Serdica

Mathematical Journal

Сердика

Математическо списание

Serdica Math. J. **22** (1996), 307-330

Serdica
Mathematical Journal

Institute of Mathematics
and Informatics, Bulgarian
Academy of Sciences

STABILITY OF SUPPORTING AND EXPOSING ELEMENTS OF CONVEX SETS IN BANACH SPACES

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Communicated by A. L. Dontchev

Abstract. To a convex set in a Banach space we associate a convex function (the separating function), whose subdifferential provides useful information on the nature of the supporting and exposed points of the convex set. These points are shown to be also connected to the solutions of a minimization problem involving the separating function. We investigate some relevant properties of this function and of its conjugate in the sense of Legendre-Fenchel. Then we highlight the connections between set convergence, with respect to the slice and Attouch-Wets topologies, and convergence, in the same sense, of the associated functions. Finally, by using known results on the behaviour of the subdifferential of a convex function under the former epigraphical perturbations, we are able to derive stability results for the set of supported points and of supporting and exposing functionals of a closed convex subset of a Banach space.

1. Introduction. In this paper, we work with a function characterizing convex sets which is neither the indicator function nor the support function. This function, which we call the separating function of the convex set C, is defined in the following way: For all $x \in X$, and setting $\inf \emptyset = +\infty$, let $f_{[C,u]}$ be defined as

$$f_{[C,u]}(x) = \inf\{t \in \mathbb{R} : x + tu \in C\},$$

1991 *Mathematics Subject Classification:* primary 54A20, 52A40, 52A05, secondary 46B20

Key words: convex sets, convex functions, supported and exposed points, slice topology, Attouch-Wets topology, convex optimization

*This work was supported by the CNR while the author was visiting the University of Milan.

where u is a norm one vector.
The subdifferential of this function carries on some information on the exposed points and exposing functionals of the closed convex set. Moreover, the mapping which assigns to the set its separating function enjoys some bicontinuity properties with respect to the slice and Attouch-Wets topologies. This observation, along with continuity properties of the subdifferential with respect to the quoted variational convergences, leads to stability results for the sets of support points, exposed points and supporting functionals when the closed convex set moves along these topologies. The paper is organized as follows. In section 2 we precise our notations and recall some results that will be used in the sequel. In section 3 the separating function is introduced and some of its properties are reviewed. We also give a quantitative version of the celebrated Bishop-Phelps Theorem ([16]). In section 4 we include all the results related to stability. In particular we prove a continuity result for the subdifferential of convex functions in Asplund spaces, using the Attouch-Wets topology, and we establish stability properties for the set of supporting and exposing elements of a closed convex set of a Banach space.

2. Preliminaries and notations. Let us begin with some definitions. Given a normed vector space $(X, \|.\|)$, we shall indicate by B_X the closed unit ball and by S_X the unit sphere. The closed ball with center x and radius r is denoted by $B(x, r)$, but when $x = 0$, we shall also write rB_X. The product of normed spaces $X \times Y$ is endowed with the norm

$$\|(x, y)\| = \max(\|x\|, \|y\|).$$

We shall indicate by X^* the (continuous) dual of the Banach space X and by $\langle \cdot, \cdot \rangle$ the usual pairing between X and X^*.

Given a function $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ and given $\lambda \in \mathbb{R}$ we denote by $[f \leq \lambda]$ the set of those $x \in X$ with $f(x) \leq \lambda$, and by epi f the set

$$\text{epi } f = \{(x, t) \in X \times \mathbb{R} : f(x) \leq t\}.$$

As it is easy to show, epi f is closed if and only if f is lower semicontinuous and convex if and only if f is convex. In the sequel, as it is usual in an optimization setting, we shall often identify a function with its epigraph: in particular, when we want to topologize the set of the lower semicontinuous (and convex) functions $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$, we intend to define a topology on the closed (and convex) subsets of $X \times \mathbb{R}$.

We denote by $\mathcal{F}(X)$ the set of the closed subsets of X, by $\mathcal{C}(X)$ (resp. $\mathcal{C}(X^*)$) the set of the subsets of X (resp. X^*) which are convex and $\sigma(X, X^*)$-closed (resp. $\sigma(X^*, X)$-closed). Conv(X) will be the set of the extended real-valued (i.e. valued in $\mathbb{R} \cup \{+\infty\}$) convex functions defined on X and $\Gamma_0(X)$ (resp. $\Gamma_0(X^*)$) the set of the proper (i.e. not identically equal to $+\infty$) functions on X (resp. X^*) whose epigraphs belong to $\mathcal{C}(X \times \mathbb{R})$, (resp. $\mathcal{C}(X^* \times \mathbb{R})$). The sets $\Gamma_0(X)$ and $\Gamma_0(X^*)$ are connected by one to one mappings $\mathcal{L}$ and $\mathcal{L}_*$, the Legendre-Fenchel transforms, defined for all

$f \in \Gamma_0(X)$, $g \in \Gamma_0(X^*)$, $x \in X$, $y \in X^*$, by $\mathcal{L}(f) = f^*$ and $\mathcal{L}_*(g) = g^*$, where

$$f^*(y) = \sup\{\langle x, y\rangle - f(x) : x \in X\}$$

and

$$g^*(x) = \sup\{\langle x, y\rangle - g(y) : y \in X^*\}.$$

Given $\varepsilon \geq 0$ the ε-subdifferential $\partial_\varepsilon f$ of a function $f \in \operatorname{Conv}(X)$ is the set

$$\partial_\varepsilon f = \{(x, y) \in X \times X^* : f(x) + f^*(y) - \langle x, y\rangle \leq \varepsilon\}.$$

The set $\partial_0 f$ will be simply denoted by ∂f.

We introduce now some ways of associating functions to sets (and vice-versa). The indicator function of the subset $C \subset X$ is the function i_C from X into $\mathbb{R} \cup \{+\infty\}$ defined by $i_C(x) = 0$ if $x \in C$ and $i_C(x) = +\infty$ if $x \in X \setminus C$. Given $C \in \mathcal{C}(X)$ we denote by σ_C the support function of C. It is defined as:

$$\sigma_C(x^*) = \sup\{\langle c, x^*\rangle \ : \ c \in C\} = i_C^*.$$

$C^\circ \subset X^*$ is the polar set of C, defined by

$$C^\circ = \{y \in X^* : \sigma_C(y) \leq 1\}.$$

The recession cone 0^+C of a closed convex set C is the set of those $u \in X$ such that for all $x \in C$ one has $x + \mathbb{R}_+ u \subset C$. It is also equal to

$$0^+C = \{u \in X : \text{ for all } u^* \in \operatorname{dom} \sigma_C, \ \langle u, u^*\rangle \leq 0\}. \tag{1}$$

Given $C \in \mathcal{C}(X)$, an element $x \in C$ is said to be a support point (for C) if there exists $u^* \in X^* \setminus \{0\}$, which is called a support functional for C at x, such that

$$\sigma_C(u^*) = \langle x, u^*\rangle. \tag{2}$$

The point $x \in C$ is said to be exposed if x is the only element of C satisfying (2), it is said to be strongly exposed if every sequence $(x_n) \subset C$ converges to x whenever $(\langle x_n, u^*\rangle)$ converges to $\sigma_C(u^*)$. Given $(C, u) \in \mathcal{C}(X) \times S_X$, and $\varepsilon \geq 0$ we denote by ε-$\mathrm{Supp}_u C$ the set

$$\varepsilon\text{-}\mathrm{Supp}_u C = \{(x, w^*) \in C \times X^* : \sigma_C(w^*) - \varepsilon \leq \langle x, w^*\rangle, \ \langle u, w^*\rangle = -1\}.$$

When $\varepsilon = 0$ we set

$$\mathrm{Supp}_u C = 0\text{-}\mathrm{Supp}_u C.$$

We also set

$$\mathrm{Exp}_u C = \{(x, w^*) \in \mathrm{Supp}_u C : (\mathrm{Supp}_u C)^{-1}(w^*) = \{x\}\},$$

where

$$(\mathrm{Supp}_u C)^{-1}(w^*) = \{z \in X : (z, w^*) \in \mathrm{Supp}_u C\}.$$

We shall see later that a support point can be characterized by being solution of a certain minimization problem. Moreover, a strongly exposed point is a solution of a problem which has a particular nature, described by the next definition.

An optimization problem $\inf_{x \in X} f(x)$, where $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is defined on a metric space (X, d), is said to be well posed if it admits a (unique) solution which is the limit of each sequence (x_n) in X with $\lim_{n \to \infty} f(x_n) = \inf_{x \in X} f(x)$.

The epigraphical sum $f \underset{e}{+} g$ (or inf-convolution) of two functions f, g from X into $\overline{\mathbb{R}}$ is the function defined for all $x \in X$ by

$$(f \underset{e}{+} g)(x) = \inf_{z \in X} (f(z) + g(x - z)).$$

It is said to be exact at x if $\inf_{z \in X} (f(z) + g(x - z)) = \min_{z \in X} (f(z) + g(x - z))$.

