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NONTRIVIAL SOLUTIONS OF QUASILINEAR EQUATIONS IN *BV*

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Abstract. The existence of a nontrivial critical point is proved for a functional containing an area-type term. Techniques of nonsmooth critical point theory are applied.

1. Introduction. Let Ω be a bounded open subset of $\mathbb{R}^n$ $(n \geq 3)$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ a Carathéodory function with $g(x, 0) = 0$. A classical result of Ambrosetti and Rabinowitz [1, 12, 13] says that the semilinear problem

$$\begin{cases} -\Delta u = g(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

admits a nontrivial solution u, provided that the following conditions are satisfied:

(C1) there exist $a \in L^{\frac{2n}{n+2}}(\Omega)$, $b \in \mathbb{R}$ and $p \in \left]2, \frac{2n}{n-2}\right[$ such that

$$|g(x, s)| \leq a(x) + b\,|s|^{p-1}\,;$$

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(C2) there exist $q > 2$ and $R > 0$ such that

$$|s| \geq R \Longrightarrow 0 < qG(x,s) \leq sg(x,s),$$

where $G(x,s) = \int_0^s g(x,t)\,dt$;

(C3) it is

$$\lim_{s\to 0} \frac{g(x,s)}{s} = 0$$

uniformly with respect to x.

Such a nontrivial solution u is found as a mountain pass point of the functional $f : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$f(u) = \frac{1}{2}\int_\Omega |Du|^2\,dx - \int_\Omega G(x,u)\,dx.$$

Our aim is to get a similar result for a class of functionals which contains, as a model example, the functional

$$f(u) = \int_\Omega |Du|\,dx - \int_\Omega G(x,u)\,dx.$$

The correct expression of f, which requires a relaxation procedure, will be given in section 4. Here we want to observe that the natural adaptation of condition (C1) would be

$$|g(x,s)| \leq a(x) + b\,|s|^{p-1}$$

with $a \in L^n(\Omega)$ and $p \in \left]1, \frac{n}{n-1}\right[$. On the other hand, the natural domain of f is now the space $BV(\Omega)$. In such a space also nonsmooth versions of critical point theory cannot be directly applied, as the Palais-Smale condition fails (see [11]). To overcome this difficulty, it is possible to consider the functional f on $L^p(\Omega)$ (with value $+\infty$ outside its natural domain). If we add the stronger condition that $a \in L^{p'}(\Omega)$, then f is the sum of a convex term and a functional of class C^1, and the expected result can be obtained. Such a strategy has been applied in [11], to treat the case where f is even. However, this further condition on a seems to be merely technical. Our aim is to show that the assumption $a \in L^n(\Omega)$ is in fact sufficient. As in [11], we apply the nonsmooth critical point theory developed in [4, 6], which provides general results for continuous functionals defined on metric spaces. Among lower semicontinuous functionals (as f on

$L^p(\Omega)$), some particular classes can be treated. The main part of this paper, namely section 3, is devoted to the study of a class of lower semicontinuous functionals, which contains f and for which the theory of [4, 6] can be applied. Then, in the last section, we prove the existence of a mountain pass point for f.

2. Some notions of nonsmooth critical point theory. Let us recall some notions of nonsmooth critical point theory from [4, 6]. A similar approach to nonregular functionals can be found also in [10, 9]. In the following of this section, X will denote a metric space endowed with the metric d.

Definition 2.1. *Let $f : X \to \mathbb{R}$ be a continuous function and let $u \in X$. We denote by $|df|(u)$ the supremum of the σ's in $[0, +\infty[$ such that there exist $\delta > 0$ and a continuous map $\mathcal{H} : \mathrm{B}(u, \delta) \times [0, \delta] \to X$ such that*

$$\forall v \in \mathrm{B}(u,\delta), \forall t \in [0,\delta] : \qquad d(\mathcal{H}(v,t), v) \le t,$$
$$\forall v \in \mathrm{B}(u,\delta), \forall t \in [0,\delta] : \quad f(\mathcal{H}(v,t)) \le f(v) - \sigma t.$$

The extended real number $|df|(u)$ is called the weak slope *of f at u.*

The above notion can be extended also to lower semicontinuous functions, by means of a tool introduced for the first time in [5].

Definition 2.2. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and $b \in \mathbb{R}$. We set*

$$\mathcal{D}(f) = \{u \in X : f(u) < +\infty\},$$
$$f^b = \{u \in X : f(u) \le b\},$$
$$\mathrm{epi}(f) = \{(u,\xi) \in X \times \mathbb{R} : f(u) \le \xi\}.$$

We define the function $\mathcal{G}_f : \mathrm{epi}(f) \to \mathbb{R}$ putting $\mathcal{G}_f(u,\xi) = \xi$.

In the following $\mathrm{epi}(f)$ will be endowed with the metric

$$d((u,\xi),(v,\mu)) = (d(u,v)^2 + (\xi - \mu)^2)^{\frac{1}{2}},$$

so that $\mathcal{G}_f$ is Lipschitz continuous of constant 1. Therefore Definition 2.1 can be applied to $\mathcal{G}_f$ and $|d\mathcal{G}_f|(u,\xi) \le 1$ for every $(u,\xi) \in \mathrm{epi}(f)$.

Definition 2.3. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and let $u \in \mathcal{D}(f)$. We set*

$$|df|(u) = \begin{cases} \dfrac{|d\mathcal{G}_f|(u, f(u))}{\sqrt{1 - (|d\mathcal{G}_f|(u, f(u)))^2}} & \text{if } |d\mathcal{G}_f|(u, f(u)) < 1, \\ +\infty & \text{if } |d\mathcal{G}_f|(u, f(u)) = 1. \end{cases}$$

It is shown in [6, Proposition 2.3] that the above definition is consistent with Definition 2.1.

