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CALCULATION OF RELIABILITY CHARACTERISTICS FOR REGENERATIVE MODELS

Vladimir Kalashnikov

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ABSTRACT. If a regenerative process is represented as semi-regenerative, we derive formulae enabling us to calculate basic characteristics associated with the first occurrence time starting from corresponding characteristics for the semi-regenerative process. Recursive equations, integral equations, and Monte-Carlo algorithms are proposed for practical solving of the problem.

1. Introduction. Paying tribute to Academician Nikola Obreshkov, one must remember that not only he was an outstanding expert in algebra and calculus but he had a deep interest to probability theory, numerical methods, mathematical statistics, numbers theory, etc.

This contribution deals with topics which were in the sphere of action of Nikola Obreshkov: random processes, numerical methods, mathematical statistics.

The setup arose in reliability theory. Although the results can be immediately applied in risk theory and the theory of queues we will use reliability terminology, just for definiteness.

Assume that we consider a redundant system and the *reliability function* (probability $\mathbf{P}(\tau > x)$ that the *first break-down time* τ exceeds x, $x > 0$) should be either calculated or estimated. Very often, the dynamics of the system can be described as a

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regenerative process (RP). This means that there exists a sequence $S_0 < S_1 < \dots$ of random times (*regeneration times*) which divide the process into independent identically distributed (i.i.d.) *cycles* (see Kalashnikov [6]). In this case, τ can be represented as a random sum of lengths of i.i.d. cycles (the number of summands has a geometric distribution) and, perhaps, one additional summand. Such a representation is typical in reliability; we refer to Brown [3] and Kalashnikov [6, Chapter 4] for further references. Owing to the representation of τ as a *geometric sum* one can approximate $\mathbf{P}(\tau > x)$ by various methods (see Kalashnikov [6, 7]). But every approximation is stated in terms of the underlying RP. It is noteworthy that the desired characteristics of the RP are not easy to obtain. Until now, there is no general method to solve this practical problem.

This paper shows that such a problem is fully tractable, assuming that the underlying RP can be treated as a *semi-regenerative process* (SRP). In this case, we propose various calculation methods based on solutions of relevant integral equations and Monte-Carlo algorithms.

The paper is organized as follows. In Section 2, basic definitions associated with RP and SRP are given. In particular, three main characteristics for estimation of first occurrence times are stated. They are a probability q to fail during a regeneration cycle, a conditional distribution function (d.f.) F of a length of a regeneration cycle given there is no fail within it, and a conditional d.f. F_1 of the first occurrence time within a failed cycle. Similar characteristics are stated for SRP. If a RP is regarded as a SRP, then the relations enabling us to find q, F, and F_1 in terms of corresponding characteristics of the SRP are derived in Section 3. With the relationships in the base, Section 4 contains recurrence equations and Section 5 integral equations for finding the characteristics indicated. Monte-Carlo algorithms to estimate q, F, and F_1 are proposed in Section 6.

2. Regenerative and semi-regenerative processes. We start with well-known notions associated with regenerative processes (see Kalashnikov [6]). Let (z, S) be a pair where z is a random process with a complete separable metric state space $(\mathcal{Z}, \mathcal{B})$ evolving in time $t \in T = [0, \infty)$ and $S = (S_0, S_1, \dots)$ is an increasing sequence of non-negative random times called *renewals*.

Let

$$N(t) = \#\{n : S_n < t, n \geq 0\} \tag{2.1}$$

be a number of renewals occurring within $[0, t)$ and

$$D_n = S_n - S_{n-1}, \quad n \geq 1, \tag{2.2}$$

be successive *inter-renewal times*. Define a *shift operator* θ_s for (z, S) on time s onward:

$$\theta_s(z, S) = \Big((z(s + t, \omega))_{t \geq 0}, (S_{N(s)+k} - s)_{k \geq 0}\Big). \tag{2.3}$$

Definition 1 . *Random pair* (z, S) *is a RP if, for any* $n \geq 0$,
(*i*) $\theta_{S_n}(z, S)$ *are identically distributed;*
(*ii*) $\theta_{S_n}(z, S)$ *does not depend on the prehistory* $(z(t))_{t<S_n}$, $S_0, \ldots, S_n$.

In the present form, Definition 1 was proposed by Thorisson [12], though it is equivalent to the Smith classic definition (see Smith [10]). It follows that sequence D_n, $n \geq 1$, defined in (2.2) consists of i.i.d.r.v.'s that is $(S_n)_{n\geq 0}$ comprises a renewal process. Instants S_n are also called *regeneration times* and r.v.'s D_n, $n \geq 1$, *inter-regeneration times*.

Let us assume, for simplicity, that (z, S) is a zero-delayed RP that is $S_0 = 0$. Then regeneration times S_n, $n \geq 0$, partition process (z, S) into a sequence of i.i.d. cycles

$$C_n \equiv C_n(\omega) = \left(\left(\theta_{S_{n-1}} z(t, \omega)\right)_{0\leq t<D_n(\omega)}, D_n(\omega) \right), \quad n \geq 1. \tag{2.4}$$

Random elements C_n take their values from state space $\mathcal{Z}^T \times T$ and a corresponding probability measure on this space

$$\mathbf{P}_C(A) = \mathbf{P}\left(\omega : C_n(\omega) \in A\right), \quad A \subset \mathcal{Z}^T \times T,\, n \geq 1, \tag{2.5}$$

does not depend on $n \geq 1$ because all cycles are i.i.d.

Let us define the first occurrence time for RP. For this, we divide space $\mathcal{Z}^T \times T$ into two disjoint subsets

$$\mathcal{Z}^T \times T = \mathcal{C}_+ \cup \mathcal{C}_-, \tag{2.6}$$

where $\mathcal{C}_+ \cap \mathcal{C}_- = \emptyset$. A cycle C is called *bad* if $C \in \mathcal{C}_-$. Otherwise, C is called *good*. Denote

$$\Omega_-^{(n)} = \left\{\omega : C_n(\omega) \in \mathcal{C}_-\right\}, \quad \Omega_+^{(n)} = \left\{\omega : C_n(\omega) \in \mathcal{C}_+\right\} \tag{2.7}$$

and let

$$q = \mathbf{P}(\Omega_-^{(n)}) \tag{2.8}$$

be the probability that cycle C_n is bad.

Let us designate a generic cycle by $C = (z, D)$, where the meaning of the components z and D is clear from (2.4). We define, for a bad cycle $C \in \mathcal{C}_-$, a nonnegative r.v.

$$\xi(C) = \xi(z, D) \leq D, \tag{2.9}$$

naming this the *first occurrence time within cycle* C. Let

$$\xi_n = \xi(C_n(\omega)), \quad n \geq 1. \tag{2.10}$$

Evidently, all r.v.'s ξ_n are i.i.d. Moreover, they are defective, in general, because

$$\mathbf{P}(\xi_n < \infty) = \mathbf{P}(\Omega_-^{(n)}) = q \leq 1,$$

and we do not define $\xi(C)$ for good cycles $C \in \mathcal{C}_+$.

