f, v\rangle &:= \int_\Omega f \cdot v dx. \end{aligned}$$

Then the variational inequality $(\mathcal{V})$ reduces to the form:

Find $u \in X$ such that

$$V.I.\,(\lambda, f) \qquad \langle Au - \lambda Lu + Cu - f, v-u\rangle + \Psi(v) - \Psi(u) \geq 0, \text{ for all } v \in X.$$

It is proved in [5] that

- A is linear self-adjoint and coercive, i.e., there exists $\alpha > 0$ such that:

$$\langle Au, u \rangle \alpha \cdot \|u\|^2, \quad \forall u \in X;$$

- L is self-adjoint linear and compact;
- C is strongly continuous positively homogeneous of order $p = 3$;
- $\langle x, Cx \rangle > 0$ for each $x \in X \backslash \{0\}$;
- $\langle Cx - Cy, y - x \rangle \leq c^2 \cdot \max\{\|x\|^2, \|y\|^2\} \cdot \|x - y\|^2$ $(c > 0)$,

and C is a potential operator, i.e., there exists a functional $\Phi \in \mathcal{C}^1(X \times \mathbb{R})$ whose Fréchet derivative $\Phi^{'}(u)$ has the property that

$$\langle Cx, v \rangle = \langle \Phi^{'}(u), v \rangle, \qquad \forall x, v \in X.$$

More precisely, we have

$$\Phi(x) = \frac{1}{4} \langle Cx, x \rangle.$$

Let us notice that whenever $\Psi \equiv 0$ then $V.I.(\lambda, f)$ reduces to a variational equation. A complete theory for this case can be found in [5]. The unilateral case has been the subject of several approaches based on various theories. The majority of the works are devoted to the case where the plate is only subjected to a lateral load ($f \equiv 0$), which means that 0 is a solution. When Ψ is the indicator function of a closed convex cone, i.e., when we deal with a so-called complementarity problem, A. Cimetière ([7]) did an approach based on a bifurcation theory. In [6] he used a Galerkin method. These results have been recently confirmed in [18] by using a new approach based on a variational principle.

Precise results concerning the unilateral case have also been obtained in [12] by using the Leray-Schauder degree theory. If the transversal load f is such that $\langle f, v \rangle \leq 0$, for all v belonging to the cone, then some additional results can also be found in [6] and [10]. Let us mention that further studies presented in [22]–[24] are devoted to the case of a plate only subjected to a lateral load and constrained to stay between two obstacles. For each positive value that the energy of deformation associated with prebuckled displacement in the plane may assume, the authors proved the existence of a post-buckling configuration by using a Lusternik-Schnirelmann approach. Let us also point out that many generalized models similar to $V.I.(\lambda, f)$ have been studied in the literature (see for instance [8], [9] and references therein). However, as it is also specified in [10], these setting are unsuitable when dealing with problems of Von-Karman plates. Thus for the general case, the more appropriate theory seems to be the one presented in [10]. In this paper, we use a simple approach in order to present several results applicable to the general model $V.I.(\lambda, f)$ under conditions which are perfectly compatible with the Von-Karman theory of plates.

Theorem 1 is a general existence theorem applicable for a wide class of unilateral conditions defined by functionals Ψ and of transversal loads f. Proposition 1 gives an estimate for the solution and Theorem 2 is a uniqueness result. However, if $\Psi(v) - \langle f, v\rangle \geq 0$, on X, then 0 is always a solution and in this case, Theorem 1 does not give any relevant information. Theorems 2–3–5 deal with this case where bifurcation from the line of trivial solutions could occur. The paper is divided into two sections. One is devoted to establish existence, unicity and bifurcation results while the other catalogs the diverse situations where the theory applies.