Definition 2.4. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. We say that $u \in X$ is a* (lower) critical point *for f, if $|df|(u) = 0$. A real number c is said to be a* (lower) critical value, *if there exists $u \in \mathcal{D}(f)$ such that $|df|(u) = 0$ and $f(u) = c$.*

Definition 2.5. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and $c \in \mathbb{R}$. We say that f satisfies the* Palais-Smale condition *at level c ($(PS)_c$ for short), if from every sequence (u_h) in $\mathcal{D}(f)$ with $|df|(u_h) \to 0$ and $f(u_h) \to c$ it is possible to extract a subsequence (u_{h_k}) converging in X.*

3. Some abstract results. As pointed out in [6], the essential difficulty when dealing with lower semicontinuous functions is that we do not know in general the behaviour of $|d\mathcal{G}_f|(u, \xi)$ at the points with $\xi > f(u)$.

Therefore, the main result of this section is a theorem in the spirit of [6, Theorem 3.13] and [4, Theorem 4.4].

Theorem 3.1. *Let X be a linear space, $\|\cdot\|$, $\|\cdot\|_0$ two norms on X and $c > 0$ such that $\|\cdot\|_0 \leq c\|\cdot\|$. Let X_0 (resp. X_1) be the space X endowed with the norm $\|\cdot\|_0$ (resp. $\|\cdot\|$).*

Let $f : X \to \mathbb{R} \cup \{+\infty\}$, $f = f_0 + f_1$, such that:

(*a*) *$f_0 : X_0 \to \mathbb{R} \cup \{+\infty\}$ is convex, lower semicontinuous and for every $u_0 \in X$ there exists $r > 0$ such that*

$$\lim_{\substack{\|u\| \to \infty \\ \|u - u_0\|_0 \leq r}} f_0(u) = +\infty;$$

(*b*) *$f_1 : X_1 \to \mathbb{R}$ is of class C^1;*

(*c*) *for every* $\varepsilon > 0$ *there exist* $\varphi : X_1 \to \mathbb{R}$ *Lipschitz of constant* ε *and* $\psi : X_0 \to \mathbb{R}$ *of class* C^1 *such that* $f_1 = \varphi + \psi$.

Then the following facts hold:

(*i*) $f : X_0 \to \mathbb{R} \cup \{+\infty\}$ *is lower semicontinuous and for every* $u_0 \in X$ *there exists* $r > 0$ *such that*

$$\liminf_{\substack{\|u\| \to \infty \\ \|u-u_0\|_0 \le r}} \frac{f(u)}{\|u\|} > 0; \tag{3.1}$$

(*ii*) $f_1 : X_0 \to \mathbb{R}$ *is continuous on* X_1*-bounded subsets;*

(*iii*) $\xi > f(u) \Longrightarrow |d_0 \mathcal{G}_f|(u, \xi) = 1$;

(*iv*) *if* $u \in \mathcal{D}(f)$, *then* $|df|(u) \le c\,|d_0 f|(u)$, *where* $|d_0 f|$ *(resp.* $|df|$*) denotes the weak slope of* f *in* X_0 *(resp. in* X_1*).*

P r o o f. (i) Let $u_0 \in X$, $r > 0$ according to (a). Without loss of generality, we can suppose there exists $v_0 \in \mathcal{D}(f_0)$ such that $\|v_0 - u_0\|_0 \le r$. Let $r_1 > 0$ be such that

$$\forall u \in X : \|u - u_0\|_0 \le r, \|u - v_0\| \ge r_1 \Longrightarrow f_0(u) \ge f_0(v_0) + 1.$$

If $\|u - u_0\|_0 \le r$ and $\|u - v_0\| \ge r_1$, taking into account the convexity of f_0, we deduce that

$$f_0(v_0) + 1 \le f_0\left(v_0 + \frac{r_1}{\|u - v_0\|}(u - v_0)\right) \le f_0(v_0) + \frac{r_1}{\|u - v_0\|}(f_0(u) - f_0(v_0)),$$

hence

$$f_0(u) \ge f_0(u_0) + \frac{1}{r_1}\|u - v_0\|.$$

Thus, we have shown that, for every $u_0 \in X$ with corresponding r according to (a), it is

$$m_{u_0,r} := \liminf_{\substack{\|u\| \to \infty \\ \|u-u_0\|_0 \le r}} \frac{f_0(u)}{\|u\|} > 0. \tag{3.2}$$

Let $u_0 \in X$ and $r > 0$ according to (a). Let $\varepsilon \in]0, m_{u_0,r}[$ and $f_1 = \varphi + \psi$ according to hypothesis (c). Then, for every $u \in X$ it is

$$f(u) = f_0(u) + \varphi(u) + \psi(u) \ge f_0(u) + \psi(u) + \varphi(0) - \varepsilon\|u\|.$$

Unless reducing r, we can suppose that ψ is bounded on $\mathrm{B}_0(u_0, r)$. Therefore, from (3.2) it follows that for every $u_0 \in X$ and $r > 0$ according to (a) it is

$$\liminf_{\substack{\|u\| \to \infty \\ \|u-u_0\|_0 \le r}} \frac{f(u)}{\|u\|} \ge m_{u_0,r} - \varepsilon > 0.$$

Now, if (u_h) is a sequence convergent to u in X_0 with $f(u_h) \le c$, it follows that (u_h) is bounded also in X_1. From assumptions (a) and (b) we deduce that

$$f(u) \le \liminf_h f(u_h),$$

namely that $f : X_0 \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous.

(ii) Let $\varepsilon = 1/h$ and let φ_h, ψ_h as in (c). Then we have

$$\|u\| \le K \Longrightarrow |\varphi_h(u) - \varphi_h(0)| \le \frac{1}{h} \|u\| \le \frac{1}{h} K.$$

It follows that $(\psi_h(\cdot) + \varphi_h(0))$ is uniformly convergent to f_1 on X_1-bounded subsets, whence the assertion.

