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Serdica Math. J. **22** (1996), 91-108

Serdica
Mathematical Journal

Institute of Mathematics
and Informatics, Bulgarian
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LIMIT CYCLES OF PERTURBATIONS OF A CLASS OF QUADRATIC HAMILTONIAN VECTOR FIELDS

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Communicated by S. L. Troyanski

Abstract. We prove that in quadratic perturbations of generic Hamiltonian vector fields with two saddle points and one center there can appear at most two limit cycles. This bound is exact.

1. Preliminaries and statement of the results. The problem we solve in this paper belongs to a circle of problems arising from attempts to find an answer to the Hilbert's 16th problem (or, more precisely, to its second part). It asks about *the maximum number and the positions of the limit cycles of Poincaré* for polynomial vector fields

$$\dot{x} = X(x,y), \qquad \dot{y} = Y(x,y). \tag{1}$$

We consider a weaker (infinitesimal) version of this problem. Let $H(x,y)$ be a polynomial in $\mathbb{R}^2$ of degree $n+1$ and f, g be polynomials in $\mathbb{R}^2$ of degree less than or equal to n. Denote by Δ the set of values h of $H(x,y)$ for which the real algebraic curve $\{H(x,y)=h\}$ has a compact component $\gamma(h)$. Define the function

$$I(h) = \int\limits_{\gamma(h)} -f dy + g dx, \qquad h \in \Delta. \tag{2}$$

1991 *Mathematics Subject Classification:* 58F21, 34C05
Key words: Limit cycles, Abelian integrals, Picard-Lefschetz formula

Then the weak Hilbert problem is as follows: *find an upper bound $Z(n)$ for the number of zeros of $I(h)$ in Δ.* It is closely related to the question about the number of limit cycles of the perturbed Hamiltonian system

$$\dot{x} = H_y + \varepsilon f, \qquad \dot{y} = -H_x + \varepsilon g, \tag{3}$$

Consider the system (3) where $H(x, y)$ is a cubic Hamiltonian and f, g are quadratic perturbations:

$$f(x, y) = \sum_{i+j \le 2} a_{ij} x^i y^j, \qquad g(x, y) = \sum_{i+j \le 2} b_{ij} x^i y^j.$$

Definition. *We say that H is* generic *if there exists no coordinate system in which $H(x, y)$ has an axis of symmetry.*

Remark. It is readily seen that H is generic if and only if no level set $\{H(x, y) = h : \quad h \in \mathbb{R}\}$ contains a straight line. The unique (modulo a linear change of variables) and well-known exception to this rule is the Hamiltonian with normal form $y^2 - x^3 + x$, which has an axis of symmetry and no straight line contained in any level set. (The straight line has escaped to infinity in this case.)

The main result in this paper is the following:

Theorem 1.1. *Let H be a generic cubic Hamiltonian with two saddles and one center, let X_H be the corresponding Hamiltonian vector fields, and let $K \subset \mathbb{R}^2$ be a compact. Then there exists a neighborhood U of X_H in the space of all quadratic vector fields such that each $V \in U$ has at most two limit cycles in K.*

This result is exact as without difficulty, one can construct a perturbation for which (3) has two limit cycles.

We want to mention that the same problem was considered earliar (1993) by Zhi–fen Zhang and Chengzhi Li in [8]. The technique used there is different and in our opinion their proof contains some gaps.

According to [4] the generic Hamiltonian H, for which the corresponding vector field $(H_y, -H_x)$ has at least one center as a critical point has the following normal form:

$$H(x, y) = \frac{1}{2}(x^2 + y^2) - \frac{1}{3}x^3 + axy^2 + \frac{b}{3}y^3. \tag{4}$$

The Hamiltonians with two saddles and one center correspond to the curve $\Gamma_\infty = \{b^2 - 4a^3 = 0\}$, for $a > 0$. Such a Hamiltonian is generic **iff** its critical values are all different (or $(a, b) \neq \left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$). It is enough to consider Hamiltonians H for

$a \in \left(0, \dfrac{1}{2}\right)$ (One can obtain the rest of them by linear change of coordinates). So we can reformulate Theorem 1.1 as follows:

Theorem 1.2. *In* (3) *let the Hamiltonian* H *be given by* (4) *where* $(a,b) \in \Gamma_\infty$, *for* $a \in \left(0, \dfrac{1}{2}\right)$ *and let* $K \subset \mathbb{R}^2$ *be a compact. Then there exists* ε_0 *such that for all* $|\varepsilon| \le \varepsilon_0$ *and for any quadratic perturbations* f, g *with* $\sum |a_{ij}| + |b_{ij}| \le 1$ *the system* (3) *has at most two limit cycles in* K.

Following [4] we recall the definition of *Centroid curve.* Denote by $\gamma(h)$ the oval of the curve $H(x,y) = h$ whenever it exists. In what follows, h_s denotes the value of H for which a saddle–connection exists, and h_c denotes the value of H corresponding to the center inside this saddle-connection. Let $h_S < h_C$ for definiteness. We have the equality

$$(5)\quad I(h) = \int\limits_{\gamma(h)} -f dy + g dx = -\iint\limits_{\operatorname{Int}\gamma(h)} (f_x + g_y) dxdy = \iint\limits_{\operatorname{Int}\gamma(h)} (\alpha x + \beta y + \gamma) dxdy,$$

where $\alpha = -(2a_{20} + b_{11})$, $\beta = -(a_{11} + 2b_{02})$, $\gamma = -(a_{10} + b_{01})$. Define the functions (mechanical momenta)

$$X(h) = \iint\limits_{\operatorname{Int}\gamma(h)} x dxdy, \quad Y(h) = \iint\limits_{\operatorname{Int}\gamma(h)} y dxdy, \quad M(h) = \iint\limits_{\operatorname{Int}\gamma(h)} dxdy.$$

In this notation $I(h) = \alpha X(h) + \beta Y(h) + \gamma M(h)$. Note that $M(h)$ gives the area of $\operatorname{Int}\gamma(h)$, and also $M'(h_C) < 0$. We can write the coordinates of the *centroid point* of $\operatorname{Int}\gamma(h)$ as $\xi(h) = X(h)/M(h)$, $\eta(h) = Y(h)/M(h)$. We denote by L the curve formed by the centroid points and we shall refer to L as the *centroid curve.* $L = \{(\xi(h), \eta(h)) : h \in [h_S, h_C]\}$. From the mean–value theorem, it follows that $(\xi(h_C), \eta(h_C)) = (x_C, y_C)$, so the endpoints of the centroid curve L are the centroid of the loop area $Z = (\xi(h_S), \eta(h_S))$ and the center C lying inside. Clearly L is analytic in $(h_S, h_C]$ and affine invariant. Let us assume that:

(a) the centroid curve L is regular, that is, $(\xi'(h))^2 + (\eta'(h))^2 > 0$ for $h \in (h_S, h_C]$;

(b) each line l intersects L in at most three points (counted with multiplicities).