2. Main results. Let $(X, \langle\cdot,\cdot\rangle)$ be a real Hilbert space and $\Psi : X \to [0, +\infty]$ be a positive lower semicontinuous convex functional such that $\Psi(0) = 0$. We suppose given two linear operators $A, L : X \to X$ and a nonlinear operator $C : X \to X$. For f be fixed in X, we consider the problem:

$V.I.(\lambda, f)$ find $\lambda > 0$, $u \in X$ such that

$$\langle Au - \lambda Lu + Cu - f, v - u\rangle + \Psi(v) - \Psi(u) \;\geq\; 0, \quad \text{for all } v \in X,$$

and we assume that the following assumptions hold:

$[H_1]$ $A : X \to X$ is linear, self-adjoint and α-coercive, i.e.,

$$\langle Ax, x\rangle \;\geq\; \alpha \cdot \|x\|^2, \text{ for all } x \in X;$$

$[H_2]$ $L : X \to X$ is linear, self-adjoint and compact;

$[H_3]$ $C : X \to X$ is positively homogeneous of order $p > 1$, strongly continuous, positive, i.e.,

$$\langle Cx, x\rangle > 0, \text{ for all } x \in X\backslash\{0\},$$

which derives from a potentiel, i.e.,

$$Cx = \left(\frac{1}{p+1} < Cx, x >\right)'.$$

$[H_4]$ there exist $c > 0$, such that

$$< Cx - Cy, y - x > \leq c^2 \cdot \max\{\|x\|^2, \|y\|^2\} \cdot \|x - y\|^2.$$

As already mentioned in the introduction, problem $V.I.(\lambda, f)$ contains, as a particular case, the mathematical model used in the study of the post-critical equilibrium state of a thin plate subjected to some unilateral conditions.

Let us state the first result:

Theorem 1. *Suppose that assumptions $[H_1]$ through $[H_3]$ hold. Then for each $\lambda > 0$ and for each $f \in X$, the solution set of $V.I.(\lambda, f)$ is nonempty and weakly compact.*

Proof. Let $\Phi : X \to \mathbb{R}$, be the functional defined by

$$\Phi(u) := \frac{1}{2}\langle Au, u\rangle - \frac{\lambda}{2}\langle Lu, u\rangle + \frac{1}{p+1}\langle Cu, u\rangle - \langle f, u\rangle.$$

Set

$$X_n := \{x \in X \mid \|x\| > n\}.$$

X_n is a nonempty weakly compact convex set in X. Since the functional $\Phi + \Psi$ is weakly lower semicontinuous, it reaches its minimum on each X_n, let say at $x_n \in X_n$:

$$(\Phi + \Psi)(x_n) \leq (\Phi + \Psi)(v), \quad \text{for each } v \in X_n.$$

Let $v \in X_n$. Using the convexity of X_n and Ψ we get

$$\Psi(v) - \Psi(x_n) + \langle \Phi'(x_n), v - x_n\rangle \geq 0, \quad \text{for each } v \in X_n.$$

Assume for contradiction that the sequence $\{x_n \mid n \in \mathbb{N}\}$ is unbounded. Then, using the convexity of X_n, we may assume by considering possibly a subsequence that

$$(1) \qquad \langle Ax_n - \lambda Lx_n + Cx_n - f, v - x_n\rangle + \Psi(v) - \Psi(x_n) \geq 0, \quad \text{for all } v \in X_n,$$

and

$$\lim_{n\to+\infty} \|x_n\| = +\infty.$$

We claim that there exists $\varepsilon > 0$ such that

$$(2) \qquad \langle Cx_n, x_n\rangle \geq \varepsilon \cdot \|x_n\|^{p+1}.$$

If this fails to be true, by considering possibly a subsequence, and by setting $v_n := x_n/\|x_n\|$, we may suppose that

$$\lim_{n\to\infty}\langle C(v_n), v_n\rangle = 0,$$

and

$$w - \lim_{n\to\infty} v_n = v_0,$$

where $w - \lim_{n\to\infty} v_n = v_0$ means that the sequence $\{v_n|\ n \in \mathbb{N}\}$ converges to v_0 for the weak topology on X. Thus, since C is strongly continuous, we would obtain $\langle Cv_0, v_0\rangle = 0$, and therefore $v_0 = 0$ by virtue of $[H_3]$.