We shall need a result of [8] on the computation of the conjugate of the sum of two convex functions in general normed spaces.

Theorem 2.1. *Let X be a normed space and let f, $g \in Conv(X)$ be proper convex functions. Assume that for some real numbers λ, $s > 0$, $r > 0$ one has*

$$sB_X \subset [f \le \lambda] \cap rB_X - [g \le \lambda] \cap rB_X. \tag{3}$$

Then for all $y \in X^*$

$$(f + g)^*(y) = (f^* \underset{e}{+} g^*)(y)$$

and the epigraphical sum is exact.

To conclude our preliminaries, let us introduce the set topologies we shall use in the paper. Given $x \in X$ and given subsets A, C of X we set

$$d(x, A) = \inf\{\|x - z\| : z \in A\},$$

with the convention $d(x, \emptyset) = +\infty$,

$$D(A, C) = \inf\{\|x - z\| : (x, z) \in A \times C\}$$

and

$$e(C, D) = \sup\{d(x, D) : x \in C\},$$

with the conventions $e(\emptyset, D) = 0$ and $e(C, \emptyset) = +\infty$ if $C \neq \emptyset$. The Hausdorff distance between C and D is defined by

$$h(C, D) = \max(e(C, D), e(D, C)).$$

For each $r \in \mathbb{R}_+, \varepsilon \in \mathbb{R}_+^*$ we define

$$\begin{aligned} e_r(C,D) &= e(C \cap rB_X, D), \\ h_r(C,D) &= \max(e_r(C,D), e_r(D,C)), \\ \mathcal{U}^-_{r,\varepsilon} &= \{(C,D) \in 2^X : e_r(C,D) \le \varepsilon\}, \\ \mathcal{U}^+_{r,\varepsilon} &= \{(C,D) \in 2^X : e_r(D,C) \le \varepsilon\}, \\ \mathcal{U}_{r,\varepsilon} &= \{(C,D) \in 2^X : h_r(C,D) \le \varepsilon\} = \mathcal{U}^-_{r,\varepsilon} \cap \mathcal{U}^+_{r,\varepsilon}. \end{aligned}$$

The family of sets $\mathcal{U}^-_{r,\varepsilon}$ (resp. $\mathcal{U}^+_{r,\varepsilon}$) is a basis for a quasi-uniformity on the hyperspace $\mathcal{F}(X)$. We denote by τ_{AW_-} (resp. τ_{AW_+}) the topology induced by this quasi-uniformity. We denote by τ_{AW} the supremum of τ_{AW_-} and τ_{AW_+}. This topology, the Attouch-Wets topology (see [2]), is associated with a metrizable uniformity on $\mathcal{F}(X)$ whose basis is constituted by the sets $\mathcal{U}_{r,\varepsilon}$ when (r,ε) ranges over $\mathbb{R}_+ \times \mathbb{R}_+^*$. In terms of sequences, $C = \tau_{AW}\text{-}\lim_{n\to\infty} C_n$ if and only if $\lim_{n\to\infty} h_\rho(C, C_n) = 0$ for all (large) ρ.

The inferior limit of a sequence (C_n) of closed subsets of a metric space (X,d) is the set $\operatorname{Li}_{n\to\infty} C_n$ of those $x \in X$ such that $\lim_{n\to\infty} d(x, C_n) = 0$. Equivalently, it is the set of $x \in X$ for which there exists a sequence (x_n) converging to x such that $x_n \in C_n$ eventually. Instead the superior limit $\operatorname{Ls}_{n\to\infty} C_n$ is the set of $x \in X$ for which there exists a sequence (x_k) converging to x such that $x_k \in C_{n_k}$, where $\{n_k\}$ is a subsequence of the integers. We shall say that the sequence (C_n) converges to C in Kuratowski sense if $\operatorname{Ls}_{n\to\infty} C_n \subset C \subset \operatorname{Li}_{n\to\infty} C_n$.

Together with the Attouch-Wets topology, we shall consider another topology on the subset $\mathcal{C}(X)$ of the hyperspace $\mathcal{F}(X)$ of the closed subsets of the normed space X: the slice topology. To briefly introduce it, let us define the family of sets

$$\mathcal{O}^- = \{\{C \in \mathcal{C}(X) : O \cap C \neq \emptyset\} : O \text{ runs over the family of the open subsets of } X\}$$

and

$$(\mathcal{B}^c)^{++} = \{\{C \in \mathcal{C}(X) : D(C, X \setminus A) > 0\} : A \text{ runs over the convex bounded sets}\}.$$

Then the slice topology is defined as the smallest topology containing the families $\mathcal{O}^-$ and $(\mathcal{B}^c)^{++}$. More precisely, $\mathcal{O}^-$ gives rise to the *lower* slice topology and $(\mathcal{B}^c)^{++}$ generates the *upper* slice topology. For more information on this hypertopology, we refer to [11], [13], [27], [28], [10]. Here we just mention the following facts, that will be used in the sequel:

- A sequence (C_n) of closed convex sets converges for the lower slice topology to a closed convex set C if and only if $C \subset \operatorname{Li}_{n\to\infty} C_n$;

- a sequence (C_n) of closed convex sets converges for the slice topology to a closed convex set C if and only if the sequence (σ_{C_n}) slice converges to σ_C;
- given $\{f, f_n : n \in \mathcal{N}\} \subset \Gamma_0(X)$, then (f_n) slice converges to f if and only if the two following conditions hold:

a) for all $u \in X$, there exists a sequence $(u_n) \longrightarrow u$ such that

$$\limsup_{n\to\infty} f_n(u_n) \leq f(u),$$

b) for all $u^* \in X^*$, there exists a sequence $(u_n^*) \longrightarrow u^*$ such that

$$\limsup_{n\to\infty} f_n^*(u_n^*) \leq f^*(u^*).$$

3. Separating functions of convex sets. In the sequel C will be a convex subset of a normed vector space X different from $\emptyset$ and X [1]. We shall study the supported points of C by associating to it a direction u and a closed hyperplane H, not containing u, which allow defining a function, that carries useful information about C. To do this, let us set the following notations:

$$(\mathcal{A}) \qquad \begin{cases} u \in S_X \text{ and } -u \notin 0^+C \\ H = \{x \in X : \langle x, u^* \rangle = 0\} \text{ for some } u^* \in X^* \text{ with } \langle u, u^* \rangle = -1. \end{cases}$$

We shall indicate by π the projection on H in the direction u, namely $\pi(x) = x + \langle x, u^* \rangle u$ for all $x \in X$.

Remark 3.1. It is useful for the sequel to observe the following: the dual space H^* of H is isomorphic to the subspace $\widehat{H}^* \subset X^*$ defined as $\widehat{H}^* = \{x^* \in X^* : \langle u, x^* \rangle = 0\}$. A natural isomorphism $j : H^* \longrightarrow \widehat{H}^*$ is for instance $j(a^*) = x^*$, where $x^*_{|H} = a^*$ and $\langle u, x^* \rangle = 0$. Moreover the dual norm of the norm of X restricted to H is equivalent to the restriction of the dual norm of X on H^*. Thus in the sequel we shall identify H^* with the hyperplane $\{x^* \in X^* : \langle u, x^* \rangle = 0\}$.

Following [20] and [29], we can now introduce the function which will play a fundamental role in the study of the convex set C. For all $x \in X$, and setting $\inf \emptyset = +\infty$, let $f_{[C,u]}$ be defined as

$$f_{[C,u]}(x) = \inf\{t \in \mathbb{R} : x + tu \in C\} = (i_C \underset{e}{+} (i_{\mathbb{R}u} - \langle ., x^* \rangle))(x),$$

[1] This assumption guarantees in particular the existence of a unit vector $u \in X$ such that $-u \notin 0^+C$, which is all we need when in our statements we shall assume subsequent hypothesis $(\mathcal{A})$.

where $x^* \in S_{X^*}$ is such that $1 = \|u\| = \langle u, x^* \rangle$. We say that $f_{[C,u]}$ is the separating function of C with respect to the direction u. We also introduce the restriction $f_{[C,u,H]}$ of $f_{[C,u]}$ to H. Let us observe here that the assumption of existence of a vector u such that $-u \notin 0^+C$ is necessary to avoid the trivial situation when the separating function $f_{[C,u]}$ never assumes real values. Moreover, when $-u \in 0^+C$ it may happen that the function $f_{[C,u]}$ is not lower semicontinuous even when C is closed. Indeed let $C \subset \mathbb{R} \times \mathbb{R}$ be the epigraph of the convex function $h(x) = (|x| - 1)^{-2} + i_{[-1,1]}(x)$ and $u = (0, -1)$. We obtain

$$f_{[C,u]}(x + tu) = \begin{cases} -\infty & \text{if } (x,t) \in]0,1[\times\mathbb{R} \\ \\ +\infty & \text{if } (x,t) \notin]0,1[\times\mathbb{R}. \end{cases}$$

On the other hand, as we shall see later (see Proposition 3.1), the function $f_{[C,u]}$ has nice properties whenever $-u \notin 0^+C$.

