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APPLICATION OF LYAPUNOV'S DIRECT METHOD TO THE EXISTENCE OF INTEGRAL MANIFOLDS OF IMPULSIVE DIFFERENTIAL EQUATIONS

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Abstract. The present paper investigates the existence of integral manifolds for impulsive differential equations with variable perturbations.

By means of piecewise continuous functions which are generalizations of the classical Lyapunov's functions, sufficient conditions for the existence of integral manifolds of such equations are found.

1. Preliminary notes and definitions. In recent years, the impulsive differential equations have been an object of numerous investigations (Bainov and Simeonov, 1989; Dishliev and Bainov, 1989; Lakshmikantham, Bainov and Simeonov, 1989; Kulev and Bainov 1990; Simeonov and Bainov, 1991, 1993)

In the present paper, some problems related to the existence of an integral manifold are considered. The main results are obtained by means of piecewise continuous functions which are analogous to the classical Lyapunov's functions.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with norm $\|\cdot\|$ and scalar product $\langle\cdot,\cdot\rangle$, and let $I=[0,\infty)$. We denote by $PC^k(J,\mathbb{R}^n)$, where $J\subset I$, $k=1,2,\ldots$, the space of all piecewise continuous functions $x:J\to\mathbb{R}^n$ such that:

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1. The set $D = \{t_i \in J, i = 1, 2, \ldots\}$ of all points of discontinuity of x has no finite point of accumulation.

2. For any $t_i \in D$, $x(t_i - 0) = x(t_i)$ and the limit $x(t_i + 0)$ is finite.

3. x is a C^k-continuous function in $J \setminus D$.

Let Ω be a domain in $\mathbb{R}^n$, $f : I \times \Omega \to \mathbb{R}^n$, $\Phi_i : \Omega \to \mathbb{R}^n$, $i = 1.2 \ldots$, $\tau_i : \Omega \to \mathbb{R}$.

Introduce the following assumptions:

H1. $f \in C^1(I \times \Omega, \mathbb{R}^n)$.

H2. $\Phi_i \in C^1(\Omega, R^n)$, $i = 1, 2, \ldots$.

H3. If $x \in \Omega$, then $x + \Phi_i(x) \in \Omega$, $L_i(x) = x + \Phi_i(x)$ are invertible in Ω, and $(L_i(x))^{-1} \in \Omega$ for $i = 1, 2, \ldots$.

H4. $\tau_i(x) \in C^1(\Omega, I)$ and $0 < \tau_1(x) < \tau_2(x) < \cdots$, $\lim_{i\to\infty} \tau_i(x) = \infty$ uniformly on $x \in \Omega$.

H5. The following inequalities hold

$$\sup\{\|f(t,x)\| : (t,x) \in I \times \Omega\} \leq A < \infty,$$

$$\sup\left\{\left\|\frac{\partial \tau_i(x)}{\partial x}\right\| : x \in \Omega, i = 1, 2, \ldots\right\} \leq B < \infty, \quad AB \leq 1,$$

$$\sup\left\{\left\langle \frac{\partial \tau_i(x + s\Phi_i(x))}{\partial x}, \Phi_i(x)\right\rangle : s \in [0,1],\ x \in \Omega,\ i = 1, 2, \ldots\right\} \leq 0.$$

Let the conditions **H1** – **H5** be satisfied. Consider the system of impulsive differential equations

$$\dot{x}(t) = f(t, x), \quad t \neq \tau_i(x), \tag{1}$$

$$\Delta x = \Phi_i(x), \quad t = \tau_i(x), \quad i = 1, 2, \ldots \tag{2}$$

We recall (see Bainov and Simeonov, 1989) and (Lakshmikantham, Bainov and Simeonov, 1989) that for any point $(t_0, x_0) \in I \times \Omega$ which is not in hypersurfaces $\sigma_i = \{(t,x) \in I \times \Omega : t = \tau_i(x)\}$, the function $x(t) = x(t; t_0, x_0)$ is called a solution of the system (1), (2) with initial condition $x(t_0 + 0) = x_0$, if:

1. $x(t; t_0, x_0) \in PC^1(J, \mathbb{R}^n)$ and, for any $t_i \in J$,

$$x(t_i + 0; t_0, x_0) = x(t_i; t_0, x_0) + \Phi_i(x(t_i; t_0, x_0))$$

2. $x(t)$ satisfies in $J \setminus D$ equation (1).

We also note too that from **H5** it follows that the phenomenon "beating" is absent for the system (1), (2) i.e. the integral orbit of any solution of the system (1), (2) meets each hypersurface σ_i, $i = 1, 2, \ldots$ at most once (Dishliev and Bainov, 1989).