Now we will prove our main theorem modulo these two assumptions. Let us fix a generic Hamiltonian H_0 with two saddles and one center. Since L is regular one can consider its curvature K at $(\xi(h_0), \eta(h_0))$ for $h_0 \in (h_S, h_C)$. Applying Corollary 2.1 from [4] subject to the second assumption we conclude that K can have only simple zeros in (h_S, h_C). In other words, K changes the sign in a neighbourhood of any of its zeros.

Generic Hamiltonians with three saddles and one center correspond to the set

$$\Omega = \{(a,b) : 0 < a < 1, \quad 0 < b < (1-a)\sqrt{(1+2a)}, \quad b^2 < 4a^3\}.$$

Let us choose continuous family of centroid curves $L_{(a(s),b(s))}$ for $s \in [0,1]$ such that $(a(s), b(s)) \in \Omega$ for $s \in (0,1]$ and $(a(0), b(0)) \in \Gamma_\infty$ represents the Hamiltonian H_0. By [4] $L_{(a(s),b(s))}$ is strictly convex for $s \in (0,1]$. Thus we can conclude that $L = L_{(a(0),b(0))}$ is (non–strictly) convex. This means that the curvature K does not change its sign in the interval (h_S, h_C). As we state in the previous paragraph this implies that K has no zeros in that interval, i.e. L is strictly convex in the internal points. Theorems 3.1 and 3.2 from [4] prove that L is strictly convex near the endpoints (these theorems consider an arbitrary generic Hamiltonian[1]). Hence, we prove that L is strictly convex. In other words each line l intersects L in at most two points (counted with multiplicities). It follows from [4] (see the proof of Theorem 1) that if:
a) H is generic, b) L is regular, c) l intersects L in n points (n=0,1,2),
then at most n cycles are produced. Theorem 1.2 is proved.

Now we consider two main assumptions. In our case we have $\Delta = (h_S, h_C)$. Following [3] we introduce $\mathrm{Ind}_G I$ where $G \subset [h_S, h_C]$. If in the neighborhood of $h_0 \in (h_S, h_C)$ holds

$$I(h) = c_k(h-h_0)^k + \cdots, \quad c_k \neq 0, \quad k \in \mathbb{Z}, \tag{6}$$

then we define $\mathrm{Ind}_{h_0} I = k$. If $h_0 \in \{h_S, h_C\}$ then either

$$I(h) = c_1(h-h_C) + c_2(h-h_C)^2 + \cdots, \qquad \text{or} \tag{7}$$

$$I(h) = c_1 + c_2(h-h_S)\log(h-h_S) + c_3(h-h_S) + c_4(h-h_S)^2\log(h-h_S) + \cdots \tag{8}$$

(see Section 2). In both cases we define $\mathrm{Ind}_{h_0} I = k$, where $c_1 = \cdots = c_k = 0$, $c_{k+1} \neq 0$. At last if G is a subset of $\overline{\Delta}$ we define $\mathrm{Ind}_G I = \sum\limits_{h \in G} \mathrm{Ind}_h I$.

According to [4] $\mathrm{Ind}_{\overline{\Delta}} I$ is equal to the number of intersections (with multiplicities) of the line $l : \alpha x + \beta y + \gamma = 0$ and the centroid curve L. First of all we evaluate the number of zeros of $\dfrac{d}{dh} I$ in the interval $(h_S, h_C]$.

Theorem 1.3. *Suppose that $I(h)$ does not vanish identically in Δ. Then $\dfrac{d}{dh} I$ has at most three zeros in $(h_S, h_C]$.*

[1] In fact Theorem 3.2 (ii) was stated for generic Hamiltonians with three saddles and one center but the autors use only the fact that an arbitrary line intersects centroid curve in at most three poins (which has been established earliar in [3]).

As an application of the method we use we prove the regularity of the curve L in Section 4, i.e. we prove our first assumption.

In Section 5 we explain (following [3]) how one can prove Theorem 1.4, or in other words our second assumption.

Theorem 1.4. *Suppose that $I(h)$ does not vanish identically in Δ. Then* $\mathrm{Ind}_{\overline{\Delta}} I \leq 3$.

2. Normal forms. After suitable rotation of (4) and linear change of h we get the following

Lemma 2.1. *Let the Hamiltonian system $(\dot{x}, \dot{y}) = X_H$ has two saddles and one center, and the corresponding critical values of H are all different. Then there exists $\mathbb{R}$–linear change of the coordinate system such that*

$$H(x,y) = \frac{x^2+y^2}{2} + x(x+\mu y)^2, \tag{9}$$

for some $\mu > 0$. Three critical values of such Hamiltonian satisfy inequalities $0 = \tilde{h}_1 < \tilde{h}_2 < \tilde{h}_3$.

Let Ω be $\mathbb{R}$–linear space of all quadratic perturbations ω of the Hamiltonian system modulo exact forms. Then $\dim \Omega = 3$ and $\Omega = \mathbb{R}\{ydx, xydx, y^2dx\}$.

Consider now the Riemann surface $\Gamma_h = \{(x,y) \in \mathbb{C}^2 \colon H(x,y) = h\}$. Each oval of Γ_h (i.e. a compact smooth real curve contained in $\mathrm{Re}\,\Gamma_h$) is a closed orbit of the Hamiltonian system. For a later use we shall put the curve into an elliptic normal form. Let $z = (1+2\mu^2 x)y + 2\mu x^2$. In (x,z) coordinates Γ_h takes the form

$$\Gamma_h = \{z^2 = -2(1+\mu^2)x^3 - x^2 + 4h\mu^2 x + 2h\}; \tag{10}$$

and, modulo the forms $\mathbb{R}\left\{\frac{dx}{1+2\mu^2 x}, \frac{dx}{(1+2\mu^2 x)^2}, \frac{hdx}{1+2\mu^2 x}\right\}$

$$\Omega = \mathbb{R}\left\{\frac{zdx}{1+2\mu^2 x}, zdx, \frac{zdx}{(1+2\mu^2 x)^2}\right\}.$$

Let us notice that $ydx = \frac{zdx}{1+2\mu^2 x} - \frac{2\mu x^2 dx}{1+2\mu^2 x}$. Each closed orbit of the Hamiltonian system which is contained in the level set $\{H(x,y) = h\}$ is an oval of the algebraic curve (10). Note that the opposite is not true – an oval of the algebraic curve (10) does not come necessary from a closed orbit of the Hamiltonian system.