Let us set $C_u^+ = C + \mathbb{R}_+ u$. Observe that $C_u^+ = \{x : f_{[C,u]}(x) \leq 0\}$ and that $f_{[C,u]}$ is proper whenever there exists $u^* \in C^\circ$ with $\langle u, u^* \rangle < 0$, a condition equivalent to $-u \notin 0^+C$. Observe also that $x + f_{[C,u]}(x)u \in C$ whenever $(x + \mathbb{R}u) \cap C$ is closed and that $f_{[C,u]} = f_{[C_u^+,u]}$. Moreover, for all $x \in X$ and $\mu \in \mathbb{R}$ one has

$$f_{[C,u]}(x + \mu u) = f_{[C,u]}(x) - \mu. \tag{4}$$

Recalling that x and its projection are related by the formula $\pi(x) = x + \langle x, u^* \rangle u$ we then have, for all $x \in X$

$$f_{[C,u,H]}(\pi(x)) = f_{[C,u]}(x) - \langle x, u^* \rangle. \tag{5}$$

Moreover

$$x = \pi(x) + (f_{[C,u,H]}(\pi(x)) - f_{[C,u]}(x))u,$$

whenever $x \in \operatorname{dom} f_{[C,u]}$, and for all $y \in X$

$$f_{[C,u]}(x) = f_{[(C+y),u]}(x + y).$$

Finally, we observe that, in the case where $C = \operatorname{epi} h$ with $h \in \operatorname{Conv}(X)$ and $u = (0, 1)$ we get, for all $x \in X$:

$$f_{[C,u]}(x, 0) = h(x)$$

and for all $(x, t) \in X \times \mathbb{R}$

$$f_{[C,u]}(x, t) = h(x) - t.$$

Proposition 3.1. *Let C be a closed subset of X. Then for all $u \in S_X$ for which $-u \notin 0^+C$ one has*

$$f_{[C,u]} \in \Gamma_0(X).$$

Proof. We only need to prove lower semicontinuity of $f_{[C,u]}$. Let $x \in X$ and let (x_n) be a sequence converging to x and such that

$$\lim_{n\to\infty} f_{[C,u]}(x_n) = l,$$

with $l = \liminf_{z\to x} f_{[C,u]}(z)$. There is nothing to prove if $l = \infty$. Otherwise, observe that $l > -\infty$, as there is $v^* \in C^\circ$ such that $\langle u, v^*\rangle < 0$. Thus there is a sequence $t_n \to 0$ such that $x_n + (l + t_n)u \in C$. As C is closed, $x + lu \in C$. Thus $f_{[C,u]}(x) \le l$ which ends the proof of the proposition. $\square$

The function $f_{[C,u]}$ provides a complete description of the set C. Indeed if C is closed convex and if $u \in S_X$ four cases occur

$u \notin 0^+C, -u \notin 0^+C$	$C = \{x + tu : \ x \in H, \ f_{[C,u,H]}(x) \le t \le -f_{[C,-u,H]}(x)\}$
$u \in 0^+C, -u \notin 0^+C$	$C = \{x + tu : \ x \in H, \ f_{[C,u,H]}(x) \le t\}$
$u \notin 0^+C, -u \in 0^+C$	$C = \{x + tu : \ x \in H, \ t \le -f_{[C,-u,H]}(x)\}$
$u \in 0^+C, -u \in 0^+C$	$C = \{x + tu : \ x \in H, \ f_{[C,u,H]}(x) \le t \le -f_{[C,-u,H]}(x)\}$

Proposition 3.2. *Let C be a convex subset of a normed vector space X. Then a) for all $v^* \in X^*$*

$$f^*_{[C,u]}(v^*) = \begin{cases} \sigma_C(v^*) & \text{if } \langle u, v^*\rangle = -1 \\ +\infty & \text{if } \langle u, v^*\rangle \ne -1 \end{cases}.$$

Moreover, assuming $(\mathcal{A})$, one has
b) for all $w^* \in H^* = \{w^* \in X^* : \langle u, w^*\rangle = 0\}$:

$$f^*_{[C,u,H]}(w^*) = \sigma_C(w^* + u^*).$$

Proof. Let $x^* \in X^*$ be such that $\langle u, x^*\rangle = 1$. As

$$f_{[C,u]}(x) = (i_C \underset{e}{+} (i_{\mathbb{R}u} - \langle ., x^*\rangle))(x),$$

we derive that

$$\begin{aligned} f^*_{[C,u]}(v^*) &= \sigma_C(v^*) + \sigma_{\mathbb{R}u}(v^* + x^*) \\ &= \sigma_C(v^*) + i_{\{x^*:\langle u,x^*\rangle=0\}}(v^* + x^*) \\ &= \sigma_C(v^*) + i_{\{x^*:\langle u,x^*\rangle=-1\}}(v^*), \end{aligned}$$

which is the asserted expression.

Let us now compute $f^*_{[C,u,H]}$. Let $x_0 \in H \cap \operatorname{dom} f_{[C,u]}$. For all $x \in X$ one has

$$x = x_0 + tu + \pi(x - x_0)$$

with $t = -\langle x - x_0, u^* \rangle$, which leads to

$$B_X \subset [f_{[C,u]} \leq \lambda] \cap rB_X - [i_H \leq \lambda] \cap rB_X,$$

where $r \geq \max(1 + \|x_0\|, \|\pi\|(1 + \|x_0\|))$, and $\lambda \geq 0$ is such that $\lambda \geq f(x_0) + r$. Thus we can apply Theorem 2.1 yielding, for all $w^* \in H^*$

$$\begin{aligned} f^*_{[C,u,H]}(w^*) &= (f_{[C,u]} + i_H)^*(w^*) \\ &= \inf_{v^* \in X^*} (f^*_{[C,u]}(w^* - v^*) + \sigma_H(v^*)) \\ &= \inf_{t \in \mathbb{R}} f^*_{[C,u]}(w^* - tu^*). \end{aligned}$$

As $w^* - tu^* \in H^*$ (only) for $t = -1$, we obtain $f^*_{[C,u,H]}(w^*) = \sigma_C(w^* + u^*)$ which concludes the proof of the proposition. □

The following result shows the main connections between supporting points of the convex set C and a minimum problem associated to the separating function.

Theorem 3.3. *Let $C \subset X$ be a closed convex subset.*

a) Let $x \in C$ be a support point, let $\widehat{H}$ be the corresponding supporting hyperplane:

$$\widehat{H} = \{z \in X : \langle z, w^* \rangle = \langle x, w^* \rangle\},$$

for some $w^* \in X^*$. Finally, let H be the hyperplane parallel to $\widehat{H}$ through the origin. Then, for each $u \in S_X$ such that $\langle u, w^* \rangle < 0$, one has

$$\inf_{a \in H} f_{[C,u,H]}(a) = f_{[C,u,H]}(\pi(x)).$$

Moreover, if x is exposed, the problem $\inf_H f_{[C,u,H]}$ has unique solution $\pi(x)$, and if x is strongly exposed, the problem $\inf_H f_{[C,u,H]}$ is well posed.

b) Conversely, let $w^* \in X^* \setminus \{0\}$ and let $\widehat{H}$ be a closed affine hyperplane through $x \in C$ parallel to the linear hyperplane $H \ni 0$ of the form

$$\widehat{H} = \{z \in X : \langle z, w^* \rangle = \langle x, w^* \rangle\}.$$

If there is $u \in S_X$ such that $\langle u, w^* \rangle < 0$ and $\inf_{a \in H} f_{[C,u,H]}(a) = f_{[C,u,H]}(\pi(x))$, then x is a support point for C, with supporting functional w^*. Moreover, if the problem $\inf_H f_{[C,u,H]}$ is well posed, then x is strongly exposed.

Proof. a) Assume that $\langle ., w^*\rangle$ attains its maximum over C at x. One has $f_{[C,u]}(x) = 0$, so that $x = \pi(x) + f_{[C,u,H]}(\pi(x))u$ and $f_{[C,u,H]}(\pi(x)) = \dfrac{\langle x, w^*\rangle}{\langle u, w^*\rangle}$. Now let $a \in H$. For each $t \in \mathbb{R}$ such that $a + tu \in C$, one has $\langle a + tu, w^*\rangle \leq \langle x, w^*\rangle$, implying $t\langle u, w^*\rangle \leq \langle x, w^*\rangle$, so that $t \geq \dfrac{\langle x, w^*\rangle}{\langle u, w^*\rangle}$, hence $f_{[C,u,H]}(a) \geq f_{[C,u,H]}(\pi(x))$ and then

$$\inf_{a\in H} f_{[C,u,H]}(a) = f_{[C,u,H]}(\pi(x)).$$

Assume now that $\widehat{H}$ strongly exposes $x \in C \cap \widehat{H}$. Let (a_n) be a sequence in H such that $(f_{[C,u,H]}(a_n))$ converges to $f_{[C,u,H]}(\pi(x))$. Setting $x_n = a_n + f_{[C,u,H]}(a_n)u$ we get $x_n \in C$ and

$$\langle x_n, w^*\rangle = f_{[C,u,H]}(a_n)\langle u, w^*\rangle,$$

hence the sequence $(\langle x_n, w^*\rangle)$ converges to $f_{[C,u,H]}(\pi(x))\langle x, w^*\rangle = \langle x, w^*\rangle$. Thus (x_n) converges to x so that (a_n) converges to $\pi(x)$.

