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SHARP BOUNDS ON THE NUMBER OF RESONANCES FOR SYMMETRIC SYSTEMS II. NON-COMPACTLY SUPPORTED PERTURBATIONS

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Abstract. We extend the results in [5] to non-compactly supported perturbations for a class of symmetric first order systems.

The purpose of this note is to extend the results obtained in [5] to non-compactly supported perturbations. Consider in $\mathbf{R}^n$, $n \geq 3$ odd, a first order matrix-valued differential operator of the form $\sum_{j=1}^n A_j^0 D_{x_j}$, A_j^0 being constant Hermitian $d \times d$ matrices, and denote by G_0 its selfadjoint realization on $H = L^2(\mathbf{R}^n; \mathbf{C}^d)$. Suppose that the matrix $A(\xi) = \sum_{j=1}^n A_j^0 \xi_j$, $\xi \in \mathbf{R}^n \setminus 0$, is invertible for all ξ, i.e. the operator G_0 is an elliptic one. Consider the operator $\sum_{j=1}^n A_j(x) D_{x_j} + B(x)$, where $A_j(x) \in C^1(\mathbf{R}^n, \mathbf{C}^d)$, $B(x) \in C^0(\mathbf{R}^n, \mathbf{C}^d)$ satisfy for $|x| \gg 1$:

$$\sum_{j=1}^n |A_j(x) - A_j^0| + |B(x)| \leq C e^{-\gamma \langle x \rangle^{1+\varepsilon}}, \quad C, \gamma, \varepsilon > 0,$$

where $\langle x \rangle = (1 + |x|^2)^{1/2}$. Suppose that this operator admits an unique elliptic selfadjoint extension (denoted by G) on H. As we shall see later on, for any $a \gg 1$, the modified resolvent

$$R_a(z) = e^{-a\langle x \rangle}(G - z)^{-1} e^{-a\langle x \rangle} : H \to H$$

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admits a meromorphic continuation from $\Im z < 0$ to $\Im z < C_1 a$ with some constant $C_1 > 0$ independent of a. The poles of this continuation are called resonances and the multiplicity of a resonance $\lambda \in \mathbf{C}$, $\Im\lambda < C_1 a$, is defined as the rank of the residue of $R_a(z)$ at $z = \lambda$. As in [1], one sees that these definitions are independent of a. Denote by $N(r)$ the number of the resonances of G, counted with their multiplicities, in $\{z \in \mathbf{C} : |z| \le r\}$. Our main result is the following theorem.

Theorem. *Under the above assumptions, the counting function $N(r)$ satisfies the bound*

$$N(r) \le C r^{n(1+\varepsilon)/\varepsilon} + C, \tag{1}$$

with some constant $C > 0$ independent of r.

Remark. Such a bound has been recently obtained by Sa Barreto and Zworski [1] for the Shrödinger operator, while for metric perturbations of the Laplacian they have obtained a more rough but still polynomial upper bound. Their method however uses essentially the fact that the free operator is the Laplacian and no longer works for general symmetric systems.

Proof. Denote by $R_0(z)$ the outgoing free resolvent of G_0 defined for $\Im z < 0$ by

$$R_0(z) = -i \int_0^\infty e^{-itz} e^{itG_0} \mathrm{d}t.$$

Let us first see that the operator

$$R_{0,a}(z) = e^{-a\langle x\rangle}(G_0 - z)^{-1} e^{-a\langle x\rangle} : H \to H$$

admits an analytic continuation from $\Im z < 0$ to $\Im z < C_1 a$ with some $C_1 > 0$ independent of a, and

$$\|R_{0,a}(z)\| \le C/a, \quad \Im z < C_1 a, \tag{2}$$

where $\|\cdot\|$ denotes the norm in $\mathcal{L}(H,H)$. In view of Huygens' principle there exist $C', C'' > 0$ such that

$$\chi(|x| \le C't) e^{itG_0} \chi(|x| \le C''t) = 0$$

for $t > 0$ where $\chi(M)$ denotes the characteristic function of M. Hence,

$$\|e^{-a\langle x\rangle} e^{itG_0} e^{-a\langle x\rangle}\| \le \|e^{-a\langle x\rangle}\chi(|x| \ge C't) e^{itG_0} e^{-a\langle x\rangle}\|$$

$$+ \quad \|e^{-a\langle x\rangle}\chi(|x| \le C't) e^{itG_0} \chi(|x| \ge C''t) e^{-a\langle x\rangle}\| \le 2e^{-Cat},$$

which gives

$$\|R_{0,a}(z)\| \le 2\int_0^\infty e^{t\Im z - Cat} \mathrm{d}t \le 2\int_0^\infty e^{-Cat/2} \mathrm{d}t \le C'''/a,$$

provided $\Im z < Ca/2$, which implies (2) with $C_1 = C/2$.