We now introduce the *first occurrence time* $\tau = \tau(\omega)$ for RP (z, S) as follows. Let

$$\nu(\omega) = \min\{n : \omega \in \Omega_-^{(n)},\ n \geq 1\}. \tag{2.11}$$

Since all cycles are i.i.d. and because of (2.8),

$$\mathbf{P}(\nu = k) = q(1-q)^{k-1}, \quad k \geq 1. \tag{2.12}$$

Put

$$\tau(\omega) = S_{\nu(\omega)-1}(\omega) + \xi_{\nu(\omega)}.$$

It is possible to represent τ as a geometric sum of i.i.d.r.v.'s. For this, we consider independent r.v.'s Z, X_1, X_2, ... and suppose that all X_i, $i \geq 1$, are i.i.d. Let X stands for a generic r.v., $X \stackrel{d}{=} X_1$ ($\stackrel{d}{=}$ denotes the equality in distribution). Let all indicated r.v.'s do not depend on ν and have the following distribution functions

$$F_1(x) = \mathbf{P}(Z \leq x) = \mathbf{P}(\xi(C) \leq x \mid C \in \mathcal{C}_-) \tag{2.13}$$

and

$$F(x) = \mathbf{P}(X \leq x) = \mathbf{P}(D \leq x \mid C = (z, D) \in \mathcal{C}_+). \tag{2.14}$$

Obviously,

$$\tau \stackrel{d}{=} Z + X_1 + \cdots + X_{\nu-1}. \tag{2.15}$$

Then

$$\mathbf{P}(\tau \leq x) = qF_1(x) + (1-q)F_1 * W_q(x),$$

where W_q is the d.f. of a geometric sum $X_1 + \cdots + X_\nu$ that is

$$W_q(x) = \sum_{k=1}^{\infty} q(1-q)^{k-1} F^{*k}(x). \tag{2.16}$$

Example 1. Let a redundant system consist of an operating element, N unloaded redundant elements, and a repairing unit. When operating, each element has random lifetime with d.f. $A(u) = 1 - \exp(-\lambda u)$. Upon failure, it joins the queue before the repairing unit if it is busy or starts repairing if the repairing unit is idle. After restoration, the element joins the group of redundant elements if one of the elements is operating or starts operating if all other elements failed. Repair times are i.i.d. having the d.f. $B(u)$. A failed element is instantly replaced by one of the redundant elements (if any).

Let $z(t)$ be a number of failed elements at time t. Assume that $z(0) = 0$ and

$$S_0 = 0, \quad S_{n+1} = \min\{t : z(t) = 0,\ z(t-0) \neq 0,\ t > S_n\},\ n \geq 0.$$

Then (z, S) is a RP. Define

$$\tau = \min\{t : z(t) = N + 1\}$$

as the first break-down time of the system. This defines the partition (2.6) and all relations (2.8) through (2.16) are valid. But d.f.'s F and F_1 as well as probability q are difficult to define in terms of d.f.'s A and B.

A number of methods were developed to estimate d.f. W_q; see Brown [3], Kalashnikov [6, 7], and Soloviev [11]. But any bound should be necessarily expressed in terms of d.f.'s F and F_1 and probability q. The main purpose of this paper is to propose a way of finding these parameters in the case where each regeneration cycle consists of a sequence of simpler *semi-regeneration cycles* that is in the case where (z, S) can be regarded as a SRP. The concept of semi-regeneration relates closely to the concept of Markov renewal processes. The latter is discussed in Disney and Kiessler [4].

Let us construct a SRP with the help of a semi-Markov process. Suppose $(E, \mathcal{E})$ is a complete separable metric space and each $\alpha \in E$ is equipped with a r.v. Y_α having the d.f.

$$G_\alpha(x) = \mathbf{P}(Y_\alpha \le x). \tag{2.17}$$

Let

$$K(\alpha, x;\, A) = \mathbf{P}(\alpha_{n+1} \in A \in \mathcal{E} \mid \alpha_n = \alpha, Y_\alpha = x) \tag{2.18}$$

be a transition probability that defines the dynamics of a random sequence

$$(\alpha_n, Y^{(n)}) \equiv (\alpha_n, Y_{\alpha_n}), \quad n \ge 0,$$

as follows. If $\alpha_n = \alpha$ and $Y^{(n)} = x$, then state α_{n+1} is a r.v. having the probability distribution (2.18), while $Y^{(n+1)}$ is a r.v. independent of other characteristics and distributed as

$$G_{\alpha'}(x) = \mathbf{P}(Y_{\alpha_{n+1}} \le x \mid \alpha_{n+1} = \alpha').$$

In order to complete the definition of the sequence, it is sufficient to fix the initial state α_0. The sequence (α_n, Y_{α_n}) thus defined is called a *semi-Markov process*.

Semi-Markov process can be viewed as a continuous time random process with piecewise-constant paths occupying state $\alpha_n = \alpha$ for random time Y_α after which it jumps to a new state α_{n+1} and r.v. α_{n+1} depends on only α and Y_α. In each state α, occupation time Y_α depends on α only and has d.f. G_α, successive occupation times (for a given state) being independent.

Apparently, $(\alpha_n)_{n\ge 0}$ is a time-homogeneous Markov chain taking its values in $(E, \mathcal{E})$ and having a transition kernel

$$K(\alpha; A) = \int_0^\infty K(\alpha, x; A)\, dG_\alpha(x). \tag{2.19}$$

Assume the chain to have a *proper atom* in the following sense: there exists a state $a \in E$ such that it is positive recurrent; that is, $\mathbf{E}\psi_a < \infty$ where

$$\psi_a = \min\{n:\ \alpha_n = a,\ n \geq 1\}, \quad \alpha_0 = a, \tag{2.20}$$

is the first recurrence time to state a.

We now construct a SRP. For this, assume that each state $\alpha \in E$ is equipped with a random pair (Y_α, z_α) where Y_α is an occupation time of this state and $z_\alpha(t)$, $0 \leq t < Y_\alpha$, is a random process evolving over the occupation time. Both random elements are dependent, in general. This means that there is a family of probability measures

$$\mathbf{P}_\alpha(B) = \mathbf{P}\left((Y_\alpha, z_\alpha) \in B\right), \quad \alpha \in E, \tag{2.21}$$

serving as probability distributions of generic pairs (Y_α, z_α). In addition, for each $\alpha \in E$ and each "value" of the pair (Y, z) we define a probability measure

$$\mathbf{Q}_\alpha(A \mid Y, z),\ A \in \mathcal{E}. \tag{2.22}$$

Construct a sequence of random triples

$$(\alpha_n, Y_{\alpha_n}, z_{\alpha_n}) \equiv \left(\alpha_n, Y^{(n)}, z^{(n)}\right), \quad n \geq 0,$$

as follows. Let $\alpha_0 = a$ and $(Y^{(0)}, z^{(0)})$ be a random pair having a distribution $\mathbf{P}_a$. Given $(\alpha_k, Y^{(k)}, z^{(k)})$, $0 \leq k \leq n$, r.v. α_{n+1} depends on $(\alpha_n, Y^{(n)}, z^{(n)})$ only and has a conditional distribution

$$\mathbf{P}\left(\alpha_{n+1} \in A \mid (\alpha_k, Y^{(k)}, z^{(k)})_{0 \leq k \leq n}\right) = \mathbf{Q}_{\alpha_n}\left(A \mid Y^{(n)}, z^{(n)}\right).$$