We shall denote by $J^+ = J^+(t_0, x_0)$ (resp. $J^- = J^-(t_0, x_0)$) the maximal forward interval (t_0, ω) (resp. backward interval (α, t_0)) of existence of $x(t; t_0, x_0)$.

Next, by $\theta_+(t_0, x_0)$, $\theta_-(t_0, x_0)$, $\theta(x_0, t_0)$ we shall denote the integral orbit of the solution $x(t; t_0, x_0)$ for $t \in J^+$, $t \in J^-$ and $t \in J$ respectively.

Definition 1. *We call an arbitrary manifold M in the extended phase space:*
a) r-integral manifold, if from $(t_0, x_0) \in M$ it follows that $\theta_+(t_0, x_0) \subset M$;
b) l-integral manifold, if from $(t_0, x_0) \in M$ it follows that $\theta_-(t_0, x_0) \subset M$;
c) integral manifold, if M is r-integral manifold and l-integral manifold.

In this paper there are formulated and proved sufficient conditions for the existence of an integral manifolds of the system (1), (2).

The following is the main assumption in this paper:

H6. $J^+(t_0, x_0) = [t_0, \infty)$.

Definition 2. *The impulsive system* (1), (2) *is said to be complete if for any $(t_0, x_0) \in I \times \Omega$ such that $(t_0, x_0) \notin \sigma_i$, $i = 1, 2, \ldots$, it follows $J(t_0, x_0) = I$.*

It is easily verified that if the system (1), (2) is complete then conditions **H5** and **H6** hold.

Example 1. We consider the impulsive system (1), (2) with $\tau_i(x) = \tau_i$ i. e. the functions $\tau_i(x)$, $i = 1, 2, \ldots$ are independent from x. If the sequence $\{\tau_i\}$, $i = 1, 2, \ldots$ is a strictly increasing sequence, then the system (1), (2) is complete.

In what follows we shall use the class V_M of partially continuous auxiliary functions $V : I \times \Omega \to \mathbb{R}$ which are analogue to Lyapunov's functions.

Let M be an arbitrary manifold on the extended phase space of (1), (2). Put $\tau_0(x) = 0$ for $x \in \Omega$. Consider the sets

$$G_i = \{(t, x) \in I \times \Omega : \tau_{i-1} < t < \tau_i(x)\}, \quad i = 1, 2, \ldots.$$

Definition 3. *We shall say that the function $V : I \times \Omega \to \mathbb{R}$ belongs to the class V_M which kernel is the manifold M in the extend phase space of* (1), (2), *if the following conditions hold:*

1. $V(t, x) \in C^1 \left(\bigcup_{i=1}^{\infty} G_i, \mathbb{R} \right)$.

(3) 2. $V(t, x) = 0$, $(t, x) \in M$; $V(t, x) > 0$, $(t, x) \in (I \times \Omega) \setminus M$,

3. *For any* $i = 1, 2, \dots$ *and each point* $(\xi, \eta) \in \sigma_i$ *the following finite limits exist* $V(\xi - 0, \eta) = \lim_{\substack{(t,x)\to(\xi,\eta)\\(t,x)\in G_i}} V(t, x)$, $V(\xi + 0, \eta) = \lim_{\substack{(t,x)\to(\xi,\eta)\\(t,x)\in G_{i+1}}} V(t, x)$ *and moreover the equality* $V(\xi - 0, \eta) = V(\xi, \eta)$ *holds.*

Note that if $x = \varphi(t)$ is a solution of system (1), (2), then for $(t, x) \in \bigcup_{i=1}^{\infty} G_i$ (i.e. $t \neq \tau_i(x)$), the equality $\dot{V}(t, x) = D^+V(t, x)$ is satisfied, where

$$D^+V(t, x) = \limsup_{\Delta\to 0^+} \Delta^{-1} \{V(t + \Delta, \varphi(t + \Delta)) - V(t, \varphi(t))\}$$

is the upper right derivative of Dini of the function $V(t, \varphi(t))$.

Finally denote by K the class of all continuous and strictly increasing functions $a : I \to I$ such that $a(0) = 0$.

