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ANALYTIC RENORMINGS OF $C(K)$ SPACES

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The aim of our present note is to show the strength of the existence of an equivalent analytic renorming of a Banach space, even compared to C^∞-Fréchet smooth renormings.

It was Haydon who first showed in [8] that $C(K)$ spaces for K countable admit an equivalent C^∞-Fréchet smooth norm. Later, in [7] and [9] he introduced a large clams of tree-like (uncountable) compacts K for which $C(K)$ admits an equivalent C^∞-Fréchet smooth norm.

Recently, it was shown in [3] that $C(K)$ spaces for K countable admit an equivalent analytic norm. Our Theorem 1 shows that in the class of $C(K)$ spaces this result is the best possible.

Theorem 1. *Let $C(K)$ be a real Banach space. Then the following are equivalent:*

(i) *K is countable.*

(ii) *$C(K)$ has an equivalent analytic norm.*

Proof.

$(i) \Rightarrow (ii)$ can be found in [3].

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$(ii) \Rightarrow (i)$.

Since $C(K)$ has an equivalent analytic (hence C^1) norm, the space $C(K)$ is an Asplund space, so K is scattered and the dual of $C(K)$ is isometric to $\ell_1(K)$. Let $P(\cdot)$ be an arbitrary polynomial on $C(K)$ with values in $\ell_1(K)$. It is shown in [5], p.146 that $P(\cdot)$ is weakly-sequentially continuous (although they use the term "completely continuous" in their note). Since our $C(K)$ is an Asplund space, we have according to Proposition 2.12 of [1] (via an easy complexification argument) that $P(\cdot)$ is weakly uniformly continuous on bounded sets.

Let $\phi(\cdot)$ be an arbitrary real valued analytic function defined on the neighbourhood of $f_0 \in C(K)$. Then on some bounded neighbourhood U of f_0, ϕ is a uniform limit of polynomials:

$$\phi(\cdot) = \sum_{i=1}^{\infty} P_i(\cdot), \quad \text{and}$$

$$d\phi(\cdot) = \sum_{i=1}^{\infty} dP_i(\cdot)$$

where dP and $d\phi$ stand for the derivatives of the functions, mapping $C(K)$ into $\ell_1(K) = C(K)^*$. The function $d\phi$ is analytic in its domain. Thus $d\phi$ is weakly uniformly continuous when restricted to U. Lemma 2.2 of [2] shows that $d\phi(U)$ is norm-relatively compact set in $\ell_1(K)$ (again, a standard argument of passing to the complexified space is needed).

Let us suppose by contradiction that $\|\cdot\|$ is an equivalent analytic norm on $C(K)$ where K is an uncountable compact. Let $0 \neq f \in C(K)$. By the above considerations there exists some bounded open neighbourhood U of f in $C(K)$ such that $d\|\cdot\|(U)$ is norm relatively compact in $\ell_1(K)$. In particular, there exists a countable set $S \subset K$ such that $\operatorname{supp}(d\|g\|) \subset S$ for every $g \in U$. Choose $x_0 \in K \setminus S$, denote by $\phi_{x_0}(\cdot)$ the x_0-coordinate of an element in $\ell_1(K)$. We have:

$$\phi_{x_0}(d\|g\|) = 0 \text{ for every } \ g \in U.$$

From the analyticity of $d\|\cdot\|$ away from the origin we obtain:

$$\phi_{x_0}(d\|g\|) = 0 \text{ for every } 0 \neq g \in C(K). \tag{1}$$

Denote $e_{x_0} \in \ell_1(K)$ the evaluation map at x_0. By the Bishop-Phelps theorem, the set $\{d\|g\|, 0 \neq g \in C(K)\}$ is dense in the dual unit sphere of $\|\cdot\|$ and $\|e_{x_0}\|^{-1}e_{x_0}$ belongs to this unit sphere. However (1) implies:

$$\|d\|g\| - \|e_{x_0}\|^{-1}e_{x_0}\|_1 \geq \|e_{x_0}\|^{-1}\|e_{x_0}\|_1$$

for every $0 \neq g \in C(K)$, contradiction. The proof is finished. $\square$

It is clear that an analytic norm on a Banach space is always rotund. However, it was proved in [6] that the existence of an equivalent C^2-Fréchet smooth and rotund norm on $c_0(\Gamma)$ implies that Γ is countable. It is therefore natural to ask whether one can get similar results is the class of $C(K)$ spaces. Since $c_0(K \setminus K')$ is a closed linear subspace of $C(K)$, it follows that for a space $C(K)$ the existence of an equivalent C^2-Fréchet smooth and rotund norm implies that $K \setminus K'$ is countable. This is equivalent to K being separable. However, there are examples of uncountable and separable compacts for which $K^{(\omega_0)} = \emptyset$ (e.g. [4], p. 260). The following simple proposition shows that on the corresponding $C(K)$ spaces there exist C^∞-smooth and rotund norms.

Proposition 2. *Let K be a separable and scattered compact such that $K^{(\omega_0)} = \emptyset$. Then $C(K)$ admits an equivalent C^∞-smooth and rotund norm.*

P r o o f. By Theorem 4.1.8 of [4], these spaces admit an equivalent C^∞-smooth norm $\|\cdot\|$. Denote for every $x_i \in K \setminus K'$, $i \in \mathbb{N}$ by δ_{x_i}, the corresponding Dirac functional from $\ell_1(K)$. Put:

$$|||\cdot|||^2 = \|\cdot\|^2 + \sum_{i=1}^{\infty} \frac{1}{2^i} \delta_{x_i}^2(\cdot).$$

It is easy to cheek that $|||\cdot|||$ has the required properties.

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REFERENCES

[1] R. M. Aron, C. Hervés, M. Valdivia. Weakly Continuous Mappings on Banach Spaces. *J. Funct. Anal.*, **52** (1983), 189-204.

[2] R. M. Aron, J. B. Prolla. Polynomial Approximation of Differentiable Function on Banach Spaces. *J. Reine Angew. Math.* (Crelle), **313** (1980), 195-216.

[3] R. Deville, V. Fonf, P. Hájek. Analytic and Polyhedral Approximations of Norms in Separable Banach Spaces. Preprint.

[4] R. Deville, G. Godefroy, V. Zizler. Smoothness and Renormings in Banach Spaces. Pitman Monographs and Surveys in Pure and Applied Mathematics, 64, 1993.

[5] M. González, J. M. Gutiérrez. When Every Polynomial is Unconditionally Converging. *Arch. Math.*, **63** (1994), 145-151.

[6] P. Hájek. On Convex Functions in $c_0(\omega_)$. Preprint.

[7] R.G. Haydon. Infinitely Diferentiable Norms on Certain Banach Spaces. Preprint.

[8] R. G. Haydon. Normes et partitions de l'unité indéfiniment différentiables sur certains espaces de Banach. *C. R. Acad. Sci. Paris*, to appear.

[9] R. G. Haydon. Trees in Renorming Theory. Preprint.

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