In turn, $(Y^{(n+1)}, z^{(n+1)})$ is a random pair only depending on α_{n+1} and having distribution

$$\mathbf{P}\left((Y^{(n+1)}, z^{(n+1)}) \in B \mid (\alpha_k, Y^{(k)}, z^{(k)})_{0 \leq k \leq n},\ \alpha_{n+1} = \alpha\right) = \mathbf{P}_\alpha(B).$$

These constructions define a random sequence $(\alpha_n, Y^{(n)}, z^{(n)})$, $n \geq 0$, uniquely in distribution. Sequence $(\alpha_n, Y^{(n)})$, $n \geq 0$, forms a semi-Markov process while α_n, $n \geq 0$, is a Markov chain. Let

$$\sigma_0 = 0, \quad \sigma_1 = Y_{\alpha_0}, \ldots, \sigma_n = Y_{\alpha_0} + \cdots + Y_{\alpha_{n-1}}$$

and

$$\sigma = (\sigma_0, \sigma_1, \ldots). \tag{2.23}$$

Random times σ_n, $n \geq 0$, can be called *semi-regeneration times*. Define a random process

$$z(t) = z_{\alpha_n}(t - \sigma_n), \quad \sigma_n \leq t < \sigma_{n+1},\ n \geq 0. \tag{2.24}$$

Definition 2 . *Random pair* (z, σ) *with sequence* σ *defined in* (2.23) *and process* z *defined in* (2.24) *is called a SRP.*

Let us call the pair (Y_α, z_α) a *generic semi-regeneration cycle* associated with state α or simply α-*cycle*. Markov chain $(\alpha_n)_{n\geq 0}$ is called *accompanying*. Denote by

$$S_0 = \sigma_0 = 0, \quad S_{n+1} = \min\{\sigma_k : \sigma_k \geq S_n,\, \alpha_k = a\}, \quad n \geq 0, \tag{2.25}$$

successive semi-regeneration times at which the accompanying Markov chain returns to state a. Apparently, (z, S) is a RP because of the imposed restrictions. Therefore, (z, S) is a RP induced by SRP (z, σ). Should state space E consist of a single element a, SRP (z, σ) is reduced to a RP (z, S). In this case, $\sigma_i \equiv S_i$, $i \geq 0$.

We now introduce the first occurrence time for SRP (z, σ). Let us view $\mathcal{Z}^T \times T$ as a space of possible "values" of semi-regeneration cycles. For each $\alpha \in E$, we partition $\mathcal{Z}^T \times T$ into two subsets

$$\mathcal{Z}^T \times T = \mathcal{C}_+^{(\alpha)} \cup \mathcal{C}_-^{(\alpha)}$$

and call α-cycle $C_\alpha = (z_\alpha, Y_\alpha)$ *bad* if it belongs to $\mathcal{C}_-^{(\alpha)}$ and *good* otherwise. Let

$$q(\alpha) = \mathbf{P}\left(C_\alpha \in \mathcal{C}_-^{(\alpha)}\right) \tag{2.26}$$

be the probability that an α-cycle is bad. Given bad C_α, define a r.v. $\xi_\alpha \leq Y_\alpha$ treated as the first occurrence time within α-cycle C_α. If

$$\kappa = \min\left\{n : C_{\alpha_n} \in \mathcal{C}_-^{(\alpha_n)}\right\} \tag{2.27}$$

is the number of the earliest bad cycle, then the first occurrence time is equal to

$$\tau = \sigma_{\kappa-1} + \xi_{\alpha_\kappa}. \tag{2.28}$$

SRP (z, σ) is more convenient for analysis than RP (z, S) just because quantities (2.17), (2.18), and (2.26) can easily be calculated in many practical cases.

Example 2. Let us return to the system considered in Example 1 and define

$$\sigma_0{=}0, \quad \sigma_{n+1}{=}\min\left\{t : \{z(t){=}0, z(t-0){\neq}0\} \cup \{z(t){=}z(t-0)-1\},\, t > \sigma_n\right\}, \quad n \geq 0.$$

Then, evidently, (z, σ) is a SRP, where $E = \{0, 1, \dots\}$. One can take $a = 0$ as a proper atom. Then

$$G_0(x) = 1 - \exp(-\lambda x); \quad G_i(x) = B(x), \quad 1 \leq i \leq N+1;$$

$$K(0,x;j)=\delta_{j,1};\quad K(i,x;j)=\frac{(\lambda x)^{j-i+1}}{(j-i+1)!}\,e^{-\lambda x},\quad i-1\le j\le N,\ 1\le i\le N;$$

$$K(i,x;N+1)=\sum_{j\ge N+1}\frac{(\lambda x)^{j-i+1}}{(j-i+1)!}\,e^{-\lambda x},\quad 1\le i\le N;\quad K(N+1,x;j)=\delta_{j,N};$$

$$q(0)=0;\quad q(i)=\sum_{j\ge N+1}\int_0^\infty\frac{(\lambda x)^{j-i+1}}{(j-i+1)!}e^{-\lambda x}\,dB(x),\quad 1\le i\le N,$$

where $\delta_{i,j}$ is the Kronecker delta. Thus, all "governing parameters" of SRP is written in an explicit form. And the problem is how to express the desired characteristics of RP from Example 1 in terms of the above characteristics of SRP. The paper is devoted exactly to this problem.

3. RP in terms of SRP. In contrast to r.v. ν from (2.11) and (2.12), r.v. κ (see (2.27)) does not have a geometric distribution. Nevertheless, it is still possible to transform the first occurrence time τ from (2.28) to a geometric sum. Such a transformation is useful for mathematical treating the problem. To this end, we can use a representation of (z,σ) as a RP (z,S). Let us say that a generic regeneration cycle is *good* if all semi-regeneration cycles comprising it are good and *bad* otherwise. According to this, we can define quantities of RP (z,S) in terms of characteristics of SRP (z,σ) in the following way.

For each $n\ge 0$, we divide a bad regeneration cycle into $n+1$ semi-regeneration cycles. First n of these cycles are supposed to be good and the $(n+1)th$ bad. With the help of the total probability formula, we arrive at the relation

$$q=\sum_{n=0}^{\infty}q^{(n)}, \tag{3.1}$$

where

$$q^{(0)}=q(a);\quad q^{(n)}=\int\limits_{\alpha\ne a}q(\alpha)\,\overline{K}_n(a;d\alpha),\quad n\ge 1, \tag{3.2}$$

probabilities $q(\alpha)$ are defined in (2.26) and

$$\overline{K}_0(a;A)=\mathbf{1}_A(a),\quad \overline{K}_1(a;A)=\overline{K}(a;A), \tag{3.3}$$

$$\overline{K}_n(a;A)=\int\limits_{\alpha_1\ne a}\cdots\int\limits_{\alpha_{n-1}\ne a}\overline{K}(\alpha_{n-1};A)\prod_{j=0}^{n-2}\overline{K}(\alpha_j;d\alpha_{j+1}),\quad n\ge 1. \tag{3.4}$$