(iii) Let $(u, \xi) \in \mathrm{epi}(f)$ with $\xi > f(u)$. Without loss of generality, we can assume that $f_1(u) = 0$. By (i), there exists $r \in]0, 1[$ and $K > 0$ such that

$$\forall v \in X : \begin{cases} f(v) \le \xi + 1 \\ \|v - u\|_0 \le r \end{cases} \Longrightarrow \|v - u\| \le K. \tag{3.3}$$

Let $\delta \in \left]0, \min\{\frac{\xi - f_0(u)}{2}, r\}\right[$ and $\varepsilon > 0$ such that $\varepsilon K < \delta/4$. Choose φ and ψ according to (c) with $\varphi(u) = \psi(u) = 0$ and set, for every $v \in X$,

$$\begin{aligned} \tilde{f}_0(v) &= f_0(v) + \langle d\psi(u), v - u \rangle_0, \\ \tilde{\psi}(v) &= \psi(v) - \langle d\psi(u), v - u \rangle_0, \end{aligned}$$

so that $f = \tilde{f}_0 + \varphi + \tilde{\psi}$ and $d\tilde{\psi}(u) = 0$ in X_0'.

Let $\mathcal{H} : (\mathrm{B}((u, \xi), \delta) \cap \mathrm{epi}(\tilde{f}_0)) \times [0, \delta] \to \mathrm{epi}(\tilde{f}_0)$ be defined by

$$\mathcal{H}((v, \mu), t) = \left(v + \frac{t(u - v)}{\sqrt{\|v - u\|_0^2 + \left|\mu - \tilde{f}_0(u)\right|^2}}, \mu - (\mu - \tilde{f}_0(u)) \frac{t}{\sqrt{\|v - u\|_0^2 + \left|\mu - \tilde{f}_0(u)\right|^2}} \right).$$

As in the proof of [6, Theorem 3.13], it follows that for every $(v,\mu)\in \mathrm{B}_0\left((u,\xi),\delta\right)\cap \mathrm{epi}(\tilde f_0)$ and every $t\in[0,\delta]$, it is

$$d\left(\mathcal{H}\left((v,\mu),t\right),(v,\mu)\right)\le t,$$

$$\mathcal{G}_{\tilde f_0}\left(\mathcal{H}\left((v,\mu),t\right)\right)\le \mathcal{G}_{\tilde f_0}(v,\mu)-\frac{\xi-\delta-\tilde f_0(u)}{\sqrt{\delta^2+(\xi+\delta-\tilde f_0(u))^2}}t.$$

Let $\delta'\in]0,\delta/2[$ such that $\left|\tilde\psi(v)\right|<\delta/2$ if $v\in \mathrm{B}_0\left(u,\delta'\right)$. Then, if $(v,\mu)\in \mathrm{B}_0\left((u,\xi),\delta'\right)\cap \mathrm{epi}(\tilde f_0+\varphi)$ it is, taking into account (3.3),

$$|\mu-\varphi(v)-\xi|\le|\mu-\xi|+|\varphi(v)|\le\frac{\delta}{2}+\varepsilon\|v-u\|\le\frac{\delta}{2}+\varepsilon K\le\frac{\delta}{2}+\frac{\delta}{4}=\frac{3}{4}\delta$$

so that it is easy to check that $(v,\mu-\varphi(v))\in \mathrm{B}_0\left((u,\xi),\delta\right)\cap\mathrm{epi}(\tilde f_0)$.

If $\rho>0$, by the definition of $\mathcal{H}$ we can deduce that, for every $(v,\mu)\in \mathrm{B}_0\left((u,\xi),\delta'\right)\cap\mathrm{epi}(\tilde f_0+\varphi)$

$$\|\mathcal{H}_1\left((v,\mu-\varphi(v)),\rho t\right)-v\|=\frac{\|v-u\|}{\sqrt{\|v-u\|_0^2+\left|\mu-\varphi(v)-\tilde f_0(u)\right|^2}}\rho t\le\frac{\rho K}{\delta}t$$

since $\left|\mu-\varphi(v)-\tilde f_0(u)\right|\ge\left|\xi-\tilde f_0(u)\right|-|\mu-\xi|-|\varphi(v)|>2\delta-\frac{\delta}{2}-\frac{\delta}{4}>\delta$.

Let now $\rho=\left(1+\frac{\varepsilon K}{\delta}\right)^{-1}$, and define $\tilde{\mathcal{H}}:\left(\mathrm{B}_0\left((u,\xi),\delta'\right)\cap\mathrm{epi}(\tilde f_0+\varphi)\right)\times[0,\delta']\to\mathrm{epi}(\tilde f_0+\varphi)$ setting

$$\tilde{\mathcal{H}}\left((v,\mu),t\right)=\Big(\mathcal{H}_1((v,\mu-\varphi(v)),\rho t),\mathcal{H}_2((v,\mu-\varphi(v)),\rho t)+\varphi(\mathcal{H}_1((v,\mu-\varphi(v)),\rho t))\Big).$$

It is readily seen that $\tilde{\mathcal{H}}$ actually takes his values in $\mathrm{epi}(\tilde f_0+\varphi)$.

Furthermore, since φ is continuous in X_0 on X_1-bounded sets, (3.3) implies that $\tilde{\mathcal{H}}$ is continuous.

It is

$$\left\|\tilde{\mathcal{H}}\left((v,\mu),t\right)-(v,\mu)\right\|^2_{X_0\times\mathbb{R}}=$$

$$=\|\mathcal{H}_1((v,\mu-\varphi(v)),\rho t)-v\|_0^2+\Big(\mathcal{H}_2((v,\mu-\varphi(v)),\rho t)+\varphi(\mathcal{H}_1((v,\mu-\varphi(v)),\rho t))-\mu\Big)^2=$$

$$=\|\mathcal{H}_1((v,\mu-\varphi(v)),\rho t)-v\|_0^2+\Big(\mathcal{H}_2((v,\mu-\varphi(v)),\rho t)-(\mu-\varphi(v))\Big)^2+$$

$$+2\Big(\mathcal{H}_2((v,\mu-\varphi(v)),\rho t)-(\mu-\varphi(v))\Big)\Big(\varphi(\mathcal{H}_1((v,\mu-\varphi(v)),\rho t))-\varphi(v)\Big)+$$

$$+\Big(\varphi(\mathcal{H}_1((v,\mu-\varphi(v)),\rho t))-\varphi(v)\Big)^2\leq$$

$$\leq\rho^2t^2+2\rho t\varepsilon\,\|\mathcal{H}_1((v,\mu-\varphi(v)),\rho t)-v\|+\varepsilon^2\,\|\mathcal{H}_1((v,\mu-\varphi(v)),\rho t)-v\|^2\leq$$

$$\leq\rho^2t^2+2\rho^2\frac{\varepsilon K}{\delta}t^2+\rho^2\frac{\varepsilon^2K^2}{\delta^2}t^2=\rho^2t^2\left(1+\frac{\varepsilon K}{\delta}\right)^2=t^2.$$