If we put $v = 0$ in (1), we obtain

$$\langle Ax_n - \lambda Lx_n + Cx_n - f, x_n\rangle + \Psi(x_n) \le 0,$$

or also, since $\Psi \ge 0$,

$$\lambda\langle Lx_n, x_n\rangle \ge \langle Ax_n, x_n\rangle + \langle Cx_n, x_n\rangle - \langle f, x_n\rangle.$$

By $[H_1]$ and $[H_3]$, this yields

$$\lambda\langle Lx_n, x_n\rangle \ge \ \alpha\cdot\|x_n\|^2 - \|f\|\cdot\|x_n\|.$$

Dividing the last inequality by $\lambda\cdot\|x_n\|^2$, we get

$$\langle Lv_n, v_n\rangle \ge (\alpha/\lambda) - (\|f\|/\lambda)/\|x_n\|. \tag{3}$$

Taking the limit in (3) we obtain:

$$0 = \langle Lv_0, v_0\rangle \ge \alpha/\lambda > 0,$$

a contradiction and therefore the claim is established.

If we put $v = 0$ in (1), using (2) we obtain, since $\Psi(x_n) \ge 0$,

$$\varepsilon\|x_n\|^{p+1} + \langle Ax_n, x_n\rangle - \lambda\langle Lx_n, x_n\rangle - \langle f, x_n\rangle \le 0.$$

Hence we get

$$\varepsilon + \|x_n\|^{1-p}(\alpha - \lambda.\|L\|) - \|f\|\cdot\|x_n\|^{-p} \le 0. \tag{4}$$

Taking the limit in (4) as n goes to infinity we obtain $\varepsilon \le 0$, a contradiction.

Hence the sequence $\{x_n|\ n \in \mathbb{N}\}$ is bounded.

On relabeling if necessary, we may suppose that $\{x_n|n \in \mathbb{N}\}$ weakly converges to $x^* \in X$. Fix $y \in X$. Since for n sufficiently large $y \in X_n$, we have

$$(\Phi + \Psi)(x_n) \leq (\Phi + \Psi)(y),$$

and thanks to the weak lower semicontinuity of Φ and Ψ we derive

$$(\Phi + \Psi)(x^*) \leq \lim_{n\to\infty} \inf(\Phi + \Psi)(x_n) \leq (\Phi + \Psi)(y).$$

Therefore $(\Phi + \Psi)(x^*) = \min_X(\Phi + \Psi)(y)$, and x^* is a solution of $V.I.(\lambda, f)$.

Since $\Phi + \Psi$ is weakly lower semi continuous, the solution set of $V.I.(\lambda, f)$ is clearly weakly closed. If we suppose that the solution set of $V.I.(\lambda, f)$ is unbounded, then, by similar arguments as used above, we obtain a contradiction, and the weak compactness follows. $\square$

Proposition 1. *Suppose that assumptions $[H_1]$ through $[H_3]$ hold. Let $f \in X$, and $\lambda \in (0, \alpha/\|L\|)$ be fixed. Then each solution u of $V.I.(\lambda, f)$ satisfies the following estimate*

$$\|u\| \leq \|f\|/(\alpha - \lambda\|L\|).$$

P r o o f. Let $\lambda \in (0, \alpha/\|L\|)$ and let u be a corresponding solution. We have

$$\langle Au - \lambda Lu + Cu, v - u\rangle + \Psi(v) - \Psi(u) \geq \langle f, v - u\rangle, \quad \text{for every} \quad v \in X.$$

If we put $v = 0$ in the preceding inequality, we get, since $\Psi \geq 0$,

$$\langle Au, u\rangle - \lambda\langle Lu, u\rangle + \langle Cu, u\rangle \leq \langle f, u\rangle.$$

Since $\lambda < \alpha/\|L\|$, we obtain

$$\|u\| \leq \|f\|/(\alpha - \lambda.\|L\|). \qquad \square$$

In the sequel we define $\lambda^* := (\alpha - (c\|f\|)^{2/3})/\|L\|$. The following result is in order:

Theorem 2. *Suppose that assumptions $[H_1]$ through $[H_4]$ hold. Let $f \in X$ be fixed such that $\|f\| < (\alpha^{3/2}/c)$. Then, for all $\lambda \in (0, \lambda^*)$, $V.I.(\lambda, f)$ has a unique solution.*