Kernel $\overline{K}(\alpha;A)$ is given by

$$\overline{K}(\alpha;A)=\int_0^\infty K_+(\alpha,u;A)\,dG_\alpha(u), \tag{3.5}$$

where

$$K_+(\alpha, u; A) = \mathbf{P}\left(\alpha_{n+1} \in A,\, C_\alpha = (z_\alpha, u) \in \mathcal{C}_+^{(\alpha)} \mid \alpha_n = \alpha,\, Y_\alpha = u\right) \tag{3.6}$$

can be found from distributions $\mathbf{P}_\alpha$ and $\mathbf{Q}_\alpha$ (see (2.21) and (2.22)). Kernel $\overline{K}(\alpha; A)$ is equal to the probability for the accompanying Markov chain to jump from state $\alpha_n = \alpha$ to $\alpha_{n+1} \in A$ in such a way that the $(n+1)$th semi-regeneration cycle is good. Apparently, $\overline{K}$ can be viewed as a contraction operator and hence, series

$$\overline{K}^\infty(\alpha; A) \equiv \sum_{n=0}^{\infty} \overline{K}_n(\alpha; A)$$

converges for each $\alpha \in E$ and represents a finite measure on $(E, \mathcal{E})$. This can be used either for numerical estimation of q in terms of eigenvalues of operator $\overline{K}$ or for statistical estimates of q by Monte-Carlo methods. Quantity $q^{(n)}$ in (3.2) is equal to the probability that a regeneration cycle is bad and consists of more than n semi-regeneration cycles (first n are good and the $(n+1)$th is bad).

Let (see (2.14))

$$R_+(x) \equiv \mathbf{P}(D \le x,\, C \in \mathcal{C}_+) = \sum_{n=0}^{\infty} R_+^{(n)}(a, x), \tag{3.7}$$

where functions $R_+^{(n)}(\alpha, x)$, $\alpha \in E$, are defined recursively:

$$R_+^{(0)}(\alpha, x) = \int_0^x K_+(\alpha, u; a)\, dG_\alpha(u), \tag{3.8}$$

$$R_+^{(n+1)}(\alpha, x) = \int_0^x \int\limits_{\beta \ne a} R_+^{(n)}(\beta, x-u)\, K_+(\alpha, u; d\beta)\, dG_\alpha(u), \quad n \ge 0. \tag{3.9}$$

Evidently, $R_+^{(n)}(\alpha, x)$ is the joint probability that a generic regeneration cycle consists of $n+1$ good semi-regeneration cycles and the total length of these cycles does not exceed x provided that the accompanying Markov chain starts from state α. Therefore, the d.f. $F(x)$ from (2.14) can be defined as

$$F(x) = \frac{R_+(x)}{1-q}. \tag{3.10}$$

We now introduce a function (cf. (2.13))

$$R_-(x) = \mathbf{P}(\xi(C) \le x,\, C \in \mathcal{C}_-) = \sum_{n=0}^{\infty} R_-^{(n)}(a, x), \tag{3.11}$$

where, for $\alpha \in E$,

$$R_-^{(0)}(\alpha, x) = \mathbf{P}(\xi_\alpha \le x,\, C_\alpha \in \mathcal{C}_-^{(\alpha)}), \tag{3.12}$$

$$R_-^{(n+1)}(\alpha, x) = \int_0^x \int_{\beta \ne a} R_-^{(n)}(\beta, x-u)\, K_+(\alpha, u; d\beta)\, dG_\alpha(u), \quad n \ge 0, \tag{3.13}$$

and kernel K_+ is given in (3.6). Function $R_-^{(n)}(\alpha, x)$ is the joint probability that a generic regeneration cycle consists of more than n semi-regeneration cycles (first n are good and the $(n+1)th$ is bad) and the first occurrence time within regeneration cycle does not exceed x given the accompanying chain starts from state α.

Evidently, d.f. $F_1(x)$ from (2.13) is equal to

$$F_1(x) = \frac{R_-(x)}{q}. \tag{3.14}$$

The relations above can be used in numerical methods for seeking q, $F(x)$, and $F_1(x)$. When seeking these quantities, we assume probabilities $q(\alpha)$ from (2.26), kernel K_+ from (3.6), and joint probability $R_-^{(0)}$ from (3.12) to be known. They all characterize the behavior of (z, σ) within separate semi-regeneration cycles.

4. Recurrence equations. Let us retrace the results of Section 3 to allow for a recursive algorithm enabling us to calculate q, $F(x)$, and $F_1(x)$. There are at least three features of formulae from Section 3 deserving additional attention. First, those relations are not equipped with a stopping rule. Second, for our purposes, it is often sufficient to know the moments of the d.f.'s F and F_1 not the d.f.'s themselves. Therefore, the relations derived can be reduced to a simpler form resulting in the desired information. Third, successive terms $q^{(n)}$, $R_+^{(n)}$, and $R_-^{(n)}$ are not monotone, and this is a drawback from a practical standpoint. In this section, we will deal with the first and third problems, and leave the second one for the following section.

In order to step to monotone characteristics, let us consider partial sums

$$\overline{q}^{(n)} = \sum_{j=0}^{n} q^{(j)}, \quad n \ge 0, \tag{4.1}$$

$$\overline{R}_+^{(n)}(\alpha, x) = \sum_{j=0}^{n} R_+^{(j)}(\alpha, x), \quad n \ge 0, \tag{4.2}$$

$$\overline{R}_-^{(n)}(\alpha, x) = \sum_{j=0}^{n} R_-^{(j)}(\alpha, x), \quad n \ge 0. \tag{4.3}$$

All sequences $\overline{q}^{(n)}$, $\overline{R}_+^{(n)}$, and $\overline{R}_-^{(n)}$, $n \geq 0$, are monotone increasing. Probability $\overline{q}^{(n)}$ can be calculated with the help of formulae (3.2) and (3.4). It follows from definition of $\overline{q}^{(n)}$ that

$$q - \overline{q}^{(n)} = p_{n+1}, \tag{4.4}$$

where p_{n+1} is the probability that a generic regeneration cycle is bad, consists of more than $n+1$ semi-regeneration cycles, and the first bad semi-regeneration cycle has the number $n+2$ or higher. In order to calculate p_{n+1} we have to process an infinite number of iterations because such a calculation requires finding probabilities $q^{(j)}$. But, it is possible to estimate this probability in terms of $n+1$ iterations only. For example, the following inequality holds true

$$p_{n+1} \leq \mathbf{P}(B_{n+1}), \tag{4.5}$$

where B_{n+1} is the event that a generic regeneration cycle consists of more than $n+1$ cycles. In turn, the probability $\mathbf{P}(B_{n+1})$ can be expressed as a taboo probability for the accompanying Markov chain:

$$\mathbf{P}(B_{n+1}) = \mathbf{P}(\alpha_k \neq a,\, 1 \leq k \leq n+1 \mid \alpha_0 = a) = \int\limits_{\alpha_1 \neq a} \cdots \int\limits_{\alpha_{n+1} \neq a} \prod_{j=0}^{n} K(\alpha_j;\, d\alpha_{j+1}). \tag{4.6}$$

Despite the fact that bound (4.5) can be inaccurate, it can be recommended for accuracy control since it is expressed explicitly in terms of the accompanying Markov chain.