Choose functions $\tilde{\chi}_1, \tilde{\chi}_2 \in C_0^\infty(\mathbf{R}^n)$, $\tilde{\chi}_1 = 1$ for $|x| \le 1$, $\tilde{\chi}_1 = 0$ for $|x| \ge 2$, $\tilde{\chi}_2 = 1$ for $|x| \le 3$, $\tilde{\chi}_2 = 0$ for $|x| \ge 4$, and set $\chi_1 = \tilde{\chi}_1(x/r)$, $\chi_2 = \tilde{\chi}_2(x/r)$, where $r = C_2 a^{1/\varepsilon}$ with $C_2 > 0$ to be chosen later on. Set also $V_j = (1-\chi_j)(G_0 - G)$, $j = 1, 2$. For $\Im z < 0$, following [2], [3], we have

$$(G-z)(1-\chi_1)R_0(z)(1-\chi_2)$$

$$= 1 - \chi_2 - [G, \chi_1]R_0(z)(1-\chi_2) - V_1 R_0(z)(1-\chi_2).$$

Combining this with the identity

$$1 = (G-z)R(z_0) + (z-z_0)R(z_0),$$

where $z_0 = -iC_3 a$, with a constant $C_3 > 0$ to be chosen later on, we obtain

$$1 = (G-z)(R(z_0) + (z-z_0)(1-\chi_1)R_0(z)(1-\chi_2)R(z_0))$$

$$+(z-z_0)\chi_2 R(z_0) + (z-z_0)[G,\chi_1]R_0(z)(1-\chi_2)R(z_0) \tag{3}$$

$$+(z-z_0)V_1 R_0(z)(1-\chi_2)R(z_0).$$

On the other hand,

$$(G_0 - z_0)(1-\chi_2)R(z_0) = 1 - \chi_2 - [G_0, \chi_2]R(z_0) - V_2 R(z_0),$$

that is,

$$(1-\chi_2)R(z_0) = R_0(z_0)(1-\chi_2) - R_0(z_0)[G_0,\chi_2]R(z_0) - R_0(z_0)V_2 R(z_0). \tag{4}$$

By (3) and (4) we get for $\Im z < 0$:

$$R(z)(1 - (z-z_0)\chi_2 R(z_0) - ([G,\chi_1](R_0(z) - R_0(z_0)) - (z-z_0)V_1 R_0(z_0)R_0(z))$$

$$(1 - \chi_2 - [G_0,\chi_2]R(z_0) - V_2 R(z_0)))$$

$$= R(z_0) + (1-\chi_1)(R_0(z) - R_0(z_0))(1 - \chi_2 - [G_0,\chi_2]R(z_0) - V_2 R(z_0)).$$

Multiplying the both sides of this identity by $e^{-a\langle x\rangle}$ gives

$$R_a(z)(1 - K(z)) = K_1(z), \tag{5}$$

where $K(z) = K_2(z) + K_3(z) + K_4(z)$,

$$K_1(z) = R_a(z_0) + (1-\chi_1)(R_{0,a}(z) - R_{0,a}(z_0))K_5,$$

$$K_2(z) = (z - z_0)\chi_2 e^{a\langle x\rangle} R(z_0) e^{-a\langle x\rangle},$$
$$K_3(z) = e^{a(1+\delta)\langle x\rangle}[G,\chi_1](R_{0,a\delta}(z) - R_{0,a\delta}(z_0))e^{-a(1-\delta)\langle x\rangle} K_5,$$
$$K_4(z) = (z - z_0)e^{a\langle x\rangle} V_1 R_0(z_0) e^{a\langle x\rangle} R_{0,a}(z) K_5,$$
$$K_5 = 1 - \chi_2 - [G_0,\chi_2]e^{a\langle x\rangle} R(z_0) e^{-a\langle x\rangle} - e^{a\langle x\rangle} V_2 R(z_0) e^{-a\langle x\rangle}.$$

Observe now that

$$\|e^{\alpha a\langle x\rangle} R(z_0) e^{-\alpha a\langle x\rangle}\|_{\mathcal{L}(H,H^s)} \le C' a^{s-1}, \quad s = 0,1, \quad \alpha = \pm 1, \tag{6}$$

provided C_3 is large enough, with some $C' > 0$ independent of a. Set

$$W(x) = [G, e^{\alpha a\langle x\rangle}]e^{-\alpha a\langle x\rangle}.$$

Clearly,

$$|W(x)| \le C'' a, \quad \forall x, \tag{7}$$

with a constant $C'' > 0$ independent of a and x. We have

$$(G - z_0)e^{\alpha a\langle x\rangle} R(z_0) e^{-\alpha a\langle x\rangle} = 1 + W e^{\alpha a\langle x\rangle} R(z_0) e^{-\alpha a\langle x\rangle},$$

and hence

$$e^{\alpha a\langle x\rangle} R(z_0) e^{-\alpha a\langle x\rangle} = R(z_0) + R(z_0) W e^{\alpha a\langle x\rangle} R(z_0) e^{-\alpha a\langle x\rangle}.$$

