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A PERIODIC LOTKA-VOLTERRA SYSTEM

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Abstract. In this paper periodic time-dependent Lotka-Volterra systems are considered. It is shown that such a system has positive periodic solutions. It is done without constructive conditions over the period and the parameters.

1. The Periodic Lotka-Volterra System. Consider the Predator-Prey model (see Volterra [1])

$$\begin{aligned} N_1' &= (\varepsilon_1 - \gamma_1 N_2)\, N_1 \\ N_2' &= (-\varepsilon_2 + \gamma_2 N_1)\, N_2. \end{aligned} \tag{1}$$

The functions N_1 and N_2 measure the sizes of the Prey and Predator populations respectively. The coefficients ε_1, ε_2, γ_1, γ_2 are assumed as nonnegative ω-periodic functions of time t. The period $\omega > 0$ is arbitrary chosen and fixed. This periodicity assumption is natural; one may see for instance the work of J. Cushing [2] in which is given a satisfactory justification on it. We still recount (due to [2]) some periodic factors like seasonal effects of weather, food supply, mating habits, hunting or harvesting seasons, etc. Here one may add any unidirectional ω-periodic influence of another predator over the prey.

We will look for ω-periodic *positive* solutions for the conservative system (1) that corresponds to the nature of N_1 and N_2.

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This work presents a result of existence. Notice that in the following theorem there are no conditions on the period and there are no constructive conditions on the parameters of the system. Our result is obtained under weak assumptions. However, it is not explicit that makes the solutions difficult to any further examination.

Theorem. *Suppose that ε_1, ε_2, γ_1, γ_2 are nonnegative continuous ω-periodic functions and that each of them is not equal to zero identically. Then there exist ω-periodic solutions with $N_1(t) > 0$ and $N_2(t) > 0$ for $t \in \mathbb{R}$. Moreover, these solutions satisfy the inequalities*

$$\min N_1 \geq e^{-\omega \max \varepsilon_1} \max N_1, \quad \min N_2 \geq e^{-\omega \max \varepsilon_2} \max N_2$$

and

$$\max N_1 \leq \frac{\int_0^\omega \varepsilon_2(s)ds}{\int_0^\omega \gamma_2(s)ds} e^{\omega \max \varepsilon_1}, \quad \max N_2 \leq \frac{\int_0^\omega \varepsilon_1(s)ds}{\int_0^\omega \gamma_1(s)ds} e^{\omega \max \varepsilon_2}.$$

The present work is related to the mentioned paper of J. Cushing [2] who considered the system

$$\begin{aligned} N_1' = &\ (b_1 - c_{11}N_1 - c_{12}N_2)\, N_1 \\ N_2' = &\ (-b_2 + c_{21}N_1 - c_{22}N_2)\, N_2 \end{aligned}$$

and has proved existence theorems. It is done under the constructive condition $c_{11}(t)c_{12}(t) > 0$ for all t that makes the addend $c_{11}N_1$ unremovable. Therefore, there is no intersect between the results of [2] and the above Theorem.

The framework of the present paper is closed to the papers of Z. Amine and R. Ortega [3] and R. Ortega and A. Tineo [4] in which the authors considered the Lotka-Volterra system

$$\begin{aligned} u' = &\ (a(t) - b(t)u - c(t)v)\, u \\ v' = &\ (d(t) \pm f(t)u - g(t)v)\, v \end{aligned}$$

under the condition that the coefficients are *strictly* positive.

In this connection notice the the paper of A. Tineo and C. Alvarez [5] in which the authors, due to K. Gopalsamy, studied the periodic solutions of competing systems

$$u_i' = u_i\Big[b_i - \sum_{j=1}^n a_{ij}u_j\Big], \quad 1 \leq i \leq n,$$

$(n \geq 2)$ under the conditions

$$\min(b_i) > \sum_{j \neq i} \frac{\max(a_{ij})}{\min(a_{jj})} \max(b_j), \quad 1 \leq i \leq n,$$

that implies $\min(a_{jj}) > 0$. So there is no overlap between the approach of the works [2]–[5] and the approach of the present paper.