b) Assume that $\inf_H f_{[C,u,H]} = f_{[C,u,H]}(\pi(x))$, for some $x \in C \cap \widehat{H}$ and $u \in X$ such that $\langle u, w^*\rangle < 0$. Let $z \in C$, then $z = \pi(z) + tu$ where

$$t \geq f_{[C,u,H]}(\pi(z)) \geq f_{[C,u,H]}(\pi(x)),$$

yielding

$$\langle z, w^*\rangle = t\langle u, w^*\rangle \leq f_{[C,u,H]}(\pi(x))\langle u, w^*\rangle = \langle x, w^*\rangle.$$

Thus w^* supports C at x. Now assume that $f_{[C,u,H]}$ is well posed and let (x_n) be a sequence in C such that $(\langle x_n, w^*\rangle)$ converges to $\langle x, w^*\rangle$. Setting $a_n = \pi(x_n)$ we get $x_n = a_n + \lambda_n u$ where $\lambda_n = \dfrac{\langle x_n, w^*\rangle}{\langle u, w^*\rangle}$ converges to $f_{[C,u,H]}(\pi(x))$. As

$$f_{[C,u,H]}(a_n) - \lambda_n = f_{[C,u]}(x_n) \leq 0,$$

we get $f_{[C,u,H]}(\pi(x)) \leq f_{[C,u,H]}(a_n) \leq \lambda_n$ and thus $(f_{[C,u,H]}(a_n)) \to f_{[C,u,H]}(\pi(x))$. By well posedness of $f_{[C,u,H]}$, then (a_n) converges to $\pi(x)$, thus (x_n) converges to x. We have shown that $\widehat{H}$ strongly exposes x. □

We now show how the ε-subdifferential of the function $f_{[C,u,H]}$ characterizes the set of ε-support points and support functionals of C [2].

Proposition 3.4. *Let C be a convex subset of a normed vector space X, assume $(\mathcal{A})$ and let $\varepsilon \geq 0$. Then, for all $a \in H \cap \mathrm{dom} f_{[C,u,H]}$, the translation t_{u^*} is a one to one mapping from $\partial_\varepsilon f_{[C,u,H]}(a)$ onto $(\varepsilon\text{-}\mathrm{Supp}_u C)(x) = \{u^* \in X^* : (x, u^*) \in \varepsilon\text{-}\mathrm{Supp}_u C\}$ with $x = a + f_{[C,u,H]}(a)u$. In other words,*

$$(a, a^*) \in \partial_\varepsilon f_{[C,u,H]} \Longleftrightarrow (a + f_{[C,u,H]}(a)u, u^* + a^*) \in \varepsilon\text{-}\mathrm{Supp}_u C. \tag{6}$$

[2] Remembering Remark 3.1, we are here identifying $a^* \in H^*$ with the element (still denoted by a^*) of X^* acting as a^* on H and such that $\langle u, a^*\rangle = 0$.

Moreover

$$(x, w^*) \in \mathrm{Supp}_u C \iff (\pi(x), w^* - u^*) \in \partial f_{[C,u,H]}. \tag{7}$$

P r o o f. Let $a^* \in \partial_\varepsilon f_{[C,u,H]}(a)$, $v^* = t_{u^*}(a^*) = u^* + a^*$ and $x = a + f_{[C,u,H]}(a)$. We have:

$$f^*_{[C,u,H]}(a^*) + f_{[C,u,H]}(a) \leq \langle a, a^* \rangle + \varepsilon,$$

and $\langle x, v^* \rangle = \langle a, a^* \rangle - f_{[C,u,H]}(a)$. Relying on Proposition 3.2, we derive

$$\sigma_C(v^*) - \varepsilon \leq \langle x, v^* \rangle,$$

and, as $\langle u, v^* \rangle = -1$, $v^* \in (\varepsilon\text{-}\mathrm{Supp}_u C)(x)$. Conversely let $v^* \in (\varepsilon\text{-}\mathrm{Supp}_u C)(x)$ and let $a^* = v^* - u^*$ [3]. One has:

$$\sigma_C(a^* + u^*) - \varepsilon \leq \langle x, a^* + u^* \rangle,$$

and $H \ni a = x + \langle x, u^* \rangle u$ which give

$$f_{[C,u,H]}(a) = f_{[C,u,H]}(x + \langle x, u^* \rangle u) = f_{[C,u]}(x) - \langle x, u^* \rangle = -\langle x, u^* \rangle.$$

Thus we get

$$f^*_{[C,u,H]}(a^*) + f_{[C,u,H]}(a) \leq \langle a, a^* \rangle + \varepsilon,$$

hence $a^* \in \partial_\varepsilon f_{[C,u,H]}(a)$.

Now observing that $(x, w^*) \in \mathrm{Supp}_u C$ implies $f_{[C,u]}(x) = 0$ and thus $x = \pi(x) + f_{[C,u,H]}(\pi(x))u$, we immediately derive (7) from (6). □

With the help of the Ekeland variational principle, Proposition 3.4 provides a simple proof of the Bishop-Phelps Theorem (see [16]) on density of support points and support functionals. Let us recall that given a Banach space X and $f \in \Gamma_0(X)$, it is an immediate consequence of the Ekeland variational principle applied to $f - \langle ., w^* \rangle$ that given $\varepsilon > 0$, $z \in X$ and $w^* \in \partial_\varepsilon f(z)$ there exists $x \in X$ and $v^* \in \partial f(x)$ with

$$\begin{aligned} &\|x - z\| \leq \sqrt{\varepsilon}, \\ &\|v^* - w^*\| \leq \sqrt{\varepsilon}, \\ &|f(z) - f(x)| \leq \sqrt{\varepsilon}(\|w^*\|_* + \sqrt{\varepsilon}). \end{aligned} \tag{8}$$

Theorem 3.5. *Let $C \neq X$ be a closed nonempty convex subset of a Banach space X. Then the set of support points is dense in ∂C and the set of support functionals is dense in $\mathrm{dom}\,\sigma_C$.*

[3] As $\langle u, a^* \rangle = 0$, we can consider a^* as an element of H^*.

Proof. Let $z \in \partial C$. One has $\mathbb{R}_+(z - C) \neq X$ so that there exists $u \in S_X$ with $u \notin \mathbb{R}_+(z - C)$ thus $-u \notin 0^+C$. Let H be the linear hyperplane associated to some $u^* \in \operatorname{dom} \sigma_C$ with $\langle u, u^* \rangle = -1$. Set $b = \pi(z)$. As $u \notin \mathbb{R}_+(z - C)$, we get $f_{[C,u]}(z) = 0$ thus $z = b + f_{[C,u,H]}(b)u$. Let $\varepsilon > 0$ and let $w^* \in \partial_\varepsilon f_{[C,u,H]}(b)$. Returning to (8), we get the existence of $a \in H$, $a^* \in \partial f_{[C,u,H]}(a)$ with $\|a - b\| \leq \sqrt{\varepsilon}$, $\|a^* - w^*\| \leq \sqrt{\varepsilon}$, and $|f_{[C,u,H]}(a) - f_{[C,u,H]}(b)| \leq \sqrt{\varepsilon}(\sqrt{\varepsilon} + \|w^*\|\sqrt{\varepsilon})$. From Proposition 3.4 we obtain

$$(x, v^*) \in \operatorname{Supp}_u C,$$

with $x = a + f_{[C,u,H]}(a)u$, $v^* = a^* + u^*$ and $\|z - x\| \leq \sqrt{\varepsilon}(2\sqrt{\varepsilon} + \|w^*\|\sqrt{\varepsilon})$.

Now let $w^* \in \operatorname{dom} \sigma_C$. We can choose $u \in S_X$ such that $\langle u, w^* \rangle < 0$ and $k > 0$ such that $\langle u, kw^* \rangle = -1$. Let $H = [\langle u, w^* \rangle = 0]$ and let $z \in C$ be such that $\sigma_C(w^*) - \varepsilon \leq \langle z, w^* \rangle$. Setting $b = \pi(z)$, we have $w^* - kw^* \in \partial_\varepsilon f_{[C,u,H]}(b)$, thus there exist $a \in H$ and $v^* \in \partial f_{[C,u,H]}(a)$ with $\|w^* - kw^* - v^*\| \leq \varepsilon$. From Proposition 3.4 we derive that $v^* + kw^*$ is a support functional for the set C. □

4. Stability of supported and exposed points. This section is dedicated to stability of the supported and exposed elements of a closed convex set C, under perturbations in the sense of the slice and Attouch-Wets topologies. As we want to study this with the help of the separating functions, and we learned that their subdifferentials provide information on supporting points, we start with some auxiliary results on the approximation of the points of single valuedness of the subdifferential of a convex function. Then we connect convergence of the sets to convergence of the associated separating functions and finally we provide the main stability results. Our first result is a technical lemma, the convex version of Proposition 7.1.3 in [11].