The $\mathbb{R}$–linear change of the variable $x \mapsto mx + n$, where $m = -\dfrac{1}{\sqrt[3]{2(1+\mu^2)}}$, $n = \dfrac{m^3}{3}$ brings the curve Γ_h in the form

$$\Gamma_h = \left\{ z^2 = x^3 + \Big(\frac{-2(1+2m^3)h}{m^2} - \frac{m^4}{3}\Big)x + \Big(\frac{4}{3}(1-m^3)h - \frac{2m^6}{27}\Big)\right\}.$$

At last changing the variable $z \mapsto \dfrac{z}{\sqrt{2}}$ and $h \mapsto \dfrac{-2(1+m^3)h}{m^2}$ we obtain the following normal form

$$\Gamma_h = \left\{ \frac{z^2}{2} = x^3 + p(h)x + q(h) \right\}, \tag{11}$$

where

$$p(h) = h + p_2, \quad p_2 = -\frac{m^4}{3}, \qquad q(h) = q_1 h + q_2, \quad q_2 = -\frac{2m^6}{27}, \quad q_1 = -\frac{2m^2(1-m^3)}{3(1+2m^3)}.$$

We point out that singular curves occure when $h \in \{h_1, h_2, h_3\}$, $h_i = -\dfrac{2(1+2m^3)}{m^2}\tilde{h}_i$. Since $-\dfrac{2(1+2m^3)}{m^2} = -\dfrac{4\mu^2}{\sqrt[3]{2(1+\mu^2)}} < 0$ then $0 = h_1 > h_2 > h_3$. In this notation we get

$$\Omega = \mathbb{R}\left\{ \frac{zdx}{x+q_1}, zdx, \frac{zdx}{(x+q_1)^2} \right\} \tag{12}$$

modulo the forms $\mathbb{R}\left\{ \dfrac{dx}{x+q_1}, \dfrac{hdx}{x+q_1}, \dfrac{dx}{(x+q_1)^2} \right\}$.

Consider the bifurcation diagram B of the family of curves (11), i.e. the set of values $(p,q) \in \mathbb{R}^2$, for which the corresponding curve is singular. Explicitly we have

$$B = \left\{ (p,q) \in \mathbb{R}^2 : \quad \delta(p,q) = \frac{q^2}{4} + \frac{p^3}{27} = 0 \right\}.$$

To each Hamiltonian function (9) there correspond a straight line

$$l_H = \{p(h), q(h) : h \in \mathbb{R}\} \subset \mathbb{R}^2_{p,q}. \tag{13}$$

To each level set $\{H = h\} \subset \mathbb{R}^2$ there corresponds a point $(p,q) \in \mathbb{R}^2$. The condition that H has three distinct critical values is equivalent to the condition that l_H intersects the set B in three distinct points. The set $\mathbb{R}^2 \backslash B$ consists of two unbounded components. Denote the component where $\delta(p,q) < 0$ by D. If the curve $\{H(x,y) = h\}$ contains an oval then the corresponding curve (11) contains also an oval, which is equivalent to $(p, q) \in D$. Since $q_1, q_2, p_2 < 0$ then the line l_H passess through second, third and fourth quarter of $\mathbb{R}^2_{q,p}$. Three points of intersection between the line l_H ant the set B are the critical values of the hamiltonian (9). Obviously $0 = h_1 > h_2 > h_3$, where $h_C = h_1$ and $h_S = h_2$. The oval $\gamma(h)$ of (11) vanishes **iff** (p, q) lies on the segment of B for which $q < 0$. We point out that for $h \in (-\infty, h_3)$ the curve (11) has an oval but the same curve in (x, y) coordinates has not one.

At last let us recall the classical Picard-Lefschetz formula [1]. Denote

$$D_R = \{|z| < R\} \backslash \{h_1, h_2, h_3\} \subset \mathbb{C}, \tag{14}$$

where R is a sufficiently big fixed real number. Let z_0 be a point on the boundary $\{|z| = R\}$ of D_R. Any loop $l \in \pi(D_R, z_0)$ induces an isomorphism l_* (monodromy) in the first homology group

$$l_* : H_1(\Gamma_h, \mathbb{Z}) \to H_1(\Gamma_h, \mathbb{Z}) \tag{15}$$

of the affine algebraic curve $\Gamma_h = \{H = h\} \subset \mathbb{C}^2$. Let l^i be a loop around h_i, and $\gamma_j(h) \in H_1(\Gamma_h, \mathbb{Z})$ a cycle vanishing at h_j, $\;j = 1, 2, 3$. The Picard–Lefschetz formula reads

$$l^j_*(\delta) = \delta + (\gamma_j \circ \delta)\gamma_j, \tag{16}$$

where $(\gamma_j \circ \gamma)$ is the intersection index of γ_j and γ. We have the equality $\gamma(h) = \gamma_1(h)$ in $H_1(\Gamma_h, \mathbb{Z})$. We can choose γ_2, γ_3 such that

$$(\gamma_2 \circ \gamma_1) = (\gamma_3 \circ \gamma_1) = 1, \quad (\gamma_2 \circ \gamma_3) = 0.$$

(see [2]).