Moreover,

$$\mathcal{G}_{\tilde{f}_0+\varphi}\Big(\tilde{\mathcal{H}}((v,\mu),t)\Big)=\mathcal{H}_2((v,\mu-\varphi(v)),\rho t)+\varphi(\mathcal{H}_1((v,\mu-\varphi(v)),\rho t))=$$

$$=\mathcal{G}_{\tilde{f}_0}\Big(\mathcal{H}((v,\mu-\varphi(u)),\rho t)\Big)+\varphi(\mathcal{H}_1((v,\mu-\varphi(v)),\rho t))\leq$$

$$\leq\mu-\varphi(v)-\frac{\xi-\delta-\tilde{f}_0(u)}{\sqrt{\delta^2+(\xi+\delta-\tilde{f}_0(u))^2}}\rho t+\varphi(\mathcal{H}_1((v,\mu-\varphi(v)),\rho t))\leq$$

$$\leq\mu-\frac{\xi-\delta-\tilde{f}_0(u)}{\sqrt{\delta^2+(\xi+\delta-\tilde{f}_0(u))^2}}\rho t+\varepsilon\frac{\rho K}{\delta}t=$$

$$=\mu-\left(\frac{\xi-\delta-\tilde{f}_0(u)}{\sqrt{\delta^2+(\xi+\delta-\tilde{f}_0(u))^2}}-\varepsilon\frac{K}{\delta}\right)\frac{t}{\left(1+\frac{\varepsilon K}{\delta}\right)};$$

therefore we have

$$\left|d_0\mathcal{G}_{\tilde{f}_0+\varphi}\right|(u,\xi)\geq\left(\frac{\xi-\delta-\tilde{f}_0(u)}{\sqrt{\delta^2+(\xi+\delta-\tilde{f}_0(u))^2}}-\varepsilon\frac{K}{\delta}\right)\frac{1}{\left(1+\frac{\varepsilon K}{\delta}\right)}.$$

But by [6, Proposition 2.7] it is

$$|d_0\mathcal{G}_f|\,(u,\xi)=\left|d_0\mathcal{G}_{\tilde{f}_0+\varphi+\tilde{\psi}}\right|(u,\xi)=\left|d_0\mathcal{G}_{\tilde{f}_0+\varphi}\right|(u,\xi),$$

hence by the arbitrariness of $\varepsilon\in]0,\frac{\delta}{4K}[$ it is

$$|d_0\mathcal{G}_f|\,(u,\xi)\geq\frac{\xi-\delta-f_0(u)}{\sqrt{\delta^2+(\xi+\delta-f_0(u))^2}},$$

and finally by the arbitrariness of $\delta\in\Big]0,\min\{\frac{\xi-f_0(u)}{2},r\}\Big[$ we obtain

$$|d_0\mathcal{G}_f|\,(u,\xi)\geq 1.$$

(iv) Let $u \in \mathcal{D}(f)$ and $r > 0$ corresponding to u as in (a). For every $v \in X$, let

$$\begin{aligned}\tilde{f}_0(v) &= f_0(v) + f_1(u) + \langle df_1(u), v-u\rangle_1,\\ \tilde{f}_1(v) &= f_1(v) - f_1(u) - \langle df_1(u), v-u\rangle_1.\end{aligned}$$

Then $\tilde{f}_0$ is convex; furthermore, we can choose $\varepsilon \in]0, m_{u,r}[$ and obtain the decomposition

$$\tilde{f}_0(v) = f_0(v) + f_1(u) + \langle d\varphi(u), v-u\rangle_1 + \langle d\psi(u), v-u\rangle_0$$

with $\|d\varphi(u)\|_{X_1'} \leq \varepsilon$. It follows that

$$\liminf_{\substack{\|v\|\to\infty\\ \|v-u\|_0 \leq r}} \frac{\tilde{f}_0(v)}{\|v\|} > 0.$$

As in the proof of (i), we deduce that $\tilde{f}_0 : X_0 \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, hence $\tilde{f}_0$ satisfies assumption (a). Of course, $\tilde{f}_1 : X_1 \to \mathbb{R}$ is of class C^1 with $\tilde{f}_1(u) = 0$ and $d\tilde{f}_1(u) = 0$. It is easy to see that $\tilde{f}_1$ satisfies also assumption (c) with $\tilde{\varphi}(v) = \varphi(v) - \varphi(u) - \langle d\varphi(u), v-u\rangle_1$, $\tilde{\psi}(v) = \psi(v) - \psi(u) - \langle d\psi(u), v-u\rangle_0$ and $d\tilde{\psi}(u) = 0$. Therefore, it suffices to consider the case $f_1(u) = 0$, $df_1(u) = 0$. Moreover, when we apply (c), we can ask that $d\psi(u) = 0$.