Lemma 4.1. *Let $\{f, f_n : n \in \mathbb{N}\} \subset \Gamma_0(X)$ be such that $f = \tau_{AW}\text{-}\lim_{n\to\infty} f_n$ and let $x_0 \in \operatorname{int}(\operatorname{dom} f)$. Then for all $\varepsilon > 0$ there exist $\delta > 0$ and $N \in \mathbb{N}$ such that*

$$\sup_{x \in x_0 + \delta B_X} |f_n(x) - f(x_0)| < \varepsilon$$

for all $n \geq N$.

Proof. Let $y \in \partial f(x_0)$ be arbitrary. There exists $\delta_1 > 0$ such that for all $x \in x_0 + \delta_1 U$ we have

$$\langle x - x_0, y \rangle + f(x_0) - \frac{\varepsilon}{2} > f(x_0) - \varepsilon.$$

Since the graph of $x \longmapsto \langle x - x_0, y \rangle + f(x_0) - \dfrac{\varepsilon}{2}$ lies below $\operatorname{epi} f$, by slice convergence the set

$$\left\{(x, \alpha) \in X \times \mathbb{R} : x \in x_0 + \delta_1 B_X,\ \alpha = \langle x - x_0, y \rangle + f(x_0) - \frac{\varepsilon}{2}\right\}$$

lies below $\operatorname{epi} f_n$ for all $n \in \mathbb{N}$ sufficiently large. This means that

$$\sup_{x \in x_0 + \delta B_X} f(x_0) - f_n(x) < \varepsilon.$$

By upper semicontinuity of f at x_0 there exists $0 < \delta_2 < \dfrac{\varepsilon}{4}$ such that $f(x) < f(x_0) + \varepsilon$ whenever $x \in x_0 + \delta_2 B_X$. It follows that

$$(x_0 + \delta_2 B_X) \times \left[f(x_0) + \frac{\varepsilon}{2}, f(x_0) + \varepsilon \right] \subset \operatorname{epi} f.$$

This means that the ball with center $\left(x_0, f(x_0) + \dfrac{3}{4}\varepsilon \right)$ and radius δ_2 lies in $\operatorname{epi} f$. Take μ so large that $\mu B_X \times [-\mu, \mu]$ contains this ball. Then if $h_\mu(f_n, f) < \dfrac{\delta_2}{2}$ it easily follows from the Rådström cancellation law that the ball with center $\left(x_0, f(x_0) + \dfrac{3}{4}\varepsilon \right)$ and radius δ_2 lies in $\operatorname{epi} f_n$. Thus, for each $x \in x_0 + \dfrac{\delta_2}{2} B_X$, we have

$$f_n(x) < f(x_0) + \frac{3}{4}\varepsilon + \frac{1}{2}\delta_2 < f(x_0) + \varepsilon$$

for n large. $\square$

Corollary 4.2. *Let* $\{f, f_n \; : \; n \in \mathbb{N}\} \subset \Gamma_0(X)$ *be such that* $f = \tau_{AW}\text{-}\lim_{n\to\infty} f_n$ *and let* $x_0 \in \operatorname{int}(\operatorname{dom} f)$*. Then there exist* $\delta > 0$*,* $\rho > 0$ *and* $N \in \mathbb{N}$ *such that for all* $n \geq N$*,* $x \in x_0 + \delta B_X$*,* $y \in \partial f_n(x)$*, we have* $\|y\| \leq \rho$*.*

Proof. Choose by Lemma 4.1 $\delta' > 0$ and $N \in \mathbb{N}$ such that $|f_n(x) - f(x_0)| < \varepsilon$ for all $x \in x_0 + \delta' B_X$, $n \geq N$. In particular, each f_n is uniformly bounded above and below on $x_0 + \delta' B_X$. As a result, $\{f_n : n \geq N\}$ is an equi-Lipschitzian family on $x_0 + \dfrac{3}{4}\delta' B_X$. Taking a uniform Lipschitz constant ρ for the family restricted to $x_0 + \dfrac{3}{4}\delta' B_X$ and setting $\delta = \dfrac{1}{2}\delta'$, we claim that

$$\sup\{\|y\| : y \in \partial f_n(x), \, n \geq N, \, x \in x_0 + \delta B_X\} \leq \rho.$$

Otherwise there would exist $x \in x_0 + \delta B_X$, $n \geq N$ and $y \in \partial f_n(x)$ such that $\|y\| > \rho$. Choose a unit vector w with $\langle w, y \rangle > \rho$. Since $\|x - x_0\| \leq \dfrac{1}{2}\delta'$, $x + \dfrac{1}{4}\delta' w \in x_0 + \dfrac{3}{4}\delta' B_X$ and

$$f\left(x + \frac{1}{4}\delta' w \right) - f(x) > \rho \left(x + \frac{1}{4}\delta' w - x \right),$$

contradicting Lipschitz continuity of f_n on the ball $x + \dfrac{3}{4}\delta' B_X$. $\square$

Lemma 4.1 and Corollary 4.2 fail to be true when replacing the τ_{AW} topology by the slice topology. Indeed let X be a separable Hilbert space endowed with an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ and let $f \equiv 0$, $f_n = i_{X_n}$ with $X_n = \operatorname{span}\{e_0, \cdots, e_n\}$. We have $f = \tau_S\text{-}\lim_{n\to\infty} f_n$, $0 \in \operatorname{int}(\operatorname{dom} f)$, but setting $x_n = \dfrac{e_{n+1}}{n}$ we get $(x_n) \longrightarrow 0$ and $f_n(x_n) = +\infty$.

Given $f \in \Gamma_0(X)$, we set

$$\partial_* f = \{(x,y) \in \partial f : \partial f(x) = \{y\}\},$$

$$\Delta f = \{(x, f(x), y) : (x,y) \in \partial f\},$$

and

$$\Delta_* f = \{(x, f(x), y) : (x,y) \in \partial_* f\}.$$

Observe that given $x \in C$ with $a = \pi(x)$ we derive from Proposition 3.4 that

$$v^* \in (\operatorname{Exp}_u C)(x) \text{ if and only if } \partial f^*_{[C,u,H]}(v^* - u^*) = \{a\}.$$

A normed space X is said to be an Asplund space if every continuous convex function defined on a nonempty open convex subset of X is Fréchet differentiable on a dense G_δ subset of its domain. A result due to P. Kenderov (see for example [25, Theorem 3.20]) asserts that given a maximal monotone operator $T \subset X \times X^*$ defined on an Asplund space and satisfying $\operatorname{int}(\operatorname{dom} T) \neq \emptyset$, there exists a dense G_δ subset of $\operatorname{int}(\operatorname{dom} T)$ on which T is single valued and norm-norm upper semicontinuous.

Theorem 4.3. *Let X be Asplund, let $\{f, f_n,: n \in \mathbb{N}\} \subset \Gamma_0(X)$ be such that $f = \tau_{AW}\text{-}\lim_{n\to\infty} f_n$ and let $\operatorname{int}(\operatorname{dom} f)$ be nonempty. Then there exists a subset E of $\operatorname{int}(\operatorname{dom} \partial_* f)$ that is dense and G_δ subset of $\operatorname{dom} f$ such that for all $x \in E$, $(x, f(x), y) \in \operatorname{Li}_{n\to\infty} \Delta_* f_n$, where y is the unique subgradient of f at x.*

Proof. By Kenderov's Theorem on maximal monotone operators, the points x of $\operatorname{int}(\operatorname{dom} f)$ where ∂f is single-valued and norm-norm upper semicontinuous contain a dense G_δ subset E of $\operatorname{int}(\operatorname{dom} f)$. Fix $x_0 \in E$ and let y_0 be the unique subgradient of f at x_0. Let $\varepsilon > 0$ be such that $x_0 + \varepsilon B_X \subset \operatorname{dom} f$. There exists $0 < \delta_0 < \dfrac{\varepsilon}{2}$ such that $\|y - y_0\| < \dfrac{\varepsilon}{2}$ for all $x \in x_0 + \delta_0 B_X$, $y \in \partial f(x)$. Choose, by Lemma 4.1 and Corollary 4.2, $0 < \delta_1 < \dfrac{\varepsilon}{2}$, $\rho > 0$ and $N_1 \in \mathbb{N}$ such that both

$$\sup_{x \in x_0 + \delta_1 B_X} |f_n(x) - f(x_0)| < \varepsilon \tag{9}$$

and

$$\sup_{x \in x_0 + \delta_1 B_X} \sup_{y \in \partial f_n(x)} \|y\| \leq \rho \tag{10}$$

hold. By (9), for all $n \geq N_1$ we have $x_0 + \delta_1 B_X \subset \operatorname{dom} f_n$ and since X is Asplund there exists $x_n \in x_0 + \delta_1 B_X$ such that $\partial f_n(x_n)$ is a singleton. Fix $\mu > \max(\|x_0\| + \delta_1, \rho)$. From a known result on convergence of functions and related convergence of subdifferentials (see for example [24], Proposition 1.5), there exists an index $N \geq N_1$ such that for all $n \geq N$, $h_\mu(\partial f, \partial f_n) < \delta_1$. Now let $n \geq N$ be fixed and let y_n be the unique element of $\partial f_n(x_n)$. Since $(x_n, y_n) \in \mu B_{X \times X^*} \cap \partial f_n$, there exists $(w, y) \in \partial f$ such that both $\|x_n - w\| < \delta_1$ and $\|y_n - y\| < \delta_1$ hold. Since $w \in x_0 + \delta_0 B_X$, we have $\|y - y_0\| < \dfrac{\varepsilon}{2}$ and so $\|y_n - y_0\| < \dfrac{\varepsilon}{2} + \delta_1 < \varepsilon$. By (9), $|f_n(x_n) - f(x_0)| < \varepsilon$ and since $\delta_1 < \varepsilon$, we also have $\|x_n - x_0\| < \varepsilon$. Thus for all $n \geq N$, we have found $(x_n, f(x_n), y_n) \in \Delta_* f_n$ such that

$$\|(x_0, f(x_0), y_0) - (x_n, f(x_n), y_n)\| < \varepsilon.$$

This proves that $(x_0, f(x_0), y_0) \in \operatorname*{Li}_{n\to\infty} \Delta_* f_n$ as required. $\square$