The Picard-Lefschetz formula implies a formula for branching of abelian integrals around the critical values h_j. Namely, if ω is a meromorphic one form in a neighborhood of h_j in the complex domain it holds (see [1]).

$$I(h) = \int\limits_{\gamma(h)} \omega = (\gamma_j \circ \gamma) \frac{\log(h - h_j)}{2\pi i} \int\limits_{\gamma_j(h)} \omega + M(h), \tag{17}$$

where $M(h)$ is a meromorphic function. Let ω be the holomorphic one–form with real coefficients. Since the vanishing cycles $\gamma_j(h)$ remain bounded when h belongs to some bounded domain D and ω is holomorphic then the function $M(h)$ is holomorphic. The function $\int_{\gamma_1(h)} \omega$ has an expansion with real coefficients near the point $h = h_1$ and the functions $\int_{\gamma_j(h)} \omega$ has expansions with pure imaginary coefficients near the points $h = h_k \quad j, k = 2, 3$. Note that the birational change of variables bringing the curve $\{H(x, y) = h\}$ in the form (11) does not involve h. Hence the formulae (16) and (17) remain valid for the corresponding cycles and Abelian integrals defined on the curve (11). This will be used without explicit mentioning in the following sections. To the end of the paper an important role will also be played by the Wronskians

$$\mathrm{W}_{\tau_1,\tau_2}(\omega',\omega'') = \int\limits_{\tau_1}\omega'\int\limits_{\tau_2}\omega'' - \int\limits_{\tau_1}\omega''\int\limits_{\tau_2}\omega',$$

defined for arbitrary one forms ω', $\omega'' \in H^1(\Gamma_h,\mathbb{Z})$, and τ_1, $\tau_2 \in H_1(\Gamma_h,\mathbb{Z})$.

3. Zeros of the Abelian integral $I(h)$. In this section we prove Theorem 1.3 under the restriction that the Hamiltonian H is generic and the corresponding Hamiltonian system possesses two saddle points and one center. Let us define the following one-forms on the algebraic curve (11)

$$\begin{array}{lll} \tilde{\omega}_0 = \dfrac{zdx}{x+q_1}, & \tilde{\omega}_1 = zdx, & \tilde{\omega}_2 = \dfrac{zdx}{(x+q_1)^2}, \\ \omega_0 = \dfrac{dx}{z}, & \omega_1 = \dfrac{(x+q_1)dx}{z}, & \omega_0 = \dfrac{dx}{(x+q_1)z}, \end{array}$$

The Abelian integral $I(h)$ defined in Section 1 takes the form $I(h) = \int\limits_{\gamma(h)} -fdy+gdx = \int\limits_{\gamma(h)} d_0\tilde{\omega}_0+d_1\tilde{\omega}_1+d_2\tilde{\omega}_2$, where d_0, d_1, d_2 are real constants depending on the coefficients of f and g. Denote also $J(h) = \dfrac{d}{dh}I(h)$, $\omega = d_0\omega_0 + d_1\omega_1 + d_2\omega_2$, $J_j(h) = \int\limits_{\gamma(h)} \omega_j$. Obviously $J(h) = \sum\limits_{k=0}^{2} d_kJ_k = \int\limits_{\gamma(h)} \omega$ (we differentiate (11) by h and we get $z\dfrac{dz}{dh}(h,x) = x+q_1$).

Let $0 = h_1 > h_2 > h_3$ be the critical values of the Hamiltonian function H. They correspond to the three points of intersection of the line l_H and the bifurcation set B. As the cycle $\gamma_1(h)$ vanishes at $h = h_1$, then $I(h_1) = 0$, and hence the number of zeros of $I(h)$ in the interval (h_2,h_1) is less than or equal to the number of zeros of $J(h)$ in the same interval.

Define the function $F(h) = \dfrac{J(h)}{J_0(h)}$. The zeros of $J(h)$ are obviously zeros of $F(h)$. In order to find them we exploit the idea of Petrov [6] to continue analytically the function F in the complex domain $\mathbb{C}^2\backslash\{h \le h_2\}$ (Lemma 3.1 and Lemma 3.2) and then to use the argument principle. On its hand to apply the latter we need to find the increment of the argument of F along the boundary of suitable complex domain.

Lemma 3.1. *$J_0(h) \ne 0$ for $h \in \mathbb{C}^2\backslash\{h_2,h_3\}$.*

P r o o f. Indeed, for $h \in \mathbb{C}^2 \backslash \{h_1, h_2, h_3\}$ we use the fact that the periods of the unique holomorphic one-form ω_0 on the elliptic curve $\overline{\Gamma_h}$, never vanish (see [3]). For $h = h_1$ this is obtained by simple computation. We use the fact that ω_0 is equal to $Cydx$ in (x, y) coordinates and $0 \neq C \in \mathbb{R}$. $\square$

Lemma 3.2. *$F(h)$ is a holomorphic function in the region $D = \mathbb{C} \backslash \{h \leq h_2\}$.*

P r o o f. $I(h)$ is a holomorphic function at $h = h_1$ as by definition it is an integral along a cycle $\gamma(h)$ vanishing at the critical point $(x, y) = (0, 0)$ which corresponds to the critical value h_1 (see [1]). Then $J(h)$ is also holomorphic. The result follows from Lemma 3.1. $\square$

We recall that the cycles $\gamma_j \in H_1(\Gamma_h, \mathbb{Z})$, $j = 1, 2, 3$ are the vanishing cycles which correspond to the critical values h_j. We describe their properties in Section 2.

The function F has an analytical continuation for any point $h \neq h_2, h_3$, but the continuation depends on the path it is performed along. For point on the set $M = \{h < h_2,\ h \neq h_3\}$ denote by F^+, G^+ (respectively F^-, G^-) the analytic continuation of F, G along a path on which $\operatorname{Im} h > 0$ $(\operatorname{Im} h < 0)$. The increment of the argument of F along the intervals $(-\infty, h_3) \cup (h_3, h_2)$ can be evaluated by the zeros of its imaginary part on this interval. Sometimes it is convenient to write F along $(-\infty, h_2)^{\pm}$ instead of $F^{\pm}$.

Lemma 3.3.

$$\operatorname{Im} J^{\pm} = \begin{cases} \mp\dfrac{i}{2} \displaystyle\int_{\gamma_2} \omega, & h \in (h_3, h_2) \\ \mp\dfrac{i}{2} \displaystyle\int_{\gamma_2+\gamma_3} \omega, & h \in (-\infty, h_2) \end{cases} \tag{18}$$

P r o o f. Since $J(h)$ is a real–valued function along the interval (h_2, h_1) then we get the equality $J(\overline{h}) = \overline{J(h)}$ for $h \in D$ and $J^-(h) = \overline{J^+(h)}$ for $h \in (-\infty, h_2)$ In the case $h \in (h_3, h_2)$ the Picard-Lefschetz formula (17) implies

$$J^+(h) = J^-(h) + (\gamma_2 \circ \gamma_1) \int_{\gamma_2} \omega, \quad J^+(h) - J^-(h) = J^+(h) - \overline{J^+(h)} = \int_{\gamma_2} \omega,$$

or

$$2i\operatorname{Im} J^+(h) = \int_{\gamma_2} \omega, \quad \operatorname{Im} J^+(h) = -\frac{i}{2} \int_{\gamma_2} \omega.$$

The case $h \in (-\infty, h_3)$ is treated in the same manner. $\square$

Corollary 3.4.

$$\operatorname{Im} F^{\pm} = \begin{cases} \pm\dfrac{i}{2|J_0|^2}\mathrm{W}_{\gamma_1,\gamma_2}(\omega,\omega_0) & \textit{for} \quad h \in (h_3,h_2) \\ \pm\dfrac{i}{2|J_0|^2}\mathrm{W}_{\gamma_1,\gamma_2+\gamma_3}(\omega,\omega_0) & \textit{for} \quad h \in (-\infty,h_3) \end{cases} \tag{19}$$

The proof is straightforward computation.