Finally, notice the example of J. Kolesov and D. Shvitra in the book [6] (an actual system which includes delay effects) in which the self-existed oscillations in the Prey equation force the oscillations in the Predator equation.

2. Examples.

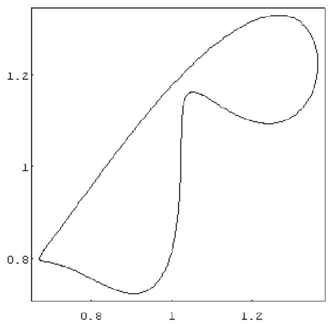

Fig. 1

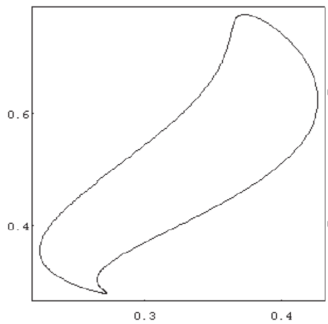

Fig. 2

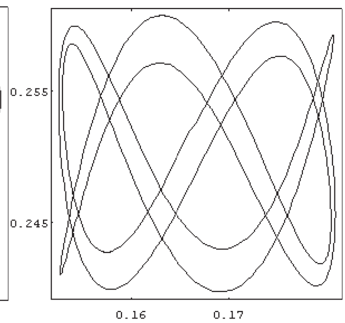

Fig. 3

Let us investigate numerically the following system

$$\begin{aligned} N_1' &= (\sin^2 t - \cos^2 2t\, N_2)N_1 \\ N_2' &= (-\cos^2 t + \sin^2 3t\, N_1)N_2. \end{aligned}$$

A π-periodic solution is found near the initial data $N_1(0) = 0.9004$ and $N_2(0) = 1.0728$. The calculations give

$$|N_1(0) - N_1(\pi)| < 0.00001, \quad |N_2(0) - N_2(\pi)| < 0.00002.$$

Its form is shown in Fig. 1. Repeat the same for the system

$$\begin{aligned} N_1' &= (\sin^2 3t - 2\cos^2 t\, N_2)N_1 \\ N_2' &= (-\cos^2 t + 3\sin^2 t\, N_1)N_2. \end{aligned}$$

A π-periodic solution is found near the initial data $N_1(0) = 0.2646$ and $N_2(0) = 0.4755$. The calculations give

$$|N_1(0) - N_1(\pi)| < 0.0005, \quad |N_2(0) - N_2(\pi)| < 0.0009.$$

Its form is shown in Fig. 2. Finally consider the system

$$\begin{aligned} N_1' &= (\sin^2 3t - 2N_2)N_1 \\ N_2' &= (-\cos^2 7t + 3N_1)N_2. \end{aligned}$$

A π-periodic solution is found near the initial data $N_1(0) = 0.166$ and $N_2(0) = 0.252$ for which

$$|N_1(0) - N_1(\pi)| < 0.00006, \quad |N_2(0) - N_2(\pi)| < 0.00057.$$

Its significant form is given in Fig. 3.

3. Proof of the Theorem. Denote

$$\overline{\varepsilon}_1 \stackrel{\text{def}}{=} \max_t \varepsilon_1(t), \quad \overline{\varepsilon}_2 \stackrel{\text{def}}{=} \max_t \varepsilon_2(t).$$

In view of our assumptions we have $\overline{\varepsilon}_1 \neq 0$ and $\overline{\varepsilon}_2 \neq 0$. Denote

$$u(t) \stackrel{\text{def}}{=} \overline{\varepsilon}_1 - \varepsilon_1(t), \quad v(t) \stackrel{\text{def}}{=} \overline{\varepsilon}_2 - \varepsilon_2(t).$$

Obviously $u(t) \geq 0,\ v(t) \geq 0,\ t \in \mathbb{R}$. Now rewrite equations (1) in the form

$$\begin{aligned} -N_1'(t) + \overline{\varepsilon}_1 N_1(t) &= (u(t) + \gamma_1 N_2(t))\, N_1(t) \\ N_2'(t) + \overline{\varepsilon}_2 N_2(t) &= (v(t) + \gamma_2 N_1(t))\, N_2(t). \end{aligned} \tag{2}$$