Since the case $|df|(u) = 0$ is obvious, let $0 < \sigma < |df|(u)$. From [6, Proposition 2.7] we deduce that $|df_0|(u) = |df|(u)$. As in the proof of [6, Theorem 2.11], we can find $w \in X$ such that

$$f_0(w) < f_0(u) - \sigma\|w-u\| \leq f_0(u) - \frac{\sigma}{c}\|w-u\|_0 .$$

Let $\delta > 0$ be such that $2\delta < \|w-u\|_0$ and

$$\forall v \in X : \|v-u\|_0 < \delta \Longrightarrow f_0(w) < f_0(v) - \frac{\sigma}{c}\|w-v\|_0 .$$

As in the proof of [6, Theorem 2.11], we can define a continuous map $\mathcal{H} : \mathrm{B}_0(u,\delta) \times [0,\delta] \to X_0$ by

$$\mathcal{H}(v,t) = v + \frac{t}{\|w-v\|_0}(w-v)$$

and we have

$$\|\mathcal{H}(v,t) - v\|_0 \leq t,$$

$$f_0(\mathcal{H}(v,t)) \leq f_0(v) - \frac{\sigma}{c}t.$$

By (i), there exists $K > 0$ such that

$$\forall v \in X : \|v-u\|_0 < \delta, f(v) < f(u)+1 \Longrightarrow \|v\| \leq K.$$

Now let $\varepsilon > 0$ and let $f_1 = \varphi + \psi$ according to (c) with $d\psi(u) = 0$. If $\|v-u\|_0 < \delta$ and $f(v) < f(u)+1$, we have

$$|\varphi(\mathcal{H}(v,t)) - \varphi(v)| \leq \varepsilon \|\mathcal{H}(v,t) - v\| = \varepsilon \frac{\|w-v\|}{\|w-v\|_0} t \leq \varepsilon \frac{\|w\|+K}{\delta} t.$$

It follows

$$(f_0+\varphi)(\mathcal{H}(v,t)) \leq (f_0+\varphi)(v) - \left(\frac{\sigma}{c} - \varepsilon\frac{\|w\|+K}{\delta}\right)t,$$

hence $|d_0(f_0+\varphi)|(u) \geq \left(\frac{\sigma}{c} - \varepsilon\frac{\|w\|+K}{\delta}\right)$. Since $d\psi(u) = 0$, from [6, Proposition 2.7] we deduce that

$$|d_0(f_0+\varphi)|(u) \geq \frac{\sigma}{c} - \varepsilon\frac{\|w\|+K}{\delta}.$$

By the arbitrariness of ε, we have $|d_0 f|(u) \geq \sigma/c$, and the assertion follows by the arbitrariness of σ. $\square$

Now we prove a theorem of saddle-point type for our class of functionals.

Theorem 3.2. *Let X and f be as in Theorem* 3.1.
Assume that

(*a*) *there exist $r > 0$ and $\alpha > 0$ such that*

$$\forall u \in X : \|u\|_0 = r \Longrightarrow f(u) \geq f(0) + \alpha;$$

(*b*) *there exists $u_1 \in X$ with $\|u_1\|_0 > r$ and $f(u_1) \leq f(0)$;*

(*c*) *for every $b \in \mathbb{R}$, f^b is complete with respect to $\|\cdot\|_0$ and $f : X_0 \to \mathbb{R} \cup \{+\infty\}$ satisfies $(PS)_c$, where*

$$c := \inf_{\gamma\in\Gamma} \sup_{0\leq t\leq 1} f(\gamma(t)),$$

$$\Gamma = \left\{\gamma \in C([0,1];X_0) : \gamma(0) = 0, \gamma(1) = u_1\right\}.$$

Then there exists $u \in X$ *such that* $f(u) = c$ *and*

$$\forall v \in X : f_0(v) \geq f_0(u) - \langle df_1(u), v - u\rangle_1.$$

Proof. We apply Theorem 4.5 of [4]. Even if X_0 is not complete, it is easy to see that $\mathrm{epi}(f)$ is complete, and this is enough.

By (i) and (iii) of Theorem 3.1, $f : X_0 \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and satisfies condition (4.1) of [4]. Since f_0 is convex, we have that $\gamma(t) = tu_1$ belongs to Γ and that

$$c \leq \sup_{0\leq t\leq 1} f(tu_1) < +\infty.$$

Then, by Theorem (4.5) of [4] there exist $u \in X$ with $f(u) = c$ and $|d_0 f|(u) = 0$; but for (iv) of Theorem 3.1 it is also $|df|(u) = 0$, hence the assertion follows from [6, Theorem 2.11]. □

4. An application. In this section we apply our abstract framework to obtain an existence result for a class of functionals containing an area-type term.

Let Ω be a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary, $\Psi : \mathbb{R}^n \to \mathbb{R}$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ two functions satisfying the following conditions:

(Ψ) the function Ψ is convex and there exist $c, d > 0$ such that

$$\forall \xi \in \mathbb{R}^n : d\,|\xi| - c \leq \Psi(\xi) \leq c(|\xi| + 1);$$

(g_1) the function g satisfies the Carathéodory conditions and there exist $a \in L^n(\Omega)$, $b \in \mathbb{R}$ and $p \in \left]1, \frac{n}{n-1}\right[$ such that

$$|g(x,s)| \leq a(x) + b\,|s|^{p-1}$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$.

Furthermore, let $G(x,s) = \displaystyle\int_0^s g(x,t)\,dt$.

According to [3, 8], we define $f : BV(\Omega) \to \mathbb{R}$ setting $f = f_0 + f_1$, where

$$f_0(u) = \int_\Omega \Psi(\nabla u^a)dx + \int_\Omega \Psi^\infty\left(\frac{\nabla u^s}{|\nabla u^s|}\right) d\,|\nabla u^s|\,(x) + \int_{\partial\Omega} \Psi^\infty(u\nu)\,d\mathcal{H}^{n-1}(x),$$

$$f_1(u) = -\int_\Omega G(x,u)\,dx,$$

$\nabla u = \nabla u^a + \nabla u^s$ is the Lebesgue decomposition of ∇u, $|\nabla u^s|$ is the total variation of ∇u^s, $\nabla u^s/|\nabla u^s|$ is the Radon-Nikodym derivative of ∇u^s with respect to $|\nabla u^s|$, Ψ^∞ is the recession functional associated with Ψ, and ν is the outer normal to Ω.