Denote $G_1 = 2|J_0|^2 \operatorname{Im} F^+$ for $h \in (h_3,h_2)$ and $G_2 = 2|J_0|^2 \operatorname{Im} F^+$ for $h \in (-\infty,h_3)$. In order to find the zeros of $\operatorname{Im} F$ on each of the intervals (h_3,h_2), $(-\infty,h_3)$ we apply the scheme we described above for F to the functions G_j. Namely we continue them in the complex domain and use the argument principle.

Define the complex domains $D_1 = \mathbb{C}\backslash(-\infty,h_3]$, $\quad D_2 = \mathbb{C}\backslash[h_3,h_2]$.

Lemma 3.5. *The functions G_1 and G_2 are holomorphic and single-valued in the complex domains D_1 and D_2 respectively.*

Proof. According to Corollary 3.4 $G_1 = i\mathrm{W}_{\gamma_1,\gamma_2}(\omega,\omega_0)$ and $G_2 = i\mathrm{W}_{\gamma_1,\gamma_2+\gamma_3}(\omega,\omega_0)$. Since the function $\int\limits_{\gamma_1}\omega$ is holomorphic and single valued in the domain D and the functions $\int\limits_{\gamma_j}\omega$, $j = 2,3$ are holomorphic and single–valued in the domain $D' = \mathbb{C}\backslash[h_1,+\infty)$, then G_1 is holomorphic and single–valued in the domain $\mathbb{C}\backslash(-\infty,h_3]\cup[h_2,+\infty)$, and G_2 in the domain $\mathbb{C}\backslash[h_3,+\infty)$.

First we prove that G_j are single–valued in the interval (h_2,h_1). The Picard-Lefschetz formula gives that after an counter–clockwise turn around the point h_2 the integrals change in the following way:

$$\int\limits_{\gamma_1^-}\omega = \int\limits_{\gamma_1^+}\omega + \int\limits_{\gamma_2}\omega, \qquad \int\limits_{\gamma_k}\omega = \int\limits_{\gamma_k}\omega \quad \text{for} \quad k = 2,3.$$

This shows that the Wronskian $\mathrm{W}_{\gamma_1,\gamma_2}(\omega,\omega_0) \mapsto \mathrm{W}_{\gamma_1,\gamma_2}(\omega,\omega_0) + \mathrm{W}_{\gamma_2,\gamma_2}(\omega,\omega_0)$. Since $\mathrm{W}_{\gamma_2,\gamma_2}(\omega,\omega_0) = 0$, then $G_1 = i\mathrm{W}_{\gamma_1,\gamma_2}(\omega,\omega_0)$ does not change and G_1 is single–valued in the domain $\mathbb{C}\backslash(-\infty,h_3] \cup [h_1,+\infty)$. We prove that $G_2 = i\mathrm{W}_{\gamma_1,\gamma_2+\gamma_3}(\omega,\omega_0)$ does not change after one turn around the interval $[h_3,h_2]$ in the same maner. Then G_2 is single-valued in the domain $\mathbb{C}\backslash(-\infty,h_3] \cup [h_1,+\infty)$. In the same way we prove that when h makes one loop around the point h_1 the functions G_1 and G_2 do not change. In the case we consider the functions $M_{\gamma_j,\omega}(h)$ from the Picard-Lefschetz formula (17) are holomorphic because $\gamma_2(h)$ and $\gamma_3(h)$ are families of bounded cycles in the neighborhood of h_1 (see [1]). Thus we obtain that in neighborhood of h_1 $\quad|\int\limits_{\gamma_k}| \le c|\log(h-h_1)|$,

$k = 2, 3$, which gives $|\mathrm{W}_{\gamma_1,\delta}(\omega, \omega_0)| \le \tilde{c}|\log(h - h_1)|$, and hence the functions G_1 and G_2 are holomorphic in the neighborhood of $h = h_1$. $\square$

Lemma 3.6.

$$\begin{aligned} \operatorname{Im} G_1^{\pm} &= \pm i \mathrm{W}_{\gamma_3,\gamma_2}(\omega, \omega_0) \qquad &&\textit{for} \quad h \in (-\infty, h_3); \\ \operatorname{Im} G_2^{\pm} &= \pm i \mathrm{W}_{\gamma_2,\gamma_3}(\omega, \omega_0) \qquad &&\textit{for} \quad h \in (h_3, h_2). \end{aligned}$$

We practically repeat the arguments of Lemma 3.3 and Lemma 3.5.

As we have remarked above, the cycles γ_2 and γ_3 are homologous on the curve Γ_h. As the residues of third kind one-form ω are constants then the quantity $\int\limits_{\gamma_2-\gamma_3} \omega = 2\pi i \sum \operatorname{Res}_{p_i} \omega$, is also constant. Straightforward computation shows that the constant is pure imaginary. Thus we conclude that each of the Wronskians defined in Lemma 3.6 either vanish identically or is equal to $C \int\limits_{\gamma_2} \omega_0$ for some (imaginary) constant C.

(**Remark.** $\int\limits_{\gamma_2} \omega_0 = \int\limits_{\gamma_3} \omega_0$, because ω_0 has no residues on the curve Γ_h).

Corollary 3.7. *The Wronskians defined in Lemma* 3.6 *either vanish identically or they have no zeros in the corresponding intervals.*

In order to find the increment of the argument along a big circle of the functions we consider one have to describe their behaviors near ∞. Let $\gamma(h)$ be an arbitrary non-zero element from $H_1(\Gamma_h, \mathbb{Z})$.