Let us sum up equations (3.8) and (3.9) to arrive at the recursive equation

$$\overline{R}_+^{(n+1)}(\alpha, x) = R_+^{(0)}(\alpha, x) + \int_0^x \int\limits_{\beta \neq a} \overline{R}_+^{(n)}(\beta, x-u)\, K_+(\alpha, u;\, d\beta)\, dG_\alpha(u), \quad n \geq 0, \tag{4.7}$$

where

$$\overline{R}_+^{(0)}(\alpha, x) = R_+^{(0)}(\alpha, x) = \int_0^x K_+(\alpha, u; a)\, dG_\alpha(u). \tag{4.8}$$

Quite similarly to (4.5) we have an accuracy estimate

$$R_+(x) - \overline{R}_+^{(n)}(a, x) \leq \mathbf{P}(B_{n+1}). \tag{4.9}$$

In a like manner, from (3.12) and (3.13),

$$\overline{R}_-^{(n+1)}(\alpha, x) = R_-^{(0)}(\alpha, x) + \int_0^x \int\limits_{\beta \neq a} \overline{R}_-^{(n)}(\beta, x-u)\, K_+(\alpha, u;\, d\beta)\, dG_\alpha(u), \ n \geq 0, \tag{4.10}$$

where

$$\overline{R}_-^{(0)}(\alpha, x) = R_-^{(0)}(\alpha, x) = \mathbf{P}\left(\xi_\alpha \leq x,\, C_\alpha \in \mathcal{C}_-^{(\alpha)}\right). \tag{4.11}$$

Because of the definition of $\overline{R}_{-}^{(n)}(\alpha, x)$ it is possible to use the same accuracy estimate

$$R_{-}(x) - \overline{R}_{-}^{(n)}(a, x) \le \mathbf{P}(B_{n+1}). \tag{4.12}$$

Let us emphasize that inequalities (4.5), (4.9), and (4.12) can be refined, but their present form is extremely simple for calculations.

The derived recurrent relations can be solved by various numerical and Monte-Carlo methods. In some cases, these relations can be simplified. For instance, if an accompanying Markov chain is finite (as in Example 2), then integrals over $\beta \neq a$ in equations (4.7) and (4.10) can be replaced by finite sums.

5. Integral Equations. We now focus on deriving integral equations containing unknown quantities q, F, and F_1 and we start with probability q. The following representation follows from (3.1) through (3.2)

$$q = \sum_{n=0}^{\infty} \int\limits_{\alpha \neq a} q(\alpha)\, \overline{K}_n(a;\, d\alpha). \tag{5.1}$$

Let an auxiliary homogeneous Markov chain $(\tilde{\alpha}_n)_{n\ge 0}$ taking its values in E be defined by the following transition probabilities

$$\mathbf{P}(\tilde{\alpha}_{n+1} \in A \mid \tilde{\alpha}_n = \alpha) = \tilde{K}(\alpha; A), \tag{5.2}$$

where

$$\tilde{K}(\alpha; A) = \overline{K}(\alpha; A) + q(\alpha)\, \mathbf{1}_A(a), \quad A \in \mathcal{E},\ \alpha \in E. \tag{5.3}$$

The fact that the right-hand side of (5.3) is a transition probability is quite evident. This Markov chain is associated with $(\alpha_n)_{n\ge 0}$ in the following manner. Let $\alpha_0 = \tilde{\alpha}_0 = a$. One can set $\tilde{\alpha}_n = \alpha_n$ until the instant φ when α_φ initiates a bad cycle of the SRP (the probability of this is equal to $q(\alpha_\varphi)$). In this case, chain $(\tilde{\alpha}_n)_{n\ge 0}$ returns to state a (that is $\tilde{\alpha}_{\varphi+1} = a$) where starts anew. It follows that

$$\tilde{K}_n(a; A) = \overline{K}_n(a; A), \quad a \notin A,\ n \ge 0. \tag{5.4}$$

Hence, we can rewrite (5.1) in the form

$$q = \sum_{n=0}^{\infty} \int\limits_{\alpha \neq a} q(\alpha)\, \tilde{K}_n(a;\, d\alpha). \tag{5.5}$$

But $\tilde{K}_n$ is an n-step taboo probability for Markov chain $(\tilde{\alpha}_n)_{n\ge 0}$. Therefore, equality (5.5) can be rewritten as

$$q = \sum_{n=0}^{\infty} \mathbf{E}\left(q(\tilde{\alpha}_n);\, \tilde{\psi}_\alpha > n\right) = \mathbf{E} \sum_{n < \tilde{\psi}_a} q(\tilde{\alpha}_n), \tag{5.6}$$

where

$$\tilde{\psi}_\alpha = \inf\{n:\ \tilde{\alpha}_n = a,\ n > 0 \mid \tilde{\alpha}_0 = \alpha\}, \quad \alpha \in E. \tag{5.7}$$

Let

$$\tilde{m}_\alpha = \mathbf{E}\tilde{\psi}_\alpha$$

be a mean recurrence time to state a for chain $(\tilde{\alpha}_n)_{n\geq 0}$. By our earlier convention, state a is positive recurrent for the accompanying Markov chain $(\alpha_n)_{n\geq 0}$. Owing to the fact that state a is even "more" recurrent for $(\tilde{\alpha}_n)_{n\geq 0}$ than for $(\alpha_n)_{n\geq 0}$,

$$\tilde{m}_a \leq m_a < \infty. \tag{5.8}$$

Regarding $(\tilde{\alpha}_n)_{n\geq 0}$ as a RP with regeneration times nested at recurrence times to state a, we have, from the Smith ergodic theorem (see Kalashnikov [6, Section 3.4])

$$q = \tilde{m}_a \int_E q(\alpha)\,\tilde{\pi}(d\alpha), \tag{5.9}$$

where $\tilde{\pi}$ is the stationary distribution of chain $(\tilde{\alpha}_n)_{n\geq 0}$ that exists due to (5.8). Distribution $\tilde{\pi}$ is defined uniquely by the following integral equation

$$\tilde{\pi}(A) = \int_E \tilde{K}(\alpha; A)\,\tilde{\pi}(d\alpha), \quad \tilde{\pi}(E) = 1, \tag{5.10}$$

which can be rewritten in terms of kernel $\overline{K}$ as

$$\tilde{\pi}(A) = \int_E \overline{K}(\alpha; A)\,\tilde{\pi}(d\alpha), \quad a \notin A; \quad \tilde{\pi}(E) = 1. \tag{5.11}$$

In turn, expectations $\tilde{m}_\alpha$ can be found from equations

$$\tilde{m}_\alpha = 1 + \int_{\beta\neq a} \tilde{m}_\beta\,\tilde{K}(\alpha;\, d\beta) = 1 + \int_{\beta\neq a} \tilde{m}_\beta\,\overline{K}(\alpha;\, d\beta), \quad \alpha \in E. \tag{5.12}$$

Equations (5.9) through (5.12) define probability q in terms of $q(\alpha)$, $\alpha \in E$.