We now turn to the problem of relating convergence of sets to convergence of the associated separating functions.

Theorem 4.4. *Let X be a normed vector space and let $\{C, C_n : n \in \mathbb{N}\} \subset \mathcal{C}(X)$. Let u, H be as in assumption $(\mathcal{A})$. If $f_{[C,u,H]} = \tau_S\text{-}\lim_{n\to\infty} f_{[C_n,u,H]}$, then*

$$C_u^+ = \tau_S\text{-}\lim_{n\to\infty} (C_n)_u^+.$$

Conversely, if $(\mathcal{A})$ holds for C and $C = \tau_S\text{-}\lim_{n\to\infty} C_n$, then

$$f_{[C,u,H]} = \tau_S\text{-}\lim_{n\to\infty} f_{[C_n,u,H]}.$$

P r o o f. Let us start with the lower part of convergence. Let $c \in C_u^+$. Then $c = \pi(c) + \lambda u$, with $(\pi(c), \lambda) \in \operatorname{epi} f_{[C,u,H]}$. Then there exists a sequence $(x_n, \lambda_n) \subset \operatorname{epi} f_{[C_n,u,H]}$ such that $(x_n) \to \pi(c)$ and $(\lambda_n) \to \lambda$. Then $x_n + \lambda_n u \in (C_n)_u^+$ and the sequence $(x_n + \lambda_n u)$ converges to c. Now suppose $D(B, C_u^+) > 0$ for some convex bounded set B. Call

$$\widehat{B} = \{(x, \lambda) : x \in H,\ x + \lambda u \in B\}.$$

It is easy to verify that $\widehat{B}$ is closed bounded in $X \times \mathbb{R}$ and that $D(\widehat{B}, \operatorname{epi} f_{[C,u,H]}) > 0$. Then $D(\widehat{B}, \operatorname{epi} f_{[C_n,u,H]}) > 0$ eventually. Suppose now, for the sake of contradiction, $(D(B, (C_n)_u^+)) \to 0$. Then there exist $b_n \in B$, $c_n \in (C_n)_u^+$ such that $(|b_n - c_n|) \to 0$. Writing $b_n = \pi(b_n) + \lambda_n u$, $c_n = \pi(c_n) + r_n u$, we have $(\pi(b_n), \lambda_n) \in \widehat{B}$, $(\pi(c_n), r_n) \in \operatorname{epi} f_{[C_n,u,H]}$ and $d[(\pi(b_n), \lambda_n), (\pi(c_n), r_n)] \to 0$, but this is impossible. So $(C_n)_u^+ \in (B^c)^{++}$ and this shows the first part of the theorem. To prove the second part, let $x \in \operatorname{dom} f_{[C,u,H]} \cap H$. As $z = x + f_{[C,u,H]}(x)u \in C$, there exists a sequence (z_n)

such that $z_n \in C_n$ and $\lim_{n\to\infty} z_n = z$. Let us write $z_n = x_n + \lambda_n u$ with $x_n \in H$; one has $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} \lambda_n = f_{[C,u,H]}(x)$ and $f_{[C_n,u,H]}(x_n) \le \lambda_n$ and thus $\limsup_{n\to\infty} f_{[C_n,u,H]}(x_n) \le f_{[C,u,H]}(x)$. Now let

$$v^* \in \operatorname{dom} f^*_{[C,u,H]} \subset \operatorname{dom} \sigma_C - u^*.$$

There exists a sequence (w_n^*) converging in X^* to $u^* + v^*$ in such a way that

$$\limsup_{n\to\infty} \sigma_C(w_n^*) \le \sigma_C(u^* + v^*).$$

Setting $\lambda_n = -(\langle u, w_n^*\rangle)^{-1}$, we have $\lim_{n\to\infty} \lambda_n = 1$. Setting $\tilde{w}_n^* = \lambda_n w_n^*$ we have $\tilde{w}_n^* = u^* + v_n^*$ with $\langle u, v_n^*\rangle = 0$ and (v_n^*) converging to v^*. Moreover

$$\limsup_{n\to\infty} f^*_{[C_n,u,H]}(v_n^*) = \limsup_{n\to\infty} \sigma_{C_n}(\lambda_n w_n^*) \le f^*_{[C,u,H]}(v^*),$$

yielding the announced result. $\square$

Corollary 4.5. *Let X be a normed vector space and let $\{C, C_n : n \in \mathbb{N}\} \subset \mathcal{C}(X)$. Let u, H be as in assumption $(\mathcal{A})$. If $f_{[C,u,H]} = \tau_S\text{-}\lim_{n\to\infty} f_{[C_n,u,H]}$ and $f_{[C,-u,H]} = \tau_S\text{-}\lim_{n\to\infty} f_{[C_n,-u,H]}$, then*

$$C = \tau_S\text{-}\lim_{n\to\infty} C_n.$$

Proof. Observe that $C = C_u^+ \cap C_{-u}^+$. Then, from Theorem 4.4 we have convergence of $(C_n)_u^+$ to C_u^+ and of $(C_n)_{-u}^+$ to C_{-u}^+. We must show that this implies convergence of the sequence (C_n) to C. Let $c \in C$. There are sequences $(c_n^+) \subset (C_n)_u^+$ and $(c_n^-) \subset (C_n)_{-u}^+$ converging to c. Write $c_n^+ = x_n + t_n u$, $c_n^- = y_n - s_n u$, with $x_n, y_n \in C$ and $t_n, s_n > 0$. Then $C_n \ni \dfrac{s_n}{s_n + t_n} x_n + \dfrac{t_n}{s_n + t_n} y_n \to c$. This proves lower convergence. Upper convergence follows from the formula $D(A, C) = \max(D(A, C_u^+), D(A, C_{-u}^+))$, which is easy to prove and is left to the reader. $\square$

Corollary 4.6. *Let X be a normed vector space and let $\{C, C_n : n \in \mathbb{N}\} \subset \mathcal{C}(X)$. Suppose moreover $C_n = (C_n)_u^+$, $C = C_u^+$ and assume $(\mathcal{A})$ for C. Then $C = \tau_S\text{-}\lim_{n\to\infty} C_n$ if and only if $f_{[C,u,H]} = \tau_S\text{-}\lim_{n\to\infty} f_{[C_n,u,H]}$.*

The assumption $-u \notin 0^+C$ plays an essential role in the former results. For, if $-u \in 0^+C$ we already remarked that it can happen that $f_{[C,u,H]}$ is not lower semicontinuous. This means that, even when $C_n = C$ for all $n \in \mathbb{N}$, we do not have in such a case $f_{[C,u,H]} = \tau_S\text{-}\lim_{n\to\infty} f_{[C_n,u,H]}$. Moreover, if $-u \in 0^+C$, even the implication

$$C = \tau_S\text{-}\lim_{n\to\infty} C_n \Longrightarrow C_u^+ = \tau_S\text{-}\lim_{n\to\infty} (C_n)_u^+$$

is no longer true, as the following example shows: $X = \mathbb{R}^2$, $C_n = \{(x,y) : y \leq nx\}$, $u = (0,1)$.

If we replace the slice topology by the τ_{AW} topology we are able to provide quantitative estimates. The first result is the following.