As a norm in $BV(\Omega)$, we shall consider

$$\|u\|_{BV} = \int_\Omega |\nabla u^a|\,dx + \int_\Omega d\,|\nabla u^s|\,(x) + \int_{\partial\Omega} |u|\,d\mathcal{H}^{n-1}(x).$$

Lemma 4.1. *Let $\gamma : [0,+\infty[\to \mathbb{R}$ be a convex function such that*

$$\forall \xi \in \mathbb{R}^n : \Psi(\xi) \geq \gamma(|\xi|).$$

Then we have

$$\forall u \in BV(\Omega) : f_0(u) \geq \mathcal{L}^n\,(\Omega)\,\gamma\left(\frac{\|u\|_{BV}}{\mathcal{L}^n\,(\Omega)}\right).$$

Proof. First of all, for any $x, y, z \in [0,+\infty[$ and $\lambda > 0$ we have

$$\lambda\gamma\left(\frac{x+y+z}{\lambda}\right) \leq \lambda\gamma\left(\frac{x}{\lambda}\right) + \gamma^\infty(y) + \gamma^\infty(z). \tag{4.1}$$

In fact, for any $\varepsilon \in]0,1/2]$ we have

$$\lambda\gamma\left(\frac{x+y+z}{\lambda}\right) = \lambda\gamma\left((1-2\varepsilon)\frac{x}{\lambda} + \varepsilon\left(\frac{x}{\lambda} + \frac{y}{\varepsilon\lambda}\right) + \varepsilon\left(\frac{x}{\lambda} + \frac{z}{\varepsilon\lambda}\right)\right) \leq$$

$$\leq \lambda(1-2\varepsilon)\gamma\left(\frac{x}{\lambda}\right) + \lambda\varepsilon\gamma\left(\frac{x}{\lambda} + \frac{y}{\varepsilon\lambda}\right) + \lambda\varepsilon\gamma\left(\frac{x}{\lambda} + \frac{z}{\varepsilon\lambda}\right) \leq$$

$$\leq \lambda(1-2\varepsilon)\gamma\left(\frac{x}{\lambda}\right) + 2\lambda\varepsilon\gamma(0) + \lambda\varepsilon\gamma^\infty\left(\frac{x}{\lambda} + \frac{y}{\varepsilon\lambda}\right) + \lambda\varepsilon\gamma^\infty\left(\frac{x}{\lambda} + \frac{z}{\varepsilon\lambda}\right) =$$

$$= \lambda(1-2\varepsilon)\gamma\left(\frac{x}{\lambda}\right) + 2\lambda\varepsilon\gamma(0) + \gamma^\infty(\varepsilon x + y) + \gamma^\infty(\varepsilon x + z).$$

Going to the limit as $\varepsilon \to 0$, we get (4.1).

Now let $u \in BV(\Omega)$. Since $\gamma^\infty(|\xi|) \leq \Psi^\infty(\xi)$, from Jensen's inequality and (4.1) we deduce that

$$f_0(u) \geq \int_\Omega \gamma(|\nabla u^a|)\,dx + \int_\Omega \gamma^\infty(1)\,d\,|\nabla u^s|\,(x) + \int_{\partial\Omega} |u|\,\gamma^\infty(1)\,d\mathcal{H}^{n-1}(x) \geq$$

$$\geq \mathcal{L}^n(\Omega)\,\gamma\left(\mathcal{L}^n(\Omega)^{-1}\int_\Omega |\nabla u^a|\,dx\right) + \gamma^\infty\left(\int_\Omega d\,|\nabla u^s|\,(x)\right) + \gamma^\infty\left(\int_{\partial\Omega} |u|\,d\mathcal{H}^{n-1}(x)\right) \geq$$

$$\geq \mathcal{L}^n(\Omega)\,\gamma\left(\frac{\|u\|_{BV}}{\mathcal{L}^n(\Omega)}\right),$$

whence the assertion. □

In particular, combining the previous lemma with assumption (Ψ), we deduce that there exist $\tilde{c}, \tilde{d} > 0$ such that

$$\forall u \in BV(\Omega) : \tilde{d}\,\|u\|_{BV} - \tilde{c} \leq f_0(u) \leq \tilde{c}(\|u\|_{BV} + 1). \tag{4.2}$$

Lemma 4.2. *For every $\varepsilon > 0$ there exist $\varphi : BV(\Omega) \to \mathbb{R}$ Lipschitz of constant ε, $\psi : L^p(\Omega) \to \mathbb{R}$ of class C^1 such that $f_1 = \varphi + \psi$.*

Proof. Let $\varepsilon > 0$ and

$$g_1(x,s) = \min\{\max\{g(x,s), -a(x)\}, a(x)\},$$

$$g_2(x,s) = g(x,s) - g_1(x,s),$$

so that $|g_1(x,s)| \leq a(x)$ and $|g_2(x,s)| \leq b\,|s|^{p-1}$.

Let $\overline{c} > 0$ be such that $\|\cdot\|_{\frac{n}{n-1}} \leq \overline{c}\,\|\cdot\|_{BV}$; if $k \in \mathbb{R}$, let $\overline{a}(x) = a(x)\chi_{\{a(x)\geq k\}}(x)$, and choose k such that $\overline{c}\,\|\overline{a}\|_n \leq \varepsilon$. Now let

$$\overline{g}_1(x,s) = g_1(x,s)\chi_{\{a(x)\geq k\}}(x),$$

$$\overline{\overline{g}}_1(x,s) = g_1(x,s)\chi_{\{a(x)< k\}}(x),$$

so that $|\overline{g}_1(x,s)| \leq \overline{a}(x)$ and $|\overline{\overline{g}}_1(x,s)| \leq k$.

Finally, let

$$G_1(x,s) = \int_0^s \overline{g}_1(x,t)\,dt,$$

$$G_2(x,s) = \int_0^s [\overline{\overline{g}}_1(x,t) + g_2(x,t)]\,dt,$$

$$\varphi(u) = \int_\Omega G_1(x,u)\,dx,$$

$$\psi(u) = \int_\Omega G_2(x,u)\,dx.$$

Since

$$|\overline{\overline{g}}_1(x,s) + g_2(x,s)| \leq k + b\,|s|^{p-1}\,,$$

it is well known that $\psi : L^p(\Omega) \to \mathbb{R}$ is of class C^1.