P r o o f. The existence follows from Theorem 1. If u_1, u_2 are two solutions for $V.I.(\lambda, f)$, we then have

$$\langle Au_1 - \lambda Lu_1 + Cu_1, u_2 - u_1\rangle + \Psi(u_2) - \Psi(u_1) \ \geq\ \langle f, u_2 - u_1\rangle$$

and

$$\langle Au_2 - \lambda Lu_2 + Cu_2, u_1 - u_2\rangle + \Psi(u_1) - \Psi(u_2) \ \geq\ \langle f, u_1 - u_2\rangle.$$

Adding the last two inequalities we derive,

$$\langle A(u_1 - u_2) - \lambda L(u_1 - u_2) + Cu_1 - Cu_2, u_2 - u_1\rangle \ \geq\ 0.$$

Thus

$$(\alpha - \lambda\|L\|)\|u_2 - u_1\|^2 \ \leq\ c^2 \cdot \max\{\|u_1\|^2, \|u_2\|^2\} \cdot \|u_2 - u_1\|^2$$

and by Proposition 1, we get

$$\|u_2 - u_1\|^2 \ \leq\ \frac{c^2 \cdot \|f\|^2}{(\alpha - \lambda\|L\|)^3}\|u_2 - u_1\|^2.$$

Since $\lambda < \lambda^*$, we obtain $u_1 = u_2$. $\square$

If $\Psi(v) - \langle f, v\rangle \geq 0$, for every $v \in X$, then $u = 0$ is solution of $V.I.(\lambda, f)$, for all $\lambda > 0$. It is then important to obtain nontrivial solutions.

We denote by Dom Ψ the *domain* of Ψ:

$$\Psi := \ \{x \in X |\ \Psi(x) < +\infty\}.$$

Set

$$\frac{1}{\rho} := \sup_{\text{Dom } \Psi\backslash\{0\}} \frac{\langle Lx, x\rangle}{\langle Ax, x\rangle},$$

and (if Ker $\Psi := \ \Psi^{-1}(0) \neq \{0\}$)

$$\frac{1}{\rho^*} := \sup_{\text{Ker } \Psi\backslash\{0\}} \frac{\langle Lx, x\rangle}{\langle Ax, x\rangle},$$

and assume that $\rho, \rho^* > 0$.

Theorem 3. *Suppose that assumptions* $[H_1]$ *through* $[H_3]$ *hold and that*

$$\Psi(v) - \langle f, v\rangle \geq 0, \quad \textit{for every} \;\; v \in X.$$

Then,

(*i*) *for all* $\lambda \in (0, \rho]$, $u = 0$ *is the unique solution for* $V.I.(\lambda, f)$;

(*ii*) *If there exists* $z^* \in \operatorname{Dom}\Psi$ *such that* $\langle Lz^*, z^*\rangle > 0$, *then there exist* $\lambda_0 > \rho$ *such that for all* $\lambda \in [\lambda_0, +\infty)$, $V.I.(\lambda, f)$ *has a nontrivial solution.*

Proof. Let $\lambda \in (0, \rho]$ be given, and suppose $u \neq 0$ is a solution for $V.I.(\lambda, f)$. We then have

$$\langle Au, u\rangle \; - \; \lambda\langle Lu, u\rangle \; \leq \; -\langle Cu, u\rangle \; + \; \langle f, u\rangle \; - \; \Psi(u) \; < \; 0.$$

Thus,

$$\frac{1}{\lambda} \; < \; \frac{\langle Lu, u\rangle}{\langle Au, u\rangle} \; \leq \; \frac{1}{\rho},$$

so that $\lambda > \rho$, a contradiction.

Theorem 1 gives the existence of a solution and this solution is obtained as the minimum of the functional $\Phi + \Psi$.