We now consider d.f.'s F and F_1. These functions are defined by equalities (3.10) and (3.14) correspondingly. Therefore, to determine them, it is sufficient to find functions $R_+(x) \equiv R_+(a, x)$ and $R_-(x) \equiv R_+(a, x)$ satisfying

$$R_+(\alpha, x) = R_+^{(0)}(\alpha, x) + \int_0^x \int_{\beta\neq a} R_+(\beta, x-u)\,K_+(\alpha, u;\, d\beta)\,dG_\alpha(u), \quad \alpha \in E, \tag{5.13}$$

and

$$R_-(\alpha,x)=R_-^{(0)}(\alpha,x)+\int_0^x\int\limits_{\beta\neq a}R_-(\beta,x-u)\,K_+(\alpha,u;\,d\beta)\,dG_\alpha(u),\quad \alpha\in E, \tag{5.14}$$

which are immediate consequences of (4.7) and (4.10) respectively. It can be easily proved by mathematical induction that the desired functions R_+ and R_- are minimal nonnegative solutions of (5.13) and (5.14) correspondingly. Those linear integral equations contain convolution terms. Equations of this type are widely used. Usually, they can be solved in terms of the Laplace-Stieltjes transform. We do not discuss such a possibility but limit ourselves to deriving equations for moments of F and F_1 (considering only first moments, for brevity). Denote

$$r_+(\alpha)=\int_0^\infty x\,d_xR_+(\alpha,x),\quad r_+^{(0)}(\alpha)=\int_0^\infty x\,d_xR_+^{(0)}(\alpha,x), \tag{5.15}$$

$$R_+(\alpha)=\lim_{x\to\infty}R_+(\alpha,x),\quad R_+^{(0)}(\alpha)=\lim_{x\to\infty}R_+^{(0)}(\alpha,x), \tag{5.16}$$

$$k_+(\alpha;A)=\int_0^\infty x\,K_+(\alpha,x;A)\,dG_\alpha(x), \tag{5.17}$$

$$r_-(\alpha)=\int_0^\infty x\,d_xR_-(\alpha,x),\quad r_-^{(0)}(\alpha)=\int_0^\infty x\,d_xR_-^{(0)}(\alpha,x), \tag{5.18}$$

$$R_-(\alpha)=\lim_{x\to\infty}R_-(\alpha,x),\quad R_-^{(0)}(\alpha)=\lim_{x\to\infty}R_-^{(0)}(\alpha,x) \tag{5.19}$$

and recall that

$$\int_0^\infty K_+(\alpha,x;A)\,dG_\alpha(x)=\overline{K}(\alpha;A). \tag{5.20}$$

Comparing (3.8) with (5.16), (5.17), and (5.20), we see that

$$R_+^{(0)}(\alpha)=\overline{K}(\alpha;a) \tag{5.21}$$

and

$$r_+^{(0)}(\alpha)=k_+(\alpha;a). \tag{5.22}$$

In these terms,

$$f_1\equiv\int_0^\infty x\,dF(x)=\frac{r_+(a)}{1-q} \tag{5.23}$$

and

$$f_{\zeta 1}\equiv\int_0^\infty x\,dF_1(x)=\frac{r_-(a)}{q}. \tag{5.24}$$

Therefore, for determining f_1 and $f_{\zeta 1}$, it is sufficient to find $r_+(a)$ and $r_-(a)$. In order to do this, we note that quantities $r_+^{(0)}(\alpha)$, $R_+^{(0)}(\alpha)$, $k_+(\alpha;A)$, $\overline{K}_+(\alpha;A)$, $r_-^{(0)}(\alpha)$, $R_-^{(0)}(\alpha)$

can be calculated from initial data and must be regarded as known. Quantities $r_+(\alpha)$ and $r_-(\alpha)$ satisfy the following equations which are direct consequences of (5.13), (5.14), (5.21), and (5.22):

$$r_+(\alpha) = k_+(\alpha; a) + \int\limits_{\beta\neq a} r_+(\beta)\,\overline{K}(\alpha;\, d\beta) + \int\limits_{\beta\neq a} R_+(\beta)\,k_+(\alpha;\, d\beta), \quad \alpha \in E, \tag{5.25}$$

$$r_-(\alpha) = r_-^{(0)}(\alpha) + \int\limits_{\beta\neq a} r_-(\beta)\,\overline{K}(\alpha;\, d\beta) + \int\limits_{\beta\neq a} R_-(\beta)\,k_+(\alpha;\, d\beta), \quad \alpha \in E. \tag{5.26}$$

Similarly, it is possible to obtain equations for moments of higher orders but we stop here.

In order to solve equations (5.25) and (5.26), it is necessary and sufficient to know $R_+(\beta)$ and $R_-(\beta)$, $\beta \neq a$. Equations for these quantities are consequences of (5.13) and (5.14):

$$R_+(\alpha) = \overline{K}(\alpha; a) + \int\limits_{\beta\neq a} R_+(\beta)\,\overline{K}(\alpha;\, d\beta), \quad \alpha \in E, \tag{5.27}$$

$$R_-(\alpha) = R_-^{(0)}(\alpha) + \int\limits_{\beta\neq a} R_-(\beta)\,\overline{K}(\alpha;\, d\beta), \quad \alpha \in E. \tag{5.28}$$

All linear integral equations (5.12) and (5.25) through (5.28) are of the Volterra type (of the second kind) with the same kernel $\overline{K}$. So, they can be written in the following general form

$$X(\alpha) = Y(\alpha) + \int\limits_{\beta\neq a} X(\beta)\,\overline{K}(\alpha;\, d\beta), \quad \alpha \in E,$$

where $Y(\alpha)$ is the only term reflecting the specific of the equation. Fitting $Y(\alpha)$, we arrive at any of the listed equations. Because of this, it is possible to determine a general solution of the equation above in the form

$$X = (I - \overline{K})^{-1} Y,$$

where inverse operator $(I - \overline{K})^{-1}$ has a standard meaning and the crucial point in the solution is the determination of this operator. A variety of numerical methods for solving such equations are covered in Baker [2] and Kantorovich and Krylov [8, Chapter 2].

Let us note, in conclusion, that one must solve the equations stated above very carefully because some terms of their solution can take extremely small values which are beyond the least significant computer digit. Because of this, it is desirable to employ numerical methods accounting for the specific features of these equations.

Example 3. For system indicated in Examples 1 and 2, the following asymptotic approximation of the probability q is often used in reliability (see Soloviev [11])

$$q \sim q_0 \equiv \frac{\lambda^{N-1} b_{N-1}}{(N-1)!}, \quad \lambda b_1 \to 0,$$

where $b_k = \int x^k \, dB(x)$.

If we use the relations presented in this section and in Example 2, then it is easy to calculate q explicitly. The following table contains values of the relative error $|q - q_0|/q$ for $N = 2$ and three forms of d.f. $B(x)$:

$$\begin{aligned} B_1(x) &= 1 - \exp\left(-\frac{x}{b_1}\right), \\ B_2(x) &= 1 - (1 + 2x/b_1)\exp\left(-\frac{2x}{b_1}\right), \\ B_3(x) &= \mathbf{1}(x - b_1). \end{aligned}$$

λb_1	$B_1(x)$	$B_2(x)$	$B_3(x)$
0.1	$1.0 \cdot 10^{-1}$	$8.0 \cdot 10^{-2}$	$5.1 \cdot 10^{-2}$
0.05	$5.0 \cdot 10^{-2}$	$3.8 \cdot 10^{-2}$	$2.5 \cdot 10^{-2}$
0.01	$1.0 \cdot 10^{-2}$	$7.5 \cdot 10^{-3}$	$5.0 \cdot 10^{-3}$
0.001	$1.0 \cdot 10^{-3}$	$7.5 \cdot 10^{-4}$	$5.0 \cdot 10^{-4}$

It is seen that the relative error is noticeable (up to 10 %) for $\lambda b_1 = 0.1$ and even for $\lambda b_1 = 0.05$. This shows that asymptotic approximations must be used very carefully in calculations.