Furthermore, it is

$$|\varphi(v) - \varphi(u)| \leq \int_\Omega \overline{a}(x)\,|v-u|\;dx \leq \|\overline{a}\|_n\,\|v-u\|_{\frac{n}{n-1}} \leq$$

$$\leq \overline{c}\,\|\overline{a}\|_n\,\|v-u\|_{BV} \leq \varepsilon\,\|v-u\|_{BV}\,,$$

namely $\varphi : BV(\Omega) \to \mathbb{R}$ is Lipschitz continuous of constant ε. □

Now we consider the space $X = BV(\Omega)$, we denote by $\|\cdot\|$ the norm of $BV(\Omega)$ and by $\|\cdot\|_0$ the norm of $L^p(\Omega)$.

Theorem 4.3. *The following facts hold:*

(i) for every $b \in \mathbb{R}$, f^b is complete with respect to $\|\cdot\|_0$;

(ii) $f_1 : X_0 \to \mathbb{R}$ is continuous on X_1-bounded subsets;

(iii) $\xi > f(u) \Longrightarrow |d_0\mathcal{G}_f|\,(u,\xi) = 1$;

(iv) if $u \in \mathcal{D}(f)$, then $|df|\,(u) \leq c\,|d_0 f|\,(u)$.

Proof. For (Ψ), (g_1) and the previous lemma, hypotheses of Theorem 3.1 are satisfied; therefore (ii), (iii) and (iv) follow by Theorem 3.1

(i) Let (u_h) be a sequence in $BV(\Omega)$ convergent to u in $L^p(\Omega)$ with $f(u_h) \leq b$. Let $\varepsilon = \tilde{d}/2$ and let φ, ψ be as in the previous lemma. Since $(\psi(u_h))$ is bounded and

$$f_0(u_h) + \varphi(u_h) - \varphi(0) \geq \frac{\tilde{d}}{2}\,\|u_h\| - \tilde{c},$$

we have that (u_h) is bounded in $BV(\Omega)$, so that $u \in BV(\Omega)$. From (i) of Theorem 3.1 assertion follows. □

Let us now assume the following superlinearity condition on G:

(g_2) there exist $q > 1$ and $R > 0$ such that for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$ with $|s| \geq R$ we have

$$0 < qG(x,s) \leq sg(x,s).$$

From (g_1) and (g_2) it follows (see e.g. [13, Theorem 6.2]) that there exists $a_0 \in L^1(\Omega)$ such that for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$ we have

$$G(x,s) \geq (R^{-q}\min\{G(x,R),G(x,-R)\})\,|s|^q - a_0(x), \tag{4.3}$$

$$qG(x,s) \leq sg(x,s) + a_0(x). \tag{4.4}$$

Let us also recall that hypothesis (Ψ) implies that Ψ is Lipschitz continuous of constant c, and therefore there exists $M \in \mathbb{R}$ such that

$$(q+1)\Psi(\xi) - \Psi(2\xi) \geq \frac{q-1}{2}\Psi(\xi) - M, \tag{4.5}$$

$$(q+1)\Psi^\infty(\xi) - \Psi^\infty(2\xi) \geq \frac{q-1}{2}\Psi^\infty(\xi). \tag{4.6}$$

Theorem 4.4. *The following facts hold:*

(*a*) *for every $u \in BV(\Omega)\setminus\{0\}$ we have*

$$\lim_{t\to+\infty} f(tu) = -\infty;$$

(*b*) *for every $c \in \mathbb{R}$, $f : X_0 \to \mathbb{R}\cup\{+\infty\}$ satisfies $(PS)_c$.*

Proof. (a) Let $u \in BV(\Omega)\setminus\{0\}$; from (4.2) and (4.3) it follows that

$$f(tu) \leq \tilde{c}\,|t|\,\|u\| - R^{-q}|t|^q\int_\Omega \min\{G(x,R),G(x,-R)\})\,|u|^q\,dx + \int_\Omega a_0\,dx,$$

hence

$$\lim_{t\to+\infty} f(tu) = -\infty.$$

(b) Let $c \in \mathbb{R}$ and let (u_h) be a sequence in $BV(\Omega)$ such that $|d_0 f|(u_h) \to 0$ and $f(u_h) \to c$.

From (iv) of Theorem 4.3 it follows that $|df|(u_h) \to 0$. From [6, Theorem 2.11], there exist $w_h \in (BV(\Omega))'$ such that $\|w_h\|_{(BV(\Omega))'} \to 0$ and

$$\forall v \in X : f_0(v) \geq f_0(u_h) + \int_\Omega g(x,u_h)(v-u_h)\,dx + \langle w_h, v-u_h\rangle.$$

Choosing $v_h = 2u_h$ we have, by (4.4),

$$f_0(2u_h) \geq f_0(u_h) + \int_\Omega u_h g(x, u_h)\, dx + \langle w_h, u_h \rangle - \int_\Omega a_0\, dx.$$

By the definition of f, we obtain

$$q f(u_h) - \langle w_h, u_h \rangle + \int_\Omega a_0\, dx \geq (q+1) f_0(u_h) - f_0(2u_h).$$

Therefore, by (4.2), (4.5) and (4.6), it follows that

$$q f(u_h) + \|w_h\|_{(BV(\Omega))'} \|u_h\| + \int_\Omega a_0\, dx \geq \frac{(q-1)}{2} f_0(u_h) - M\mathcal{L}^n(\Omega) \geq$$

$$\geq \frac{q-1}{2} \tilde{d}\, \|u_h\| - \frac{q-1}{2} \tilde{c} - M\mathcal{L}^n(\Omega).$$

Hence (u_h) is bounded in $BV(\Omega)$ and the assertion follows from the compact embedding of $BV(\Omega)$ in $L^p(\Omega)$ (see [7]). □

Now we state the last hypothesis needed for the geometrical conditions of the mountain pass theorem:

(α) there exist $\alpha \in [1, \frac{n}{n-1}[$ and $a_1 \in L^{\frac{n}{n+\alpha-n\alpha}}(\Omega)$ such that

$$\liminf_{\xi \to 0} \frac{\Psi(\xi) - \Psi(0)}{|\xi|^\alpha} > 0,$$

$$\lim_{s \to 0} \frac{G(x,s)}{|s|^\alpha} = 0,$$

$$|G(x,s)| \leq a_1(x)\, |s|^\alpha + b\, |s|^{\frac{n}{n-1}}$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$.