Clearly, if there exists $z \in X$ such that $(\Phi + \Psi)(z) < (\Phi + \Psi)(0) = 0$ then the minimum is reached on $X\backslash\{0\}$. We claim that there exists $\lambda_0 > 0$, such that for every $\lambda \geq \lambda_0$, there exists $z \in X$, such that $(\Phi + \Psi)(z) < (\Phi + \Psi)(0)$. Suppose on the contrary, we can construct a sequence $\{\lambda_n |\; n \in \mathbb{N}\}$, such that $\lim_{n\to\infty} \lambda_n = +\infty$, and for every $z \in X$.

$$\frac{1}{p+1}\langle Cz, z\rangle + \frac{1}{2}\langle Az, z\rangle - \frac{\lambda_n}{2}\langle Lz, z\rangle - \langle f, z\rangle + \Psi(z) \geq 0. \tag{5}$$

If we put $z = z^*$ in (5), we get

$$\frac{1}{p+1}\langle Cz^*, z^*\rangle + \frac{1}{2}\langle Az^*, z^*\rangle - \langle f, z^*\rangle + \Psi(z^*) \geq \frac{\lambda_n}{2}\langle Lz^*, z^*\rangle.$$

Hence,on passing to the limit, we obtain

$$\frac{1}{p+1}\langle Cz^*, z^*\rangle + \frac{1}{2}\langle Az^*, z^*\rangle - \langle f, z^*\rangle + \Psi(z^*) \geq +\infty,$$

a contradiction. By (i), clearly $\lambda_0 > \rho$. □

If $f = 0$, then we have the more precise result.

Theorem 4. *Suppose that assumptions $[H_1]$ through $[H_3]$ hold and that $\Psi(v) \geq 0$, for every $v \in X$. Then we have*

(*i*) *for all $\lambda \in (0, \rho], u = 0$ is the unique solution for $V.I.(\lambda, 0)$. Moreover, if*

$$\Psi(tx) = t\Psi(x), \quad \text{for all } t \geq 0 \quad \text{and} \quad x \in X,$$

then,

(*ii*) *for all $\lambda \in (\rho^*, +\infty)$, $V.I.(\lambda, 0)$ has a nontrivial solution;*

(*iii*) *if $\rho = \rho^*$ then ρ is a bifurcation point for $V.I.(\lambda, 0)$, i.e., there exists a sequence $\{\lambda_n | n \in \mathbb{N}\}$ such that $\lambda_n \to \rho$ and a sequence $\{u_n | n \in \mathbb{N}\}$ of solutions such that $u_n \neq 0$ and $u_n \to 0$.*

P r o o f. Apply (i) of Theorem 3, with $f = 0$.

The existence of a solution follows from Theorem 1 with $f = 0$. Furthermore, this solution is given as the minimum of the functional $\Phi + \Psi$ on X. Furthermore, if there exists $z \in X$ such that $(\Phi + \Psi)(z) < (\Phi + \Psi)(0)$, then the minimum is reached on $X \backslash \{0\}$.

If $\lambda > \rho^*$, then there exist $v \in \text{Ker } \Psi$ such that

$$\frac{\langle Lv, v\rangle}{\langle Av, v\rangle} > \frac{1}{\lambda}.$$

Equivalently,

(6) $$\langle Av, v\rangle - \lambda\langle Lv, v\rangle < 0.$$

Suppose that $(\Phi + \Psi)(z) \geq (\Phi + \Psi)(0) = 0$ for all $z \in X$ i.e.,

$$\frac{1}{p+1}\langle Cz, z\rangle + \frac{1}{2}\langle Az, z\rangle - \frac{\lambda}{2}\langle Lz, z\rangle + \Psi(z) \geq 0, \quad \text{for all } z \in X.$$

Since $z := tv \in \text{Ker}\ \Psi$, for all $t \in [0, +\infty)$, we have

$$\frac{t^{p+1}}{p+1}\langle Cv, v\rangle + \frac{t^2}{2}\langle Av, v\rangle - \frac{\lambda t^2}{2}\langle Lv, v\rangle \geq 0,$$

or also

$$\frac{t^{p-1}}{p+1}\langle Cv, v\rangle + \frac{1}{2}\langle Av, v\rangle - \frac{\lambda}{2}\langle Lv, v\rangle \geq 0. \tag{7}$$

Taking the limit as t tends to 0^+ in (7), we get

$$\langle Av, v\rangle - \lambda\langle Lv, v\rangle \geq 0,$$

a contradiction with (6).