Lemma 3.8. *There exists positive real constants K_0', K_0'', K_1, K_2; $K_0' \neq 0$ such that for all sufficiently big $|h|$ the functions J_j, $j = 0, 1, 2$ satisfy the inequalities:*

$$\begin{aligned} &(i) \qquad && K_0' \le |\int\limits_{\gamma(h)} \omega_0||h|^{\frac{1}{4}} \le K_0'', \\ &(ii) \qquad && |\int\limits_{\gamma(h)} \omega_1||h|^{-\frac{1}{4}} \le K_1, \\ &(iii) \qquad && |\int\limits_{\gamma(h)} \omega_2||h|^{\frac{3}{4}} \le K_2. \end{aligned}$$

P r o o f. In the equation (11) we make the rescaling:

$$Z = |h|^{-\frac{3}{4}} z, \qquad X = |h|^{-\frac{1}{2}} x, \qquad h = |h| \exp^{i\theta}.$$

In (X, Z) coordinates the equation (11) takes the form:

$$\left\{ \frac{Z^2}{2} = X^3 + (\exp^{i\theta} + p_2|h|^{-1})X + (q_1|h|^{-\frac{1}{2}} \exp^{i\theta} + q_2|h|^{-\frac{3}{2}}) \right\}. \tag{20}$$

The integrals we consider can be represented as:

$$\int\limits_{\gamma(h)} \frac{dx}{z} = |h|^{-\frac{1}{4}} \int\limits_{\gamma(h)} \frac{dX}{Z};$$

$$\int\limits_{\gamma(h)} \frac{(x+q_1)dx}{z} = |h|^{\frac{1}{4}} \int\limits_{\gamma(h)} \frac{(X+q_1|h|^{-1/2})dX}{Z};$$

$$\int\limits_{\gamma(h)} \frac{dx}{(x+q_1)z} = |h|^{-\frac{3}{4}} \int\limits_{\gamma(h)} \frac{dX}{(X+q_1|h|^{-1/2})Z}.$$

This together with inequality $\int\limits_{\gamma(\theta,\infty)} \frac{dX}{Z} \neq 0$ proves the lemma. $\square$

Corollary 3.9. $\forall \varepsilon > 0$ *the increase of the argument of the following functions along the sufficiently big circle* $C = \{|h| = R\}$ *satisfy the inequalities:*

$$(i) \qquad |\Delta_C J_0 + \frac{1}{4}2\pi| \leq \varepsilon, \quad \Delta_C J_1 \leq \frac{1}{4}2\pi + \varepsilon, \quad \Delta_C J_2 \leq -\frac{3}{4}2\pi + \varepsilon,$$

$$(ii) \qquad \Delta_C J = \Delta_C(d_0J_0 + d_1J_1 + d_2J_2) \leq \frac{1}{4}2\pi + \varepsilon,$$

$$(iii) \qquad \Delta_C F = \Delta_C \frac{J}{J_0} = \Delta_C J - \Delta_C J_0 \leq \frac{1}{2}2\pi + \varepsilon,$$

$$(iv) \qquad \Delta_C G_1 = \Delta_C \mathrm{W}_{\gamma_1,\gamma_2}(\omega,\omega_0) \leq \varepsilon, \quad \Delta_C G_2 \leq \varepsilon.$$

Denote by g_j the number of zeros of the function G_j in the region D_j, $j = 1, 2$. We shall say that the argument of F (G_1, G_2) has *number* k **iff** $\arg F \in [k\pi, (k+1)\pi)$. We use the notation Numb $\arg F$. If Numb $\arg F$ increases by k, then $\arg F$ increases by at most $(k+1)\pi$.

Lemma 3.10. $g_1 \leq 1$, $g_2 \leq 1$. *In the case* $\int\limits_{\gamma_2} \omega = \int\limits_{\gamma_3} \omega$ *we have* $g_1 = g_2 = 0$.

Proof. Denote by $\tilde{D}_1$ the set $(D_1 \cap \{|h| = R\}) \backslash \{|h - h_3| \leq r_3\}$, and by $\tilde{D}_2$ the set $= (D_2 \cap \{|h| = R\}) \backslash \{(|h - h_3| \leq r_3) \cup (|h - h_2| \leq r_2)\}$. According to Corollary 3.9 we choose R such that $\Delta_{|h|=R} G_1 \leq \pi$ $(\varepsilon = \pi)$.

As we have remarked above $\int\limits_{\gamma_2} \omega = \int\limits_{\gamma_3} \omega + C$, where C is pure imaginary constant.

In the case $C = 0$, Corollary 3.7 and Lemma 3.6 give that G_1 is a holomoorphic function in $\mathbb{C}$ and according to the argument principle the number of the zeros of G_1

in $\tilde{D}_1$ is less than or equal to $\left[\frac{\Delta_{|h|=R}G_1}{2\pi}\right] \leq \left[\frac{\pi}{2\pi}\right] = 0$. When R tends to ∞ we obtain $g_1 = 0$.

Let us assume that $C \neq 0$. Then near the point h_3 the function G_1 has an expansion of a type

$$G_1 = \log(h - h_3)(a_0 + a_1(h - h_3) + \cdots) + H(h), \tag{21}$$

where H is a holomorphic function in a neighborhood of h_3 and $a_0 \neq 0$. Hence the increase of the argument of G_1 along the circle $|h - h_3| = r_3$ can be made arbitrary small together with r_3 and Numb $\arg G_1$ increases at most by one. Lemma 3.6 and Corollary 3.7 give that along the intervals $(-\infty, h_3)^{\pm}$ $\operatorname{Im} G_1$ has no zeros. Thus Numb $\arg G_1$ does not change along these intervals.

Then the increase of Numb $\arg G_1$ along the contour $B_1 = \partial\tilde{D}_1 \backslash \{|h| = R\}$ is at most one. In other words $\Delta_{B_1} G_1 < 2\pi$. Finally we get $\Delta_{\partial\tilde{D}_1} G_1 < 3\pi$. According to the argument principle the number of the zeros of G_1 in $\tilde{D}_1$ is less than or equal to $\left[\frac{3\pi}{2\pi}\right] = 1$. When $R \to \infty$ and $r_3 \to 0$ we obtain $g_1 \leq 1$.

The function G_2 is treated in the same manner. $\square$

The previous lemma gives an estimate from above for the number of zeros of the functions $\operatorname{Im} F^{\pm}$ in the intervals $(-\infty, h_3) \cup (h_3, h_2)$ and this estimate is two. In the case $\int\limits_{\gamma_2} \omega = \int\limits_{\gamma_3} \omega$, the estimate is zero.