Proposition 4.7. *Let u, H be as in assumption $(\mathcal{A})$. There exists $p > 0$ such that, for all $r \geq 0$ and for all $C, D \in \mathcal{C}(X)$ the following estimate holds:*

$$h_r(C_u^+, D_u^+) \leq 2h_{rp}(f_{[C,u,H]}, f_{[D,u,H]}).$$

Proof. Take $C \in \mathcal{C}(X)$ and $x \in C_u^+ \cap rB_X$. Write $x = \pi(x) - \langle x, u^* \rangle u$. We have

$$f_{[C,u,H]}(\pi(x)) = f_{[C,u]}(x) - \langle x, u^* \rangle \leq -\langle x, u^* \rangle \leq r\|u^*\|,$$

and $\|\pi(x)\| \leq r(1 + \|u^*\|) =: p$. Thus $(\pi(x), -\langle x, u^* \rangle) \in \operatorname{epi} f_{[C,u,H]} \cap pB_{X\times\mathbb{R}}$. Then for each $\varepsilon > 0$ there is $(z,a) \in \operatorname{epi} f_{[D,u,H]}$ such that $\|(\pi(x), -\langle x, u^* \rangle) - (z,a)\|_{X\times\mathbb{R}} \leq e_{rp}(f_{[C,u,H]}, f_{[D,u,H]}) + \varepsilon$. Consider $w := z + au \in D_u^+$. To conclude, observe that $\|w - x\| \leq 2e_{rp}(f_{[C,u,H]}, f_{[D,u,H]}) + \varepsilon$ and interchange the roles of C and D. □

To obtain an inequality in the opposite sense, we need the following lemma.

Lemma 4.8. *Let $C_0 \in \mathcal{C}(X)$ be such that $(\mathcal{A})$ holds true. Then there exist a τ_{AW}-neighborhood $\mathcal{N}$ of C_0 and $p > 0$ such that for all $C \in \mathcal{C}(X)$,*

$$f_{[C,u]}(x) \geq -p(\|x\| + 1) \text{ for all } x \in X.$$

Proof. Let $v_0^* \in \operatorname{dom} f^*_{[C_0,u]}$ in such a way that $\sigma_{C_0}(v_0^*) = -1$ (see Proposition 3.2). Let us define

$$\mathcal{N} = \left\{ C \in \mathcal{C} : d((v_0^*, \sigma_{C_0}(v_0^*)), \operatorname{epi} \sigma_C) < \frac{1}{2} \right\}.$$

The set $\mathcal{N} \ni C_0$ is open due to the continuity of polarity with respect to the Attouch-Wets topology. For all $C \in \mathcal{N}$, there exists $(v^*, \lambda) \in \operatorname{epi} \sigma_C$ such that $\|v^* - v_0^*\| < \dfrac{1}{2}$ and $|\lambda - \sigma_{C_0}(v_0^*)| < \dfrac{1}{2}$. Let us set $t = -(\langle u, v^* \rangle)^{-1} \in (0,2)$. One has $(tv^*, t\lambda) \in \operatorname{epi} f^*_{[C,u]}$ and, for all $x \in X$,

$$f_{[C,u]}(x) \geq \langle x, tv^* \rangle - t\lambda \geq -p(\|x\| + 1),$$

where $p = \max(2\|v_0^*\| + 1, 2|\sigma_{C_0}(v_0^*)| + 1)$. This ends the proof. □

Proposition 4.9. *Let $C_0 \in \mathcal{C}(X)$ be such that $(\mathcal{A})$ holds true. Then there exist $p > 0$, $q > 0$ and a τ_{AW}-neighborhood $\mathcal{N}$ of C_0, such that for each C, $D \in \mathcal{N}$ and for each $r \geq 0$*

$$h_r(f_{[C,u,H]}, f_{[D,u,H]}) \leq q h_{p(r+1)+r}(C, D).$$

Proof. From Lemma 4.8, there exist a $\tau_{AW}-$neighborhood $\mathcal{N}$ of C_0 and $p > 1$ such that for all $C \in \mathcal{N}$,

$$f_{[C,u]}(x) \geq -p(\|x\| + 1) \quad \text{for all} \quad x \in X.$$

Let C, $D \in \mathcal{N}$ and let $(x, t) \in \operatorname{epi} f_{[C,u,H]} \in r B_{H \times \mathbb{R}}$. One has

$$-p(1 + r) \leq f_{[C,u,H]}(x) \leq t \leq r,$$

thus $x + f_{[C,u,H]}(x) u \in C \cap (p(1 + r) + r) B_X$. For $\varepsilon > 0$, there exists $y \in D$ with

$$\|x + f_{[C,u,H]}(x) u - y\| \leq e_{p(1+r)+r}(C, D) + \varepsilon.$$

One has $y = z + su$ with $z = \pi(y) \in H$, thus $(z, s) \in \operatorname{epi} f_{[D,u,H]}$. We get

$$|f_{[C,u,H]}(x) - s| = |\langle x + f_{[C,u,H]}(x) u - z - su, u^* \rangle| \leq \|u^*\| (e_{p(1+r)+r}(C, D) + \varepsilon),$$

and

$$\|x - z\| = \|\pi(x + f_{[C,u,H]}(x) u - y)\| \leq \|\pi\| (e_{p(1+r)+r}(C, D) + \varepsilon).$$

We derive that

$$\begin{aligned} d((x, t), \operatorname{epi} f_{[D,u,H]}) &\leq \|(x, t) - (z, s + t - f_{[C,u,H]}(x))\| \\ &\leq \max(\|\pi\|, \|u^*\|)(e_{p(1+r)+r}(C, D) + \varepsilon), \end{aligned}$$

hence letting ε go to 0

$$e_r(f_{[C,u,H]}, f_{[D,u,H]}) \leq \max(\|\pi\|, \|u^*\|) e_{p(1+r)+r}(C, D).$$

Interchanging C and D, the result follows. $\square$

From Proposition 4.7 and 4.9, we get immediately the following

Theorem 4.10. *Let X be a normed vector space and let*

$$\{C, C_n : n \in \mathbb{N}\} \subset \mathcal{C}(X).$$

Let u, H be as in assumption $(\mathcal{A})$. If $f_{[C,u,H]} = \tau_{AW}\text{-}\lim_{n\to\infty} f_{[C_n,u,H]}$, then

$$C_u^+ = \tau_{AW}\text{-}\lim_{n\to\infty} (C_n)_u^+.$$

Conversely, if $(\mathcal{A})$ *holds for* C *and* $C = \tau_{AW}\text{-}\lim_{n\to\infty} C_n$*, then*

$$f_{[C,u,H]} = \tau_{AW}\text{-}\lim_{n\to\infty} f_{[C_n,u,H]}.$$

Corollary 4.11. *Let* X *be a normed vector space and let* $\{C, C_n : n \in \mathbb{N}\} \subset \mathcal{C}(X)$*. Let* u*,* H *be as in assumption* $(\mathcal{A})$*. If* $f_{[C,u,H]} = \tau_{AW}\text{-}\lim_{n\to\infty} f_{[C_n,u,H]}$ *and* $f_{[C,-u,H]} = \tau_{AW}\text{-}\lim_{n\to\infty} f_{[C_n,-u,H]}$*, then*

$$C = \tau_{AW}\text{-}\lim_{n\to\infty} C_n.$$

Proof. As in Corollary 4.5 we must show that convergence of $(C_n)_u^+$ to C_u^+ and of $(C_n)_{-u}^+$ to C_{-u}^+ implies convergence of the sequence (C_n) to C. This follows from the formula $e_\rho(D, C) \le \max\left(e_\rho(D_u^+, C_u^+), e_\rho(D_{-u}^+, C_{-u}^+)\right)$, which is left to the reader. □

Corollary 4.12. *Let* X *be a normed vector space and let* $\{C, C_n : n \in \mathbb{N}\} \subset \mathcal{C}(X)$*. Suppose moreover* $C_n = (C_n)_u^+$*,* $C = C_u^+$ *and assume* $(\mathcal{A})$ *for* C*. Then* $C = \tau_{AW}\text{-}\lim_{n\to\infty} C_n$ *if and only if* $f_{[C,u,H]} = \tau_{AW}\text{-}\lim_{n\to\infty} f_{[C_n,u,H]}$*.*

We finally are able to prove the promised results on stability of the supporting elements under perturbations. We start with the slice topology, and we shall focus our attention to *lower* convergence of the support points, but before let us briefly explain what happens with upper convergence.

The Bishop-Phelps Theorem shows that convergence of a sequence of support points, in a *fixed* given closed convex set C, does not guarantee that the limit point is a support point for C. On the other hand, if we have convergence of pairs $(x_n, u_n^*) \in \operatorname{Supp} C_n$ to a limit pair (x, u^*), then (x, u^*) is a support point for the set C, if the sequence (C_n) converges to C in the sense of Kuratowski, as a standard direct computation shows. So, let us focus on lower convergence of support points.

