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EQUISUMMABILITY THEOREMS FOR LAGUERRE SERIES

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Abstract. Here we prove results about Riesz summability of classical Laguerre series, locally uniformly or on the Lebesgue set of the function f such that $\left(\int_0^\infty (1+x)^{mp}|f(x)|^p dx\right)^{1/p} < \infty$, for some p and m satisfying $1 \le p \le \infty$, $-\infty < m < \infty$.

1. Introduction and statement of the main results. Consider the Laguerre series in the form

$$f(y) \sim \sum_{k=0}^{\infty} f_k \Phi_k^\delta(y), \quad f_k = \int_0^\infty f(y)\Phi_k^\delta(y)dy, \quad \delta \ge -\frac{1}{2},$$

and the corresponding partial sum

$$E_\lambda f(y) = \int_0^\infty e(\lambda, x, y) f(x) dx,$$

where

$$e(\lambda, x, y) = \sum_{\mu_k < \lambda} \Phi_k^\delta(x)\Phi_k^\delta(y),$$

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and

$$\mu_k = 4k+4, \quad \Phi_k^\delta(x) = \left[\frac{\Gamma(k+1)}{\Gamma(k+\delta+1)}\right]^{1/2} e^{-x^2/2}\sqrt{2}x^{\delta+1/2}L_k^\delta(x^2)$$

are the eigenvalues and orthonormalized eigenfuctions of the operator

$$A = -\frac{d^2}{dx^2} + x^2 + (\delta^2 - \frac{1}{4})x^{-2} + 2 - 2\delta \quad \text{in} \quad L^2(0,\infty).$$

Here $L_k^\delta(x) = (k!)^{-1}e^x x^{-\delta}\left(\frac{d}{dx}\right)^k (e^{-x}x^{k+\delta})$ are the Laguerre polynomials and $e(\lambda, x, y)$ is called the spectral function of A.

The Laguerre series are investigated in the classical Szegö book [7], where sufficient conditions are given on the behaviour of the function f at infinity so that the following equiconvergence result holds:

$$E_\lambda f(y) - \int_{y-\varepsilon}^{y+\varepsilon} e^0(\lambda, x, y)f(x)dx \to 0, \quad 0 < \varepsilon < y,$$

locally uniformly on $(0,\infty)$. Here $e^0(\lambda, x, y)$ is the spectral function of the main part $-\frac{d^2}{dx^2}$.

These conditions are significantly improved in [3], where the method of the spectral function is applied. To enlarge further the classes of functions we can consider the Riesz summability method. For other results see, for example [4], [5], [9] and the bibliography in [8].

Let

$$E_\lambda^\alpha f(y) = \sum \left(1 - \frac{\mu_k}{\lambda}\right)^\alpha f_k \Phi_k^\delta(y), \quad \mu_k < \lambda$$

be the Riesz means of order α. Then

$$E_\lambda^\alpha f(y) = \int_0^\infty I^\alpha e(\lambda, x, y)f(x)dx, \tag{1.1}$$

where

$$I^\alpha e(\lambda, x, y) = \int_0^\lambda \left(1 - \frac{\mu}{\lambda}\right)^\alpha de(\mu, x, y)$$

is the Riesz kernel of order α.

The main results proved in this paper are concerned with:

a) Equisummability locally uniformly for the functions f from the space L_m^p with a norm

$$\|f\|_{m.p} = \left(\int_0^\infty (1+x)^{mp}|f(x)|^p dx\right)^{\frac{1}{p}}, \quad 1 \le p \le \infty.$$

b) Summability on the Lebesgue set of the functions from the space L^p_m for $1 \le p < \infty$, $m \ge -m_0(\alpha, p)$ if $p \ne 4/3$ and $m > -m_0(\alpha, 4/3)$ if $p = 4/3$.