Lemma 4.5. *Let $(u_h) \subseteq BV(\Omega)$ with $\|u_h\| = 1$ and $\rho_h > 0$ with $\rho_h \to 0$. Then it is*

$$\liminf_{h \to \infty} \frac{f_0(\rho_h u_h) - f_0(0)}{\rho_h^\alpha} > 0.$$

Proof. Without loss of generality, we can assume $\Psi(0) = 0$. Let $\delta > 0$ be such that

$$|\xi| \leq \delta \;\Rightarrow\; \Psi(\xi) \geq \delta\, |\xi|^\alpha.$$

Since Ψ is convex, for every $\xi \in \mathbb{R}^n$ with $|\xi| > \delta$ we have

$$\Psi\left(\frac{\delta}{|\xi|}\xi\right) \leq \frac{\delta}{|\xi|}\Psi(\xi),$$

hence

$$\Psi(\xi) \geq \frac{|\xi|}{\delta}\Psi\left(\frac{\delta}{|\xi|}\xi\right) \geq \delta^\alpha |\xi| .$$

If we define $\gamma : [0, +\infty[\to \mathbb{R}$ by

$$\gamma(s) = \begin{cases} \frac{\delta}{\alpha}s^\alpha & \text{if } 0 \leq s \leq \delta \\ \delta^\alpha s - \frac{\alpha-1}{\alpha}\delta^{\alpha+1} & \text{if } s \geq \delta \end{cases}$$

we have that γ is convex and satisfies

$$\forall \xi \in \mathbb{R}^n : \Psi(\xi) \geq \gamma(|\xi|).$$

Taking into account Lemma 4.1, we deduce that

$$f_0(\rho_h u_h) \geq \mathcal{L}^n(\Omega)\,\gamma\left(\frac{\rho_h}{\mathcal{L}^n(\Omega)}\right)$$

and the assertion follows. □

Lemma 4.6. *Let $(u_h) \subseteq BV(\Omega)$ with $\|u_h\| = 1$ and $\rho_h > 0$ with $\rho_h \to 0$. Then it is*

$$\lim_{h\to\infty} \frac{G(x, \rho_h u_h)}{\rho_h^\alpha} = 0$$

in $L^1(\Omega)$.

Proof. Up to a subsequence, we can assume that $u_h(x) \to u(x)$ for a.e. $x \in \Omega$. Moreover, (u_h) is bounded in $L^{\frac{n}{n-1}}(\Omega)$.

From (α) we deduce that

$$\lim_{h\to\infty} \frac{G(x, \rho_h u_h)}{\rho_h^\alpha} = 0$$

for a.e. $x \in \Omega$ and that

$$\frac{|G(x, \rho_h u_h)|}{\rho_h^\alpha} \leq a_1(x)\,|u_h|^\alpha + b\rho_h^{\frac{n}{n-1}-\alpha}\,|u_h|^{\frac{n}{n-1}} .$$

Since the right hand side of the last inequality is strongly convergent in $L^1(\Omega)$, the assertion follows from the Lebesgue theorem. □

Theorem 4.7. *There exists $r > 0$ such that*

$$\forall u \in BV(\Omega) : \|u\|_0 = r \Longrightarrow f(u) \geq f(0) + r^{\alpha+1}.$$

Proof. By contradiction, let (u_h) be a sequence in $BV(\Omega)$ such that $\|u_h\|_0 = 1/h$ and

$$f(u_h) < f(0) + \frac{1}{h^{\alpha+1}}.$$

From (i) of Theorem 3.1, it follows that $\|u_h\|$ is bounded; hence, from (ii) of Theorem 3.1 we deduce that

$$\lim_{h\to\infty} f_1(u_h) = 0.$$

Define $\gamma : [0, +\infty[\to \mathbb{R}$ as in Lemma 4.5. Since

$$\limsup_{h\to\infty} \mathcal{L}^n(\Omega)\, \gamma\left(\frac{\|u_h\|}{\mathcal{L}^n(\Omega)}\right) \leq \limsup_{h\to\infty} (f_0(u_h) - f_0(0)) \leq 0,$$

we have $\|u_h\| \to 0$.

Let $\rho_h = \|u_h\|$, $w_h = u_h / \|u_h\|$; applying Lemmas 4.5 and 4.6 to ρ_h and w_h we deduce that

$$\liminf_{h\to\infty} \frac{f_0(u_h) - f_0(0)}{\rho_h^\alpha} > 0,$$

$$\lim_{h\to\infty} \frac{f_1(u_h)}{\rho_h^\alpha} = -\lim_{h\to\infty} \frac{\int_\Omega G(x, u_h)\, dx}{\rho_h^\alpha} = 0.$$

It follows $\liminf_{h\to\infty} \dfrac{f(u_h) - f(0)}{\rho_h^\alpha} > 0$, whence a contradiction. □

Since $f : BV(\Omega) \to \mathbb{R}$ is the sum of a convex term and a term of class C^1 (when $BV(\Omega)$ is endowed with its natural norm), it is natural to say that $u \in BV(\Omega)$ is a (generalized) critical point for f if

$$\forall v \in BV(\Omega) : f_0(v) \geq f_0(u) - \langle df_1(u), v - u\rangle.$$

Remark 4.8. Under mild assumptions of Ψ, it is shown in [2] that the above relation implies that u satisfies a suitable Euler equation.

We may now prove the main result of this section.

Theorem 4.9. *Assume that (Ψ), (g_1), (g_2) and (α) hold. Then there exists $u \in BV(\Omega) \setminus \{0\}$ such that u is a critical point for f.*

Proof. For Theorem 4.4 and Theorem 4.7 all the hypotheses of Theorem 3.2 are satisfied. Hence the assertion follows from Theorem 3.2. □

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