2. Main results.

Theorem 1. *Assume that:*

1. *The conditions* **H1** – **H6** *hold.*

2. *For the system* (1), (2) *there exists a function* $V \in V_M$ *with kernel the manifold* M*, so that the following relations are satisfied*

(4) $$\dot{V}(t, x) \leq 0 \quad \textit{for} \quad (t, x) \in \bigcup_{i=1}^{\infty} G_i,$$

$$\left(\textit{respectively} \;\; 0 \leq \dot{V}(t, x) \quad \textit{for} \quad (t, x) \in \bigcup_{i=1}^{\infty} G_i\right),$$

(5) $$V(t + 0, x + \Phi_i(x)) \leq V(t, x) \quad \textit{for} \quad (t, x) \in \sigma_i, \quad i = 1, 2, \dots$$

$$\left(\textit{respectively} \;\; V(t, x) \leq V(t + 0, x + \Phi_i(x)) \quad \textit{for} \quad (t, x) \in \sigma_i, \quad i = 1, 2, \dots\right).$$

Then M *is an* r*-integral manifold, (respectively an* l*-integral manifold) of the system* (1), (2).

P r o o f. We shall prove Theorem 1 for the case of an r-integral manifold. For l-integral manifold the proof is similar. Suppose that M is not an r-integral manifold.

Therefore there exists t', $t' > t_0$ such that, if $(t_0, x_0) \in M$ then $(t, x(t; t_0, x_0)) \in M$ for $t_0 \leq t \leq t'$ but $(t, x(t; t_0, x_0)) \notin M$ for $t > t'$. Then $V(t', x') = 0$, where $x' = x(t'; t_0, x_0)$. Moreover, $x(t) \in PC^1(J^+(t_0, x_0), \mathbb{R}^n)$. Then for t' the following two cases are possible:

a) If $t' \in \sigma_i$, $i = l, l+1, \ldots$, then $(t', x(t'+0; t_0, x_0)) \notin M$ and from Definition 3 it follows that $V(t'+0, x(t'+0; t_0, x_0)) > 0$.

Consequently $0 = V(t', x') < V(t'+0, x(t'+0; t_0, x_0))$ which contradicts (4).

b) If $t' \notin \sigma_i$, $i = l, l+1, \ldots$ then there exists $t'' \in J^+(t', x')$, $t'' > t'$ such that $(t'', x(t''; t', x')) \notin M$. From (4) and (5) it follows that the function is not increasing in (t_0, ∞).

From (3) it follows that $V(t'', x(t''; t', x')) > 0$ so $V(t'', x(t''; t', x')) > V(t', x')$ for $t'' > t'$ which contradicts the fact that the function is not increasing in (t_0, ∞).

From a) and b), it follows that M is an r-manifold. □

Theorem 2. *Assume that:*

1. *The conditions* **H1** – **H6** *hold.*

2. *There exist a function $V \in V_M$ and a function $c \in K$ such that the following relations are satisfied*

$$
\begin{aligned}
\dot{V}(t,x) &\le -c(\|x\|) \quad \textit{for} \quad (t,x) \in \bigcup_{i=0}^{\infty} G_i \\
\Big(\textit{resp} \quad \dot{V}(t,x) &\ge c(\|x\|) \quad \textit{for} \quad (t,x) \in \bigcup_{i=0}^{\infty} G_i\Big), \\
V(t+0, x+\Phi_i(x)) &\le V(t,x) \quad \textit{for} \quad (t,x) \in \sigma_i, \ i = 1,2,\ldots \\
\Big(\textit{resp} \quad V(t,x) &\le V(t+0, x+\Phi_i(x)) \quad \textit{for} \quad (t,x) \in \sigma_i, \ i = 1,2,\ldots\Big).
\end{aligned}
$$

Then M is an r-integral manifold, (l-integral manifold) of the system (1), (2).

Proof. The proof of Theorem 2 is analogous to the proof of Theorem 1. □

Theorem 3. *Assume the following:*

1. *The conditions* **H1** – **H6** *hold.*

2. *There exist functions $V \in V_M$ and $W \in V_M$ such that the following relations are satisfied:*

$$
\begin{aligned}
\dot{V}(t,x) &\le 0 \quad \textit{for} \quad (t,x) \in \bigcup_{i=0}^{\infty} G_i, \\
\dot{W}(t,x) &\ge 0 \quad \textit{for} \quad (t,x) \in \bigcup_{i=0}^{\infty} G_i, \\
V(t+0, x+\Phi_i(x)) &\le V(t,x) \quad \textit{for} \quad (t,x) \in \sigma_i, \ i = 1,2,\ldots, \\
W(t,x) &\le W(t+0, x+\Phi_i(x)) \quad \textit{for} \quad (t,x) \in \sigma_i, \ i = 1,2,\ldots.
\end{aligned}
$$

Then M is an integral manifold of (1), (2).