Here

$$m_0(\alpha, p) = 2\alpha + \min\left(\frac{1}{p}, 1 - \frac{1}{3p}\right), \quad \alpha > 0. \tag{1.2}$$

Note that in [9] a related result is proved for $\alpha > \frac{1}{6}$ and for the case $m = 0$.

c) Summability locally uniformly for the functions f with the properties: $f(x)$ and $f'(x)$ are $O(x^\beta)$ as $x \to \infty$ for $\beta < 2\alpha + 1$. The case $\alpha = 0$ is considered in [3].

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We start with theorems about equisummability locally uniformly, which means that as $\lambda \to \infty$

$$R^\alpha_\lambda f(y) \overset{\text{def}}{=} E^\alpha_\lambda f(y) - \int_{y-\varepsilon}^{y+\varepsilon} f(x) I^\alpha e(\lambda, x, y)\, dx \to 0, \tag{1.3}$$

uniformly with respect to $y \in [c, d]$ for any compact interval $[c, d] \subset (0, \infty)$, where $\varepsilon \in (0, c)$.

Theorem 1 (equisummability locally uniformly). *If $\alpha > 0$ then the convergence* (1.3) *is fulfilled in the following cases:*

(a) $f \in L^p_m$, $1 \le p < \infty$, $p \ne \frac{4}{3}$ *if* $m \ge -m_0(\alpha, p)$

(b) $f \in L^{4/3}_m$ *if* $m > -m_0\left(\alpha, \frac{4}{3}\right)$

(c) $f \in L^\infty_m$ *if* $m > -m_0(\alpha, \infty)$

(d) $f \in C_m$ *if* $m \ge -m_0(\alpha, \infty)$.

Here C_m is the subspace of L^∞_m consisting of all continuous functions f such that $x^m f(x) \to 0$ as $x \to \infty$ and $f(x) \to 0$ as $x \to 0$.

Theorem 2. *Let $f \in L^1_{loc}[0, \infty)$ and the derivative $f'(x)$ exists for $x > A_f$. If $f(x)$ and $f'(x)$ are $O(x^\beta)$, $x \to \infty$ for $\beta < 2\alpha + 1$, $\alpha > 0$, then the convergence* (1.3) *is true.*

Corollary 1 (equisummability on the Lebesgue set). *Under the conditions of Theorems* 1 *or* 2 *we have*

$$E^\alpha_\lambda f(y) - \int_{y-\varepsilon}^{y+\varepsilon} f(x) I^\alpha e^0(\lambda, x, y)\, dx \to 0, \tag{1.4}$$

where $y \in (0, \infty)$ *is on the Lebesgue set of the function* f *and* $0 < \varepsilon < y$. *Here*

$$e^0(\lambda, x, y) = \frac{1}{\pi} \cdot \frac{\sin \sqrt{\lambda}(x-y)}{x-y}$$

and

$$I^\alpha e^0(\lambda, x, y) = \lambda^{1/2} F_\alpha(\sqrt{\lambda}|x-y|), \tag{1.5}$$

where

$$F_\alpha(s) = d_\alpha s^{-\frac{1}{2}-\alpha} J_{1/2+\alpha}(s), \quad d_\alpha = 2^\alpha (2\pi)^{-\frac{1}{2}} \Gamma(\alpha+1). \tag{1.6}$$

Corollary 2 (summability on the Lebesgue set)**.** *Under the conditions of Theorem* 1 *or* 2 , $E_\lambda^\alpha f(y) \to f(y)$ *on the Lebesgue set of the function* f.

Corollary 3 (summability in L^q_{loc})**.** *Under the conditions of Theorem* 1 *or* 2, $E^\alpha_\lambda f \to f$ *in* L^q_{loc} $1 \leq q < \infty$ *if in addition* $f \in L^q_{loc}(0, \infty)$.

Corollary 4 (summability locally uniformly)**.** *Under the conditions of Theorem* 1 *or* 2, $E^\alpha_\lambda f(y) \to f(y)$ *locally uniformly if in addition* f *is continuous.*

Corollary 5 (localization principle)**.** *Let* $y > 0$, $\varepsilon > 0$ *be