Theorem 3.11. *If the Abelian integral $I(h)$ does not vanish identically, then the function F can have no more than three zeros in the domain D. In the case $\int\limits_{\gamma_2} \omega = \int\limits_{\gamma_3} \omega$ the number of zeros of F in the same region is at most one.*

Proof. Let R be a sufficiently big number and r_2, r_3 be sufficiently small numbers (we shall choose them below). Consider the domain $\tilde{D}$, which is obtained from $D \cap \{|h| = R\}$ by removing the circles $\{|h - h_3| \leq r_3\}$ and $\{|h - h_2| \leq r_2\}$. We choose a continuous parametrization of $\arg F$ along the border of $\tilde{D}$. We again apply the argument principle to the domain $\tilde{D}$.

According to Corollary 3.9 we choose R such that $\Delta_{|h|=R} F \leq \frac{3\pi}{2}$ $(\varepsilon = \frac{\pi}{2})$. Denote $\partial\tilde{D} \backslash \{|h| = R\}$ by B. The orientation of B is such that the circle $S_2 = \{|h - h_2| = r_2\}$ and the semi-circles $S_3^{\pm} = \{|h - h_2| = r_2\}^{\pm}$ have been passed at the negative (clockwise) direction (according to h_2 and h_3 respectively). The Picard-Lefschetz formula implies the following expansions (around the points h_j, $j = 2, 3$ respectively):

$$F(h) = \frac{J(h)}{J_0(h)}, \tag{22}$$

$$\begin{aligned}
J_0(h) &= \log(h-h_2)(a_0^j + a_1^j(h-h_2)+\cdots) + (b_0^j + b_1^j(h-h_2)+\cdots),\\
J(h) &= (h-h_2)^k \log(h-h_2)(c_k^j + c_{k+1}^j(h-h_2)+\cdots) +\\
&+ (h-h_2)^l(d_l^j + d_{l+1}^j(h-h_2)+\cdots),\\
\operatorname{Im} F^+(h_3-r_3) &= \frac{(\ln r_3(f_0+f_1(-r_3)+\cdots) + \operatorname{Re} H_1(h_3-r_3))}{2|J_0(h_3-r_3)|^2},\\
\operatorname{Im} F^+(h_3+r_3) &= \frac{(\ln r_3(f_0+f_1 r_3+\cdots) + \operatorname{Re} H_1(h_3+r_3))}{2|J_0(h_3+r_3)|^2},
\end{aligned}$$

$$\operatorname{Im} F^+(h_3-r_3)\operatorname{Im} F^+(h_3+r_3) = \frac{f_0^2 \ln^2 r_3}{4|J_0(h_3-r_3)||J_0(h_3+r_3)|} + o(\ln^2 r_3), \tag{23}$$

where c_k^j, d_l^j, a_0^j, $f_0 \neq 0$ for $j=2,3$ and $f_0 \in \mathbb{R}$.

Now we shall compute the increment of Numb $\arg F$ along the contour B.

1. If $\operatorname{Im} F$ has no zeros in an interval then Numb $\arg F$ does not change in the same one.

2. There are two possible cases when h passes through a zero of $\operatorname{Im} F$:

   (i) If $F \neq 0$ at the same point Numb $\arg F$ changes by 0 or ± 1. In other words it increases by at most one;

   (ii) If $F(h_0) = 0$ for the point $h_0 \in (-\infty, h_3) \cup (h_3, h_2)$ we have to deform the contour B in such a way that F has no zeros along the new contour. Near the point h_0 $\arg F$ decreases by approximately $k\pi$, where k is the multiplicity of the zero h_0. Thus Numb $\arg F$ does not increase near the point h_0 if the deformation is sufficiently small.

3. According to expansions (22) for all $\varepsilon > 0$ if r_2 is small enough $\arg F$ increases by at most ε along the circle $\{|h-h_2| = r_2\}$ (after a clockwise turn). We choose r_2 such that $\Delta_{|h-h_2|=r_2} F < \dfrac{\pi}{2}$. Thus Numb $\arg F$ increases by at most one along the same circle.

4. We prove in the same manner that the increment of Numb $\arg F$ is less than or equal to one along the semi-circles $\{|h-h_3| = r_3\}^{\pm}$ defined by the restriction $\operatorname{Im} h \geq 0$ ($\operatorname{Im} h \leq 0$ respectively). But the formula (23) implies that the sign of $\operatorname{Im} F$ does not change after passing through this semi-circles and it follows that Numb $\arg F$ does not increase along them.

Since $\operatorname{Im} F$ has at most four (none in the case $\int\limits_{\gamma_2}\omega = \int\limits_{\gamma_3}\omega$) zeros on the intervals $(-\infty, h_3)^{\pm} \cup (h_3, h_2)^{\pm}$, then $\operatorname{Numb}\arg F$ increases by at most five (one in the other case) along the contour B. In other words $\Delta_B F < 6\pi$. Finally we get $\Delta_{\partial\tilde{D}} F < 6\pi + \dfrac{3\pi}{2}$ ($\Delta_{\partial\tilde{D}} F < \dfrac{7\pi}{2}$ correspondingly). According to the argument principle the number of zeros of F in the domain $\tilde{D}$ is less than or equal to $\left[\dfrac{15\pi}{4\pi}\right] = 3$ ($\left[\dfrac{7\pi}{4\pi}\right] = 1$ in the other case). When $R \to \infty$ and $r_3, r_2 \to 0$ we obtain the statement of the theorem. $\square$

Thus we prove

Theorem 1.3. *In the case we consider either the Abelian integral $I(h)$ vanishes identically or $\dfrac{d}{dh}I$ has at most three zeros in $(h_2, h_1]$.*

4. Regularity of the centroid curve L. In this section we prove the regularity of L in the case of a generic Hamiltonian H with two saddle points and one center (see Section 1 for definitions). We point out that $h_1 = h_C$ and $h_2 = h_S$. Our final goal is the following theorem

Theorem 4.1. *Suppose that H satisfies our restriction. Then the curve L is regular, that is*

$$(\xi'(h))^2 + (\eta'(h))^2 > 0, \ \textit{for } h \in (h_2, h_1].$$

In fact we are going to prove more than we have formulated in previous theorem.