It follows from this representation, the ellipticity of G, and the estimate

$$\|R(z_0)\| \le (C_3 a)^{-1} \tag{8}$$

that (6) with $s = 1$ is a consequence of (6) with $s = 0$. On the other hand, by the above representation we deduce

$$e^{\alpha a\langle x\rangle} R(z_0) e^{-\alpha a\langle x\rangle} = (1 - R(z_0)W)^{-1} R(z_0),$$

and hence (6) with $s = 0$ follows from (7) and (8) provided C_3 is large enough. Clearly, an analogue of (6) holds with G replaced by G_0. Moreover, K_5 is bounded on H uniformly in a, provided C_2 is large enough. Thus, we conclude that $K(z)$ is holomorphic in $\Im z < C_1 a$ with values in the compact operators on H, $K(z_0) = 0$. Hence, $(1 - K(z))^{-1}$ forms a meromorphic family which in view of (5) provides the desired meromorphic continuation of $R_a(z)$.

In view of (2) and (6) we have

$$\|K_4(z)\| \le C|z - z_0| \, \|(1 - \chi_1)e^{-\gamma\langle x\rangle^{1+\varepsilon} + 2a\langle x\rangle}\|$$

$$\|e^{\gamma\langle x\rangle^{1+\varepsilon}-a\langle x\rangle}(G_0-G)R_0(z_0)e^{a\langle x\rangle}\|\ \|R_{0,a}(z)\|$$
$$\leq Ce^{-C'a^{(1+\varepsilon)/\varepsilon}}\leq 1/4$$

for $|z-z_0|\leq \tilde{C}a$ with $\tilde{C}=C_3+C_1\delta/2$ and C_2 large enough. Similarly,

$$\|K_3(z)\|\leq \|e^{a(1+\delta)\langle x\rangle}[G,\chi_1]\|$$
$$(\|R_{0,a\delta}(z)\|+\|R_{0,a\delta}(z_0)\|)\|e^{-a(1-\delta)\langle x\rangle}K_5\|$$
$$\leq C_1e^{a(1+\delta)\langle 2r\rangle-a(1-\delta)\langle 3r\rangle}\leq 1/4,$$

for $|z-z_0|\leq \tilde{C}a$, provided $\delta>0$ is small enough, independent of a, and a large enough. Hence, for $|z-z_0|\leq \tilde{C}a$, we have

$$R_a(z)(1-P(z))=P_1(z), \tag{9}$$

where

$$P=K_2(1-K_3-K_4)^{-1},\quad P_1=K_1(1-K_3-K_4)^{-1}.$$

Denote by Δ^r the Dirichlet realization of the Laplacian in $|x|\leq r$. In view of (6) the characteristic values of K_2 satisfy

$$\mu_j(K_2(z))\leq C|z-z_0|\mu_j((1-\Delta^r)^{-1/2})\leq C'|z-z_0|rj^{-1/n},$$

and hence

$$\mu_j(P(z))\leq C''a^{(1+\varepsilon)/\varepsilon}j^{-1/n} \tag{10}$$

for $|z-z_0|\leq \tilde{C}a$. Hence, $P(z)^{n+1}$ is trace class, and by (9) and the appendix in [4] we conclude that the resonances in $|z-z_0|\leq \tilde{C}a$ are among the zeros of the function

$$h_a(z)=\det(1-P(z)^{n+1}).$$

By (10), for $|z-z_0|\leq \tilde{C}a$, we get

$$|h_a(z)|\leq \prod_{j=1}^{\infty}\Big(1+\mu_j(P(z)^{n+1})\Big)$$

$$\leq \prod_{j=1}^{\infty}(1+Ca^{(n+1)(1+\varepsilon)/\varepsilon}j^{-(n+1)/n})\leq e^{Ca^{n(1+\varepsilon)/\varepsilon}}. \tag{11}$$

Now, since $h_a(z_0)=1$, by Jensen's inequality and (11) we conclude

$$N(C'a)\leq C''a^{n(1+\varepsilon)/\varepsilon}$$

provided $C' > 0$ is small enough, which completes the proof of the theorem. $\square$

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