By the ω-periodic Green functions

$$G_1(t,\omega) = \frac{e^{\overline{\varepsilon}_1 t}}{e^{\overline{\varepsilon}_1 \omega} - 1}, \qquad G_2(t,\omega) = \frac{e^{-\overline{\varepsilon}_2 t}}{1 - e^{-\overline{\varepsilon}_2 \omega}}, \qquad t \in [0, \omega),$$

the problem for ω-periodic solutions of (2) is reduced to the problem for continuous ω-periodic solutions of the following operator system

$$\begin{aligned} N_1 = &\ \textstyle\int_0^\omega G_1(t-s,\omega)\,(u(s) + \gamma_1 N_2(s))\, N_1(s) ds \stackrel{\text{def}}{=} \mathcal{X}(N_1, N_2) \\ N_2 = &\ \textstyle\int_0^\omega G_2(t-s,\omega)\,(v(s) + \gamma_2 N_1(s))\, N_2(s) ds \stackrel{\text{def}}{=} \mathcal{Y}(N_1, N_2). \end{aligned}$$

Denote by $\mathcal{C}(\omega)$ the space of the real continuous ω-periodic functions defined on the whole axis. Let X be the Banach space $\mathcal{C}(\omega) \otimes \mathcal{C}(\omega)$ with the conventional norm

$$\|(N_1, N_2)\|_X = \max_t |N_1(t)| + \max_t |N_2(t)|.$$

It is not difficult to see that the operator

$$\mathcal{Z} \stackrel{\text{def}}{=} (\mathcal{X}, \mathcal{Y}) : X \to X$$

is completely continuous. Moreover $\mathcal{Z}$ is positive with respect to the cone

$$K \stackrel{\text{def}}{=} \{(N_1, N_2) \in X \ :\ N_1(t) \geq 0 \quad \text{and} \quad N_2(t) \geq 0; \quad t \in \mathbb{R}\}$$

i.e. $\mathcal{Z} : K \to K$. It can be shown that $\mathcal{Z}$ is positive with respect to the subcone $K^\circ \subset K$

$$K^\circ \stackrel{\text{def}}{=} \{\min_t N_1(t) \geq e^{-\overline{\varepsilon}_1 \omega} \max_t N_1(t) \ \text{and} \ \min_t N_2(t) \geq e^{-\overline{\varepsilon}_2 \omega} \max_t N_2(t)\}.$$

We pay more attention to this phenomenon in view of its importance. In fact we have $\mathcal{Z} : K \to K^\circ$ since

$$\min_t \mathcal{X}(N_1, N_2) \geq \frac{\min G_1}{\max G_1} \max_t \mathcal{X}(N_1, N_2) = e^{-\overline{\varepsilon}_1 \omega} \max_t \mathcal{X}(N_1, N_2)$$

and

$$\min_t \mathcal{Y}(N_1, N_2) \geq \frac{\min G_2}{\max G_2} \max_t \mathcal{Y}(N_1, N_2) = e^{-\overline{\varepsilon}_2 \omega} \max_t \mathcal{Y}(N_1, N_2).$$

whenever $(N_1, N_2) \in K$. One can find similar estimates in M. Krasnosel'skii, E. Lifshic and A. Sobolev [7].

The proof is based on the theory of completely continuous vector fields presented by M. Krasnosel'skii and P. Zabrejko in [8]. The following proposition is extracted from [8] in a form convenient for us.