Let $\lambda_n > \rho^* = \rho$, and $u_n \in X\backslash\{0\}$ such that

$$\langle Au_n - \lambda_n Lu_n + Cu_n, v - u_n\rangle + \Psi(v) - \Psi(u_n) \geq 0, \quad \text{for all} \quad v \in X. \tag{8}$$

We have

$$\langle Au_n - \lambda_n Lu_n + Cu_n, u_n\rangle \leq 0.$$

As in Theorem 1, we may prove the existence of $\varepsilon > 0$ such that

$$\langle Cu_n, u_n\rangle \geq \varepsilon \cdot \|u_n\|^{p+1}. \tag{9}$$

Combining (8) and (9) this yields,

$$\langle Au_n, u_n\rangle(1 - \lambda_n/\rho) + \varepsilon \cdot \|u_n\|^{p+1} \leq 0,$$

and therefore

$$\varepsilon\|u_n\|^{p-1} \leq \|A\|(\lambda_n/\rho^* - 1). \tag{10}$$

Considering possibly a subsequence, we may suppose that $\{u_n |\ n \in \mathbb{N}\}$ weakly converges to z, and taking the limit $\lambda_n \to \rho^*$ in (10), we get $\varepsilon\|z\|^{p-1} \leq 0$ and $z = 0$. □

Remark 1. Assumption iii) is satisfied if $\text{Dom}\ \Psi = \text{Ker}\ \Psi$. This case occurs if for example Ψ is the indicator function I_K of a nonempty closed convex subset K and defined as

$$I_K(z) = \begin{cases} 0 & \text{if } z \in K, \\ +\infty & \text{otherwise,} \end{cases}$$

Theorem 5. *Suppose that assumptions $[H_1]$ through $[H_3]$ hold and that $\Psi(v) \geq 0$, for every $v \in X$. We suppose also that* Dom Ψ *is closed and there exists $\beta > 2$ such that*

$$\Psi(tx) = t^\beta \Psi(x), \quad \textit{for all } \ t \geq 0 \ \textit{ and } x \in X.$$

Then we have

(*i*) *for all $\lambda \in (\rho, +\infty)$, $V.I.(\lambda, 0)$ has a nontrivial solution;*

(*ii*) *ρ is a bifurcation point for $V.I.(\lambda, 0)$.*

Proof. The existence of a solution follows from Theorem 1 with $f = 0$, and this solution is given as the minimum of the functional $\Phi + \Psi$ on X. It remains to prove the existence of $z \in X$ such that $(\Phi + \Psi)(z) < (\Phi + \Psi)(0)$.

If $\lambda > \rho$, then there exist $v \in \text{Dom } \Psi \setminus \{0\}$ such that

$$\frac{\langle Lv, v\rangle}{\langle Av, v\rangle} \ > \ 1/\lambda,$$

so that

$$\langle Av, v\rangle - \lambda\langle Lv, v\rangle \ < 0.$$

Suppose that $(\Phi + \Psi)(z) \ \geq \ (\Phi + \Psi)(0) = 0$, for all $z \in X$, i.e.,

$$\frac{1}{p+1}\langle Cz, z\rangle + \frac{1}{2}\langle Az, z\rangle - \frac{\lambda}{2}\langle Lz, z\rangle + \Psi(z) \geq 0, \quad \text{for all } \ z \in X.$$

Let $t \in [0, +\infty)$ and put $z = tv$. We get,

$$\frac{t^{p+1}}{p+1}\langle Cv, v\rangle + \frac{t^2}{2}\langle Av, v\rangle - \frac{\lambda t^2}{2} - \langle Lv, v\rangle + t^\beta \Psi(v) \geq 0,$$

or also

$$\frac{t^{p-1}}{p+1}\langle Cv, v\rangle + \frac{1}{2}\langle Av, v\rangle - \frac{\lambda}{2}\langle Lv, v\rangle + t^{\beta-2}\Psi(v) \geq 0. \tag{11}$$

Taking the limit as t tends to 0^+ in (11), we get

$$< Av, v > -\lambda < Lv, v > \geq \ 0,$$

a contradiction.