Proof. The proof of Theorem 3 follows from Theorem 1. □

Example 2. We consider the following system of impulsive differential equations

$$(6)\qquad \begin{cases} \dfrac{dy}{dt} = -y - t^2\sqrt{y}z^2, & \dfrac{dz}{dt} = t^2y^{-2}(z-2),\ t \neq i, \\ \Delta y = -\dfrac{1}{2} & \Delta z = 0,\ t = i,\ i = 1, 2, \dots \end{cases}$$

where $t \in I,\ y \in I,\ z \in I.$

Take the manifold

$$(7)\qquad M = \Big\{(t, y, z) \in I^3\ :\ z = 2,\ t > 0,\ y > 0\Big\}$$

and the functions

$$V(t, y, z) = \left(\frac{3}{4}\right)^i \exp\left\{-\left(\frac{t}{y}\right)^2\right\}(z-2)^2$$

and

$$W(t, y, z) = (z-2)^2.$$

Then

$$(8)\qquad \begin{aligned} \dot V(t, y, z) &= \left(\frac{3}{4}\right)^i \left(-2ty^{-2}\exp\left\{-\left(\frac{t}{y}\right)^2\right\}(z-2)^2\right) + \\ &+2\left(\frac{3}{4}\right)^i \exp\left\{-\left(\frac{t}{y}\right)^2\right\}(z-2)t^2y^{-2} + \left(\frac{3}{4}\right)^i 2t^2y^{-3}\exp\left\{-\left(\frac{t}{y}\right)^2\right\}(z-2)^2\times \\ &\times(-y - t^2\sqrt{y}z^2) = 2\left(\frac{3}{4}\right)^i ty^{-2}\exp\left\{-\left(\frac{t}{y}\right)^2\right\}(z-2)^2\left(-1 - t^3y^{-\frac{1}{2}}z^2\right) \le 0, \\ &i < t < i+1,\quad i = 1, 2, \dots,\quad y > 0,\quad z > 0. \end{aligned}$$

On the other hand

$$(9)\qquad \dot W(t, y, z) = 2(z-2)^2t^2y^{-2} \ge 0,\ i < t < i+1,\ i = 1, 2, \dots,\ y > 0,\ z > 0,$$

$$(10)\qquad V\left(i+0, y - \frac{1}{2}, z\right) \le V(i, y, z),\ y > 0,\ z > 0,\ i = 1, 2, \dots,$$

$$(11)\qquad W\left(i+0, y - \frac{1}{2}, z\right) = W(i, y, z),\ y > 0,\ z > 0,\ i = 1, 2, \dots.$$

From (8), (9), (10) and (11) it follows that the hypotheses of Theorem 3 are satisfied. Then (7) is an integral manifold of the system (6).

Now we consider the function

$$(12) \qquad L(t,s,x) = \begin{cases} V(t,x), \ t > s, \ (t,x) \in I \times \Omega, \ (s,x) \in I \times \Omega, \\ W(t,x), \ t < s, \ (t,x) \in I \times \Omega, \ (s,x) \in I \times \Omega, \\ \max\{V(t,x), W(t,x)\}, \ (t,x) \in I \times \Omega, \\ \max\{V(t+0, x+\Phi_i(x)), W(t+0, x+\Phi(x))\}, \\ \qquad (t,x) \in \sigma_i, \ i = 1,2,\dots, \end{cases}$$

where $V(t,x)$ and $W(t,x)$ are as in Theorem 3.

Theorem 4. *Let the conditions* **H1** – **H6** *hold. Then a manifold M from the extended phase space of* (1), (2) *is an integral manifold of* (1), (2), *if and only if there exists a function $L(t,s,x)$ of form* (12) *such that:*

$$\begin{aligned} \dot{L}(t,s,x) \le 0 \quad &\textit{for} \quad t > s, \ (t,x) \in \bigcup_{i=1}^{\infty} G_i, \ (s,x) \in \bigcup_{i=1}^{\infty} G_i, \\ L(t_i+0, s+0, x+\Phi_i(x)) \le L(t_i,s,x) \quad &\textit{for} \quad t > s, \ (t,x) \in \sigma_i, i = 1,2,\dots, \\ 0 \le \dot{L}(t,s,x) \quad &\textit{for} \quad t < s, \ (t,x) \in \bigcup_{i=1}^{\infty} G_i, \ (s,x) \in \bigcup_{i=1}^{\infty} G_i, \\ L(t_i,s,x) \le L(t_i+0, s+0, x+\Phi_i(x)) \quad &\textit{for} \quad t < s, \ (t,x) \in \sigma_i, i = 1,2,\dots. \end{aligned}$$

Proof. The proof of Theorem 4 follows from (12) and Theorem 3. □

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