Proposition [8]. *Let Y be a real Banach space with a cone Q and $L : Y \to Y$ be a completely continuous and positive ($L : Q \to Q$) with respect to Q operator. Then the following assertions are valid.*

i. *Let $L(0) = 0$. Let also L be differentable at zero with a derivative $L'(0)$ and there is no $y \in Q$, $y \neq 0$, with*

$$y \overset{\circ}{\leq} L'(0)y.$$

Then there exists $ind(0, L; Q) = 1$.

ii. *Let, for every sufficiently large R, there is no $y \in Q$ with*

$$\|y\|_Y = R \quad \text{and} \quad L(y) \overset{\circ}{\leq} y.$$

Then there exists $ind(\infty, L; Q) = 0$.

iii. *Let $L(0) = 0$ and let there exist $ind(0, L; Q) \neq ind(\infty, L; Q)$. Then L has a nontrivial fixed point in Q.*

Here $ind(\cdot, L; Q)$ denotes the index of a point with respect to L and Q. The sign $\overset{\circ}{\leq}$ denotes the semiordering generated by Q.

Of course, $\mathcal{Z}$ is differentable at zero with a derivative

$$\mathcal{Z}'(0)(N_1, N_2) = \left(\int_0^\omega G_1(t-s, \omega) u(s) N_1(s) ds, \int_0^\omega G_2(t-s, \omega) v(s) N_2(s) ds \right).$$

Let us show that there is no nontrivial $(N_1, N_2) \in K^\circ$ such that the coordinate inequality

$$(N_1(t), N_2(t)) \leq \mathcal{Z}'(0)(N_1, N_2)(t), \qquad t \in \mathbb{R},$$

holds. Otherwise there is $(\tilde{N}_1, \tilde{N}_2)$, with the mentioned property, for which, without loss of generality, we assume $\tilde{N}_1 \not\equiv 0$. Then integrating at $[0, \omega]$ we obtain

$$\int_0^\omega \tilde{N}_1(s) ds \leq \frac{1}{\overline{\varepsilon}_1} \int_0^\omega u(s) \tilde{N}_1(s) ds, \tag{3}$$

which leads to the following contradiction

$$\int_0^\omega (\overline{\varepsilon}_1 - u(s))\tilde{N}_1(s)ds \le 0$$

since, under the definition of K°, we have $\min_t \tilde{N}_1(t) > 0$ and the nonnegative difference $\overline{\varepsilon}_1 - u \equiv \varepsilon_1$ does not equal to zero identically. Thus point **i** of the cited proposition yields

$$ind(0, \mathcal{Z}; K^\circ) = 1.$$

Therefore, in accordance with point **iii**, for a proof of our theorem it is enough to show that

$$ind(\infty, \mathcal{Z}; K^\circ) = 0 \tag{4}$$

which we are going to do.

Let $\mathcal{Z}^*$ be a positive, with respect to the cone K°, operator defined as follows

$$\mathcal{Z}^*(N_1, N_2) = (\mathcal{X}^*(N_1, N_2), \mathcal{Y}^*(N_1, N_2)) =$$
$$= \left(\frac{e^{\overline{\varepsilon}_1\omega}}{\omega} \int_0^\omega N_1(s)ds + 1, \frac{e^{\overline{\varepsilon}_2\omega}}{\omega} \int_0^\omega N_2(s)ds + 1 \right).$$

At first we shall prove that, the completely continuous and positive with respect to K° fields, $\mathcal{I} - \mathcal{Z}$ and $\mathcal{I} - \mathcal{Z}^*$ are positive linear homotopic at

$$D_R \stackrel{\text{def}}{=} \{(N_1, N_2) \in K^\circ \ : \ \|(N_1, N_2)\|_X = 2R\}$$

where R is chosen arbitrary with

$$R > \omega\, e^{(\overline{\varepsilon}_1 + \overline{\varepsilon}_2)\omega} \max\left(\frac{\overline{\varepsilon}_1}{\int_0^\omega \gamma_1(s)ds}, \frac{\overline{\varepsilon}_2}{\int_0^\omega \gamma_2(s)ds} \right). \tag{5}$$