Theorem 4.13. *Let* X *be a Banach space and let* $\{C, C_n : n \in \mathbb{N}\} \subset \mathcal{C}(X)$ *be such that* $C = \tau_S\text{-}\lim_{n\to\infty} C_n$*. Then assuming* $(\mathcal{A})$ *for* C*, one has*

$$\operatorname{Supp}_u C \subset \operatorname*{Li}_{n\to\infty}(\operatorname{Supp}_u C_n).$$

Proof. Let $(x, v^*) \in \operatorname{Supp}_u C$ and let H be any closed hyperplane such that $(\mathcal{A})$ is satisfied. As $\langle u, v^* \rangle = -1$, we derive that $f_{[C,u]}(x) = 0$, thus $x = a + f_{[C,u,H]}(a)u$, where $a = \pi(x)$. Using Proposition 4.4 one gets

$$f_{[C,u,H]} = \tau_S\text{-}\lim_{n\to\infty} f_{[C_n,u,H]}.$$

From Proposition 3.4 we have $(a, v^* - u^*) \in \partial f_{[C,u,H]}$. Using Theorem 8.3.7 of [11], we get the existence of a sequence $((a_n, a_n^*)) \subset H \times H^*$ such that $((a_n, f_{[C,u,H]}(a_n), a_n^*))$ converges to $(a, f_{[C,u,H]}(a), v^* - u^*)$. Setting $x_n = a_n + f_{[C,u,H]}(a_n)u$ and $v_n^* = u^* + a_n^*$ we get $(x_n, v_n^*) \in \mathrm{Supp}_u C_n$ and the sequence $((x_n, v_n^*))$ converges to (x, v^*), yielding the announced result. □

Theorem 4.14. *Let X be a Banach space and let $\{C, C_n : n \in \mathbb{N}\} \subset \mathcal{C}(X)$ be such that $C = \tau_{AW}\text{-}\lim_{n\to\infty} C_n$. Then assuming $(\mathcal{A})$ for C, one has for all $r \geq 0$*

$$\lim_{n\to\infty} e_r(\mathrm{Supp}_u C, \mathrm{Supp}_u C_n) = 0.$$

Proof. From Proposition 4.9, we derive that

$$f_{[C,u,H]} = \tau_{AW}\text{-}\lim_{n\to\infty} f_{[C_n,u,H]}.$$

Let $r \geq 0$ and let $(x, w^*) \in \mathrm{Supp}_u C \cap rB_{X\times X^*}$. From Proposition 3.4, we have

$$(\pi(x), f_{[C,u,H]}(\pi(x)), w^* - u^*)) \in \Delta(f_{[C,u,H]}) \cap \rho B_{X\times\mathbb{R}\times X^*},$$

with $\rho = \max(r\|u^*\|, \|\pi\|r, r + \|u^*\|)$. Using a result of [15] (see for example [11], Theorem 8.3.10) one has

$$\lim_{n\to\infty} e_\rho(\Delta(f_{[C,u,H]}), \Delta(f_{[C_n,u,H]})) = 0.$$

It follows the existence of sequences $(a_n) \subset H$ and $(a_n^*) \subset H^*$ such that

$$\|(\pi(x), f_{[C,u,H]}(\pi(x)), w^* - u^*) - (a_n, f_{[C_n,u,H]}(a_n), a_n^*)\| < \varepsilon_\rho(n) + (n+1)^{-1}$$

with $\varepsilon_\rho(n) = e_\rho(\Delta(f_{[C,u,H]}), \Delta(f_{[C_n,u,H]}))$. Let us set

$$(x_n, w_n^*) = (a_n + f_{[C_n,u,H]}(a_n)u, u^* + a_n^*).$$

One has $(x_n, w_n^*) \in \mathrm{Supp}_u C_n$,

$$\|w_n^* - w^*\| = \|a_n^* - a^*\| < \varepsilon_\rho(n) + (n+1)^{-1}$$

and

$$\|x_n - x\| = \|a_n + f_{[C_n,u,H]}u - a - f_{[C,u,H]}u\| \leq 2(\varepsilon_\rho(n) + (n+1)^{-1}),$$

yielding

$$\|(x_n, w_n^*) - (x^*, w^*)\| \leq 2(\varepsilon_\rho(n) + (n+1)^{-1}),$$

thus

$$e_r(\mathrm{Supp}_u C, \mathrm{Supp}_u C_n) \leq 2(\varepsilon_\rho(n) + (n+1)^{-1}),$$

which ends the proof of the theorem. □

In order to obtain stability results involving $\mathrm{Exp}_u C$ instead of $\mathrm{Supp}_u C$, we need an additional assumption of nonflatness at infinity for the convex set C:

$$\text{there exists } v^* \in H^* \text{ such that } f^*_{[C,u,H]} \text{ is continuous at } v^*. \tag{11}$$

Observe that assumption (11) is satisfied whenever C is bounded and that obvious examples can be given in which this assumption is not satisfied and the conclusion of our next theorem is not in force. Given a set $D \subset X \times X^*$ and $w^* \in X^*$ we set

$$D(X) = \{w^* \in X^* : \text{ there exists } x \in X : (x, w^*) \in D\},$$

and

$$D^{-1}(w^*) = \{x \in X : (x, w^*) \in D\}.$$

Theorem 4.15. *Let X be a reflexive Banach space and let $\{C, C_n : n \in \mathbb{N}\} \subset \mathcal{C}(X)$ be such that $C = \tau_{AW}\text{-}\lim_{n\to\infty} C_n$. Assuming* (11) *and* ($\mathcal{A}$)*, then*

$$(\mathrm{Supp}_u C)(X) \subset \operatorname*{Li}_{n\to\infty} (\mathrm{Exp}_u C_n)(X).$$

P r o o f. Let $(z, w^*) \in \mathrm{Supp}_u C$ and let H be any closed hyperplane such that ($\mathcal{A}$) is satisfied, so that $z = \pi(z) + f_{[C,u,H]}(\pi(z))u$. Applying Theorem 4.3 to the function $f^*_{[C,u,H]}$, there exists $E \subset \operatorname{int}(\operatorname{dom} \partial_* f^*_{[C,u,H]})$, which is a G_δ dense subset of $\operatorname{dom} f^*_{[C,u,H]}$, such that for all $v^* \in E$ one has

$$(v^*, f^*_{[C,u,H]}(v^*), b) \in \operatorname*{Li}_{n\to\infty} \Delta_* f^*_{[C_n,u,H]},$$

where $\partial f^*_{[C,u,H]}(v^*) = \{b\}$. Let $((v_n^*, f^*_{[C,u,H]}(v_n^*), b_n))$ be a sequence converging to $(v^*, f^*_{[C,u,H]}(v^*), b)$ and such that $(v_n^*, f^*_{[C,u,H]}(v_n^*), b_n) \in \Delta_* f^*_{[C_n,u,H]}$. As

$$\partial f^*_{[C,u,H]}(v_n^*) = \{b_n\},$$

one gets, from Proposition 3.4,

$$(x_n, w_n^*) \in \mathrm{Exp}_u C_n,$$

where $x_n = b_n + f_{[C_n,u,H]}(b_n)u$ and $w_n^* = v_n^* + u^*$. As

$$f_{[C_n,u,H]}(b_n) = \langle b_n, v_n^* \rangle - f^*_{[C_n,u,H]}(v_n^*),$$

we derive that the sequence $(f_{[C_n,u,H]}(b_n))$ converges to $f_{[C,u,H]}(b)$ and (x_n) converges to $x = b + f_{[C,u,H]}(b)u$, yielding

$$(x, v^* + u^*) \in \operatorname*{Li}_{n\to\infty} (\mathrm{Exp}_u C_n).$$

Thus

$$v^* + u^* \in \operatorname*{Li}_{n\to\infty}(\mathrm{Exp}_u C_n)(X).$$

As $a^* = w^* - u^*$ is limit of a sequence $(v_n^*) \subset E$, we derive that $w^* := \lim_{n\to\infty}(v_n^* + u^*)$ belongs to $\operatorname*{Li}_{n\to\infty}(\mathrm{Exp}_u C_n)(X)$, since $v_n^* + u^* \in \operatorname*{Li}_{n\to\infty}(\mathrm{Exp}_u C_n)(X)$ and $\operatorname*{Li}_{n\to\infty}(\mathrm{Exp}_u C_n)(X)$ is closed. □

Example 4.1. Let $X = \mathbb{R}^2$, $C = [-1,1] \times \mathbb{R}_+$, $u = (0,1)$, $u^* = (0,-1)$, $H = \mathbb{R} \times \{0\}$ identified to $\mathbb{R}$. One has $f_{[C,u,H]}(x) = i_{[-1,1]}(x)$ and $f^*_{[C,u,H]}(y) = |y|$. One easily checks that

$$(\mathrm{Supp}_u C)(X) = \mathbb{R} \times \{-1\},$$

and

$$(\mathrm{Exp}_u C)(X) = \mathbb{R}^* \times \{-1\}.$$

We see that $\overline{(\mathrm{Exp}_u C)(X)} = (\mathrm{Supp}_u C)(X)$, as expected by applying Theorem 4.15 to the sequence $C_n \equiv C$. Nevertheless observe that

$$\mathrm{Exp}_u C = \{((1,0),(x,-1)) \ : \ x > 0\} \cup \{((-1,0),(y,-1)) \ : \ y < 0\}$$

and

$$(\mathrm{Supp}_u C)^{-1}(0,-1)) = [-1,1] \times \{0\},$$

so that one cannot expect stability for the exposed points but only for the exposed functionals.

Aknowlegement. We are indebted to G. Beer for useful discussions, concerning mainly Theorem 4.3.

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Received January 30, 1996