We prove ii) by a simple review of the proof of part (iii) in Theorem 4. □

3. Application to equilibrium state of a thin elastic plate. Let us again consider the problem defined in Section 1. For the following examples, we will assume that compressive forces are acting on Ω. Precisely, we require that:

$$\int_\Omega \sigma_{ij}^0(u) \cdot u_{,i} \cdot u_{,j} dx < 0, \quad \forall u \in X \setminus \{0\}. \tag{12}$$

Example 3.1. The plate is supported by a frictionless plan.

1. For a plate only subjected to a lateral force ($f \equiv 0$) and supported by a rigid frictionless plane, we have
$$\Psi(z) = I_K(z),$$
where K is the closed convex cone of positive displacements in X, i.e.,
$$K := \{z \in X \mid z \geq 0, \ \ a.e. \ \ \text{on } \Omega\}.$$
In this case $\rho = \rho^*$ and by Theorem 4, we can say that ρ is a bifurcation point for $VI(\lambda, 0)$, i.e., a critical load for our mechanical problem. This result covers some results which have been previously obtained in [6], [11], [18], [22], [23] and [24].

2. If the plate is pressed on the rigid plan by a transversal load then $f \neq 0$ and
$$\langle f, v\rangle \ \leq \ 0, \quad \forall v \in K.$$
By Theorem 3, we can say that if $\lambda \in (0, \rho]$ then 0 is the unique solution. Moreover, for λ is large enough we can conclude to the existence of a post-buckling configuration.

Example 3.2. The plate is only subjected to a lateral force and is restricted to stay between two obstacles. If the transversal displacement u of the plate is restricted to stay between two rigid frictionless, flat horizontal surfaces, then we have

$$\Psi(z) = I_C(z),$$

where C is the closed convex set

$$C := \{v \in X \mid \sigma_1 \leq v \leq \sigma_2, \ \text{for all } \ x \in \Omega, \ \ \sigma_1 \leq 0, \ \ \sigma_2 \geq 0, \ \ \sigma_i \neq 0, \ \ i = 1 \text{ or } 2\}$$

where σ_i are functions in $H^2(\Omega)$.

In the previous theories [22], [23], [24], it has been proved that there exists at least one post-buckling configuration for each positive value that the energy of deformation associated with prebuckled inplane displacement may assume. By Theorem 3 we prove the existence of $\lambda_0 > 0$ such that for each $\lambda \geq \lambda_0$ there exists at least one post-buckling configuration. Moreover, by Theorem 3, this last result remains true if (12) is replaced by the less restrictive condition: there exists $z \in C$ such that

$$\int_\Omega \sigma_{ij}^o(u) \cdot z_{,i} \cdot z_{,j} dx < 0. \tag{13}$$

Example 3.3. A General unilateral problem.

More generally, we may suppose that the normal loading possesses a superpotential j, which is a proper convex and lower semi-continuous function. In this case

$$\Psi(z) := \begin{cases} \int_\Omega j(z), & \text{if } j(z) \in L^1(\Omega), \\ +\infty, & \text{otherwise.} \end{cases}$$

We assume, for instance, that the plate is only subjected to a lateral stress satisfying condition (12). The problem is solved in [10] with some restrictive conditions on j. Applying our results, we are able to get specific results for many categories of functionals j. If

$$\begin{array}{ll} j(t\xi) = t \cdot j(\xi), & \text{for all } t \geq 0, \xi \in \mathbb{R} \\ j(\xi) \geq 0, & \text{for all } \xi \in \mathbb{R}, \end{array}$$

then by Theorem 4, 0 is the unique solution for each $\lambda \in (0, \rho]$ and there exists at least one post buckling configuration for each $\lambda > \rho^*$. If

$$\begin{array}{ll} j(t\xi) = t^\beta \cdot j(\xi) (\beta > 2), & \text{for all } t \geq 0, \xi \in \mathbb{R} \\ j(\xi) \geq 0, & \text{for all } \xi \in \mathbb{R}, \end{array}$$

then ρ is a critical load. For example, if $j(\xi) = |\xi|^3$ then Dom $\Psi = X$ and

$$\frac{1}{\rho} := \sup_{u \in X \setminus \{0\}} \left(\frac{\langle Au, u \rangle}{\langle Lu, u \rangle} \right)$$

is a critical load.

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