Otherwise, within the definitions (see [8]), there exist $(\tilde{N}_1, \tilde{N}_2) \in D_R$ and $\tilde{\theta} \in [0, 1]$ for which

$$\begin{array}{c} \tilde{\theta}\mathcal{X}(\tilde{N}_1, \tilde{N}_2) + (1 - \tilde{\theta})\mathcal{X}^*(\tilde{N}_1, \tilde{N}_2) = \tilde{N}_1 \\ \tilde{\theta}\mathcal{Y}(\tilde{N}_1, \tilde{N}_2) + (1 - \tilde{\theta})\mathcal{Y}^*(\tilde{N}_1, \tilde{N}_2) = \tilde{N}_2. \end{array} \tag{6}$$

Without loss of generality, we assume $\max_t \tilde{N}_2(t) \ge R$. Then

$$\min_t \tilde{N}_2(t) \ge Re^{-\overline{\varepsilon}_2\omega}.$$

At this point the first equality of (6) implies

$$\max_t \tilde{N}_1(t)\; e^{-\overline{\varepsilon}_1\omega} R\tilde{\theta}e^{-\overline{\varepsilon}_2\omega} \int_0^\omega G_1(t - s, \omega)\gamma_1(s)ds +$$

$$+\max_t \tilde{N}_1(t)\ (1-\tilde{\theta}) + (1-\tilde{\theta}) \le \max_t \tilde{N}_1(t), \quad t \in \mathbb{R}.$$

Integrating the last at $[0,\omega]$ we obtain

$$\max_t \tilde{N}_1(t)\ \tilde{\theta} \left(\frac{Re^{-\overline{\varepsilon}_2\omega} e^{-\overline{\varepsilon}_1\omega}}{\overline{\varepsilon}_1} \int_0^\omega \gamma_1(s)ds - \omega \right) + \omega(1-\tilde{\theta}) \le 0$$

which, in view of (5), may hold if and only if the function $\tilde{N}_1$ is equal to zero identically and $\tilde{\theta} = 1$. Then substituting the values found for $\tilde{N}_1$ and $\tilde{\theta}$ in the second equality of (6) we get

$$\int_0^\omega G_2(t-s,\omega)v(s)\tilde{N}_2(s)ds = \tilde{N}_2(t), \qquad t \in \mathbb{R}.$$

This leads to a contradiction in the same way as (3). Thus we prove the aforementioned homotopy.

At last we are going to show that

$$ind(\infty, \mathcal{Z}^*; K^\circ) = 0$$

which implies the validity of (4), since the homotopic fields have the same index. Here we use point **ii** of our proposition. For this purpose it is enough to observe that there is no $(\tilde{N}_1, \tilde{N}_2) \in K^\circ$ with

$$\mathcal{X}^*(\tilde{N}_1, \tilde{N}_2)(t) = \frac{e^{\overline{\varepsilon}_1\omega}}{\omega} \int_0^\omega \tilde{N}_1(s)ds + 1 \le \tilde{N}_1(t), \qquad t \in \mathbb{R}.$$

Otherwise, after integrating at $[0,\omega]$, the last gives the impossible inequality

$$e^{\overline{\varepsilon}_1\omega} \int_0^\omega \tilde{N}_1(s)ds + \omega \le \int_0^\omega \tilde{N}_1(s)ds.$$

Thus we prove that system (1) has nontrivial solutions.

The proof of the second part follows from the definition of K° and from the fact that for every solution it holds

$$\int_0^\omega \varepsilon_1(s)ds - \int_0^\omega \gamma_1(s)N_2(s)ds = \int_0^\omega \frac{N_1'(s)}{N_1(s)}ds = 0$$

and

$$\int_0^\omega \varepsilon_2(s)ds - \int_0^\omega \gamma_2(s)N_1(s)ds = \int_0^\omega \frac{N_2'(s)}{N_2(s)}ds = 0.$$

4. Notices. The most important detail in the proof was to obtain a proper growth of $\mathcal{Z}$ at infinity (in order to find $ind(\infty, \mathcal{Z}, \cdot)$) that forces the introduction of the cone K°.

It will be of certain interest to investigate the existence of positive *almost-periodic* solutions of (1) with positive almost-periodic coefficients. Perhaps this problem is much more difficult than the periodic one. In this case, together with the compactness of the solution operator, we (possibly) lose the opportunity to use a convenient cone like K°.

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