6. Monte-Carlo Algorithms. Simulation is a common instrument often used for evaluation of basic characteristics of models. Our idea is to use simulation for finding q, F, and F_1. However, it is almost impossible to do so without additional "tricks" since these values are associated with rare events which cannot be detected directly, by the inspection of paths of the model. In order for simulation to work in such situations, so-called *importance samples* are often used (see Asmussen and Rubinstein [1]).

In our case, the importance sampling means that we ought to generate paths of RP belonging to bad cycles. There is no formal criteria to compare the efficiencies of different importance samplings. Actually, the most serious problem is the choice of the probability measure in accordance with which the sampling is generated. But the greater the probability of belonging to a desired set (for example, to a bad cycle), the better the sampling is.

We propose algorithms that allow us to estimate the aforementioned characteristics and use the semi-regenerative structure of underlying processes. To start with, let probabilities $q(\alpha)$, $\alpha \in E$, and kernel $\overline{K}(\alpha; A)$, $\alpha \in E$, $A \in \mathcal{E}$, (see (3.5)) be given, kernel $\tilde{K}(\alpha; A)$ be defined by formula (5.3) and Markov chain $(\tilde{\alpha}_n)$ by formula (5.2). We assume that there exists a computer program generating paths of chain $(\tilde{\alpha}_n)_{n\geq 0}$, $\tilde{\alpha}_0 = a$. Define successive recurrence times

$$T_0 = 0, \quad T_{k+1} = \min\{n : n > T_k, \tilde{\alpha}_n = a\}, \quad k \geq 0$$

and denote

$$\hat{q}(k) = \sum_{T_k \leq n < T_{k+1}} q(\tilde{\alpha}_n), \quad k \geq 0. \tag{6.1}$$

Evidently, r.v.'s $\hat{q}(k)$ are i.i.d.,

$$\hat{q}(k) \stackrel{d}{=} \sum_{n < \tilde{\psi}_a} q(\tilde{\alpha}_n),$$

and, therefore, by (5.6),

$$\mathbf{E}\hat{q}(k) = q. \tag{6.2}$$

From the i.i.d. property of r.v.'s $\hat{q}(k)$ and equality (6.2) it follows that

$$\hat{q}_N = \frac{1}{N} \sum_{k=0}^{N-1} \hat{q}(k), \quad N \geq 1, \tag{6.3}$$

is a *strong consistent unbiased point estimate* of probability q that is $\hat{q}_N \to q$ with probability one as $N \to \infty$ and $\mathbf{E}\hat{q}_N = q$. The accuracy of this estimate can be estimated with the help of the central limit theorem. Let Φ is a standard normal d.f. and $\sigma^2(q)$ is a common variance of r.v.'s $\hat{q}(k)$. Then $100(1-2\gamma)\%$ confidence interval for q has the form

$$\hat{I}(q, N) = [\hat{q}_N - \Delta_N(q, \gamma),\ \hat{q}_N + \Delta_N(q, \gamma)], \tag{6.4}$$

where

$$\Delta_N(q, \gamma) = \frac{\sigma(q)}{\sqrt{N}} \Phi^{-1}(1-\gamma), \quad 0 < \gamma < \frac{1}{2}. \tag{6.5}$$

The relations above yield the following Monte-Carlo algorithm that builds the point estimate $\hat{q}_N$ and corresponding $100(1-2\gamma)\%$ confidence interval for given γ and N.

Algorithm 1.

(i) Set $\tilde{\alpha}_0 = a$ and $k = 0$;

(ii) simulate $\tilde{\alpha}_n$, $T_k \leq n \leq T_{k+1}$;

(iii) calculate $\hat{q}(k)$ by formula (6.1);

(iv) if $k < N-1$ then $k := k+1$ and goto (ii);
otherwise goto (v);
(v) form the point estimate $\hat{q}_N$ by formula (6.3);
form the interval $\hat{I}(q,N)$ by formulae (6.4) and (6.5);
end.

Algorithm 1 requires generating an auxiliary Markov chain $(\tilde{\alpha}_n)$ and deals with no rare event. It works if $\sigma(q)$ is known. In practice, it is possible to replace $\sigma(q)$ by a sample variance.

As we have mentioned, it is desirable to use simulation programs written for generating paths of process $z(t)$ for estimation of q, F, and F_1. Basing on Algorithm 1, let us show how we can do this with respect to q. Let $z(t)$ be a SRP with successive semi-regeneration times σ_j, $j \geq 0$ and regeneration times S_k, $k \geq 0$. Assume that probabilities $q(\alpha)$, $\alpha \in E$, are known and that there exists a simulation algorithm producing paths of $z(t)$ provided $t = 0$ is a regeneration epoch. Let us fix successive instants σ_j, $j \geq 0$, $(\sigma_0 = 0)$ as well as successive states α_j of the accompanying Markov chain in the course of the simulation. The following algorithm is designed to form both point estimate $\hat{q}_N$ by formula (6.3) and confidence interval by formulae (6.4) and (6.5). It does not require knowledge of an explicit form of $\overline{K}(\alpha; A)$. In Algorithm 2, integer k counts a number of simulated regeneration cycles while j counts a number of semi-regeneration cycles simulated within the current regeneration cycle. Constants N and γ have the same meaning as in Algorithm 1.

Denote the jth semi-regeneration cycle by $C^{(j)}$ and call it bad if $C^{(j)} \in \mathcal{C}_{-}^{(\alpha_j)}$.

Algorithm 2.
(i) Set $k = 0$;
(ii) if $k \leq N-1$ then $\hat{q}(k) := 0$; $\alpha_0 = a$ and $j = 0$;
otherwise goto (vi);
(iii) simulate $z(t)$ for $\sigma_j \leq t \leq \sigma_{j+1}$;
(iv) if $C^{(j)}$ is good then $\hat{q}(k) := \hat{q}(k) + q(\alpha_j)$;
form α_{j+1};
if $\alpha_{j+1} = a$ then set $k := k+1$ and goto (ii);
otherwise set $j := j+1$ and goto (iii);
(v) if $C^{(j)}$ is bad then set $k := k+1$ and goto (ii);
(vi) form the point estimate $\hat{q}_N$ by formula (6.3);
form the interval $\hat{I}(q,N)$ by formulae (6.4) and (6.5);
end.

We now estimate of d.f.'s F and F_1. In fact, we limit ourselves to estimation of mean values f_1 and $f_{\zeta 1}$ as defined by (5.23) and (5.24). Let μ be the mean inter-regeneration time

$$\mu = \mathbf{E}S_1 < \infty.$$

Evidently,

$$\mu = (1-q)f_1 + qf_{\zeta 1}. \tag{6.6}$$

Since probability q is small, the estimate

$$f_1 \le \frac{\mu}{1-q}$$

is accurate enough for implementation in practical calculations where μ can be estimated in the course of regenerative simulation (see Shedler [9]). The sample mean

$$\hat{\mu}_N = \frac{S_N}{N}$$

is a strong consistent unbiased estimate of μ. Its $100(1-2\gamma)\%$ confidence interval is defined by the equalities

$$\hat{I}(\mu, N) = [\hat{\mu}_N - \Delta_N(\mu,\gamma), \hat{\mu}_N + \Delta_N(\mu,\gamma)], \tag{6.7}$$

$$\Delta_N(\mu,\gamma) = \frac{\sigma(\mu)}{\sqrt{N}}\Phi^{-1}(1-\gamma), \quad 0 < \gamma < \frac{1}{2}, \tag{6.8}$$

and $\sigma^2(\mu)$ is the variance of S_1.