Theorem 4.2. *There exists a $\mathbb{R}$–linear combination $rX + sY$ of the integrals X, Y such that the function $rX' + sY' - kM'$ has no more than one zero in the interval $(h_2, h_1]$ for any real k.*

P r o o f. According to Section 2 we can change the coordinate system. In (x, z) coordinates the functions take the forms

$$M(h) = -\int\limits_{\gamma(h)} y\,dx = \int\limits_{\gamma(h)} c_0\tilde{\omega}_0 = \int\limits_{\gamma(h)} \tilde{\omega_M},$$

$$X(h) = -\int\limits_{\gamma(h)} xy\,dx = \int\limits_{\gamma(h)} \tilde{\omega}_X, \qquad Y(h) = -\int\limits_{\gamma(h)} \frac{y^2}{2}dx = \int\limits_{\gamma(h)} \tilde{\omega}_Y,$$

where c_0 is a non-zero constant and $\tilde{\omega}_X$, $\tilde{\omega}_Y$ are suitable $\mathbb{R}$-linear combinations of $\tilde{\omega}_j$, for $j = 0, 1, 2$. Obviously

$$M'(h) = \int\limits_{\gamma(h)} \omega_M, \quad X'(h) = \int\limits_{\gamma(h)} \omega_X, \quad Y'(h) = \int\limits_{\gamma(h)} \omega_Y.$$

According to Corollary 3.7 we have the equalities:
$\int\limits_{\gamma_3-\gamma_2} = C_X$, $\int\limits_{\gamma_3-\gamma_2} \omega_Y = C_Y$ where C_X and C_Y are real constants depending on ω_X and ω_Y. It follows that there exists non-zero linear combination of ω_X and ω_Y such that $\int\limits_{\gamma_3-\gamma_2} r\omega_X + s\omega_Y = 0$. Then the one-form $\omega_k = r\omega_X + s\omega_Y - k\omega_M$ have the property $\int\limits_{\gamma_3(h)-\gamma_2(h)} \omega_k = 0$, for any real k.

Theorem 3.11 implies that $\int\limits_{\gamma(h)} \omega_k = rX'(h) + sY'(h) - kM'(h)$ has at most one zero in the interval $(h_2, h_1]$ for any real k. $\square$

Corollary 4.3.

(i) – $rX + sY - kM$ has at most two zeros (counted with their multiplicities) in the interval $(h_2, h_1]$. One of them is h_1. Then $rX + sY - kM$ has at most one zero in the interval (h_2, h_1), or in other words $\dfrac{rX + sY}{M}$ is a monotone function;

(ii) – $r\xi'(h) + s\eta'(h) = \left(\dfrac{rX + sY}{M}\right)'$ has no zeros in the interval (h_2, h_1);

(iii) – $r\xi'(h_1) + s\eta'(h_1) \neq 0$.

Proof. *(ii)* Let us assume that there exists $h_0 \in (h_2, h_1)$ such that $\left(\dfrac{rX + sY}{M}\right)'(h_0) = 0$. Denote $\left(\dfrac{rX + sY}{M}\right)(h_0)$ by k. Then we have

$$\begin{aligned}\left(\frac{rX + sY}{M}\right)'(h_0) &= \frac{rX' + sY'}{M}(h_0) - \frac{(rX + sY)M'}{M^2}(h_0) \\ &= \frac{rX' + sY'}{M}(h_0) - \frac{kMM'}{M^2}(h_0) = \frac{rX' + sY' - kM'}{M}(h_0) = 0.\end{aligned}$$

It follows that $(rX' + sY' - kM')(h_0) = 0$. Then $rX + sY - kM$ has double zero at h_0 which contradict (i). We conclude that our assumption is false.

(iii) $M'(h_1) \neq 0$. The choice of coordinate system implies the following equalities: $(rX + sY)(h_1) = (rX' + sY')(h_1) = 0$. According to (i) $(rX'' + sY'')(h_1) \neq 0$. Now staightforward computation gives

$$(r\xi' + s\eta')(h_1) = \frac{(rX'' + sY'')}{2M'}(h_1) \neq 0. \quad \square$$

Combining (*ii*) and (*iii*) we obtain that $r\xi' + s\eta'$ has no zeros in the interval $(h_2, h_1]$. It follows that $(\xi'(h))^2 + (\eta'(h))^2$ has no zeros in the same interval. Thus we have proved our theorem.

5. An upper bound for the number of zeros of $I(h)$ in $\overline{\Delta} = [h_2, h_1]$. The ultimate goal of this section is to explain how one can get the statement of Theorem 1.4. In order to do that we must determine the index of the point h_2 (the value of H at separatrix loop level). The problem has been solved in the article of Horozov, Iliev [5]. In a neighborhood of h_2 in a complex domain the Abelian integral $I(h) = \int\limits_{\gamma(h)}$ has the following asymptotic expansion

$$(24)\quad I(h) = c_1 + c_2(h-h_2)\log(h-h_2) + c_3(h-h_2) + c_4(h-h_2)^2\log(h-h_2) + \cdots$$

The auxiliary result from [5] is the following

Theorem 5.1 (Horozov, Iliev [5]). *Let H be generic hamiltonian and $c_1 = c_2 = c_3 = 0$. Then ω is a Hamiltonian perturbation.*

For the proof of Theorem 1.4 we need the following lemma[2]. It allows us to push the "zeros" of $I(h)$ at h_2 in the interior of the interval (h_2, h_1) by perturbing the one–form ω.

Lemma 5.2 (Gavrilov, Horozov [3]). *Let $c_2 = 0$, $c_3 \neq 0$ in the expansion* (24). *For each sufficiently small ε there exists a sufficiently small perturbation $\omega_\varepsilon \in \Omega$ of the polynomial one-form $\omega \in \Omega$, such that the resulting Abelian integral $I_\varepsilon(h) = \int\limits_{\gamma(h)} \omega_\varepsilon$ has at least one zero in the interval $(h_2, h_2 + \varepsilon)$.*

In their article [3] Gavrilov and Horozov proved (using Theorem 5.1 and Lemma 5.2), that if "the number of zeros of $\dfrac{d}{dh}I$ in $(h_2, h_1]$" ≤ 3 then $\mathrm{Ind}_{[h_2,h_1]} I \leq 3$. We have proved the assumption from Section 3 (Theorem 1.3), so we get the statement of Theorem 1.4.

Since we have proved the two assumptions stated in the Preliminary section we obtain the statement of our main theorem.

Acknowledgments. Acknowledgments are due to Professor E. Horozov for the helpful discussion. We also wish to thank to the referee for his critical remarks which helped us to improve the text.

[2]The paper of Guckenheimer, Rand, Schlomiuk [7] essentially contains the statement of the lemma. For our puposes we use the formulation from [3].

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Received November 11, 1995
Revised April 4, 1996