Such a straightforward approach fails for estimation of $f_{\zeta 1}$ because the inequality $f_{\zeta 1} \le \mu/q$ is too crude. In order to determine $f_{\zeta 1}$ let us use (3.13) and the probability interpretation of the summands $R_-^{(n)}(\alpha, x)$ from there. Assume we have a Monte-Carlo algorithm that generates a sequence of random triples

$$(\alpha_n, Y_{\alpha_n}, z_{\alpha_n}(u)), \quad 0 \le u < Y_{\alpha_n},$$

and suppose that we can analytically obtain the following conditional distributions

$$G_+(\alpha, x) = \mathbf{P}(Y_\alpha \le x \mid C \in \mathcal{C}_+^{(\alpha)}) = \frac{R_+^{(0)}(\alpha, x)}{1-q(\alpha)};$$

$$G_-(\alpha, x) = \mathbf{P}(\xi_\alpha \le x \mid C \in \mathcal{C}_-^{(\alpha)}) = \frac{R_-^{(0)}(\alpha, x)}{q(\alpha)};$$

$$K_+(A \mid \alpha, x) = \mathbf{P}(\alpha_{n+1} \in A \mid \alpha_n{=}\alpha, Y_\alpha{=}x, (z_\alpha, x) \in \mathcal{C}_+^{(\alpha)}) \equiv \frac{K_+(\alpha, x; A)}{1-q(\alpha, x)},\ A \in \mathcal{E};$$

$${}_aK_+(A \mid \alpha, x) = \frac{K_+(A \mid \alpha, x)}{K_+(E \backslash a \mid \alpha, x)}, a \notin A;$$

$$q(\alpha, x) = \mathbf{P}(((z_\alpha, x) \in \mathcal{C}_+^{(\alpha)}) \equiv K_+(\alpha, x; E),$$

where $\alpha \in E$. We also assume that, for each $\alpha \in E$, we have generators producing the following random variables ($\overset{d}{\sim}$ stands for "distributed as"):

$$Y_+(\alpha) \overset{d}{\sim} G_+(\alpha, x), \quad x \geq 0, \tag{6.9}$$

$$\xi_-(\alpha) \overset{d}{\sim} G_-(\alpha, x), \quad x \geq 0, \tag{6.10}$$

$$\phi_+(\alpha) \overset{d}{\sim} {}_aK_+(A \mid \alpha, x), \quad A \in \mathcal{E}. \tag{6.11}$$

We propose a Monte-Carlo algorithm to estimate the quantity

$$r_-^{(n)}(\alpha) = \int_0^\infty x \, dR_-^{(n)}(\alpha, x).$$

By definition of $R_-^{(n)}(\alpha, x)$, $r_-^{(n)}(\alpha)$ is equal to a partial expectation of the sum of lengths of n successive semi-regeneration cycles and the first occurrence time within the $(n+1)$th semi-regeneration cycle. This partial expectation is taken over the event that the accompanying Markov chain starting from α does not hit state a during the defined period and that the first n cycles are good while the $(n+1)$th cycle is bad. The algorithm is based on this interpretation of $r_-^{(n)}(\alpha)$ and results in a sample value of a r.v. denoted by $\Psi^{(n)} = \Psi^{(n)}(\alpha)$ and satisfying the equality

$$\mathbf{E}\Psi^{(n)} = r_-^{(n)}(\alpha).$$

R.v. $\Psi^{(n)}$ has the form

$$\Psi^{(n)} = \Pi^{(n)} L^{(n)},$$

where $L^{(n)}$ is a sample length of the aforementioned sum of $n+1$ r.v.'s and $\Pi^{(n)}$ is a "sample probability" of the event which the partial expectation is taken over.

In the algorithm, integer $n \geq 0$ is fixed and relates to an upper index in $r_-^{(n)}(\alpha)$ and integer $k \geq 0$ counts the current number of good semi-regeneration cycles with understanding that r.v.'s from relations (6.9) through (6.11) are independent when they are generated at different steps of the algorithm. The algorithm generates lengths of n successive good semi-regeneration cycles and the first occurrence time over the $(n+1)th$ bad cycle with the help of corresponding transition probabilities. In order to obtain the estimate $\hat{r}_-^{(n)}(\alpha)$, it is necessary to get N sample values of $\Psi^{(n)}$ with the help of Algorithm 3 and take their average.

Using Algorithm 3 for different $n \geq 0$ and summing up respective results, we can estimate $f_{\zeta 1}$. Of course, it is impossible to calculate an infinite sum of $r_-^{(n)}(\alpha)$ over n. Therefore, there arises the problem of estimation of the remainder of this sum. This can be done either heuristically (stopping at such n where $\hat{r}_-^{(n)}(\alpha)$ is relative "stable" with respect to n; such an approach is widely used in simulation) or with the help of bounds such as (4.5), estimating probabilities $\mathbf{P}(B_{n+1})$ by simulation.

Algorithm 3.
(i) Set $k = 0$; $\alpha_k := a$; $L^{(n)} := 0$; $\Pi^{(n)} := 1$;
(ii) generate $Y_+(\alpha_k)$; set $L^{(n)} := L^{(n)} + Y_+(\alpha_k)$;
(iii) if $k \le n-1$ then set $\Pi^{(n)} := \Pi^{(n)}(1 - q(\alpha_k))(1 - K_+(a \mid \alpha_k, Y_+(\alpha_k)))$;
generate $\phi_+ := \phi_+(\alpha_k, Y_+(\alpha_k))$;
$k := k + 1$; $\alpha_k := \phi_+$;
goto (ii);
(iv) if $k = n$ then generate $\xi_-(\alpha_k)$;
set $L^{(n)} := L^{(n)} + \xi_-(\alpha_k)$;
set $\Pi^{(n)} := \Pi^{(n)} q(\alpha_k)$; $\Psi^{(n)} := \Pi^{(n)} L^{(n)}$;
end.

Algorithms 1 through 3 were programmed in Turbo Pascal 5.5 to be used for investigation of various redundant systems (with several repairing units, arbitrary distributions of lifetimes, etc.). Typical processing time for each algorithm was approximately 1 sec per 1000 samples (for IBM-AT/286). This time seems quite acceptable.

We regard these algorithms only as examples illuminating virtual possibilities of Monte-Carlo methods for calculation of characteristics associated with rare events. Such algorithms can be matched with the needs of specific problems in accordance with the input data available. Let us emphasize the following feature of the above algorithms: their running time does not depend on the values of the "small probabilities". So, it is possible to use them in models with as small probabilities $q(\alpha)$ as necessary.

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Institute of systems analysis
Prospect 60 let Oktyabrya, 9
117312 Moscow
Russia
e-mail: vkalash@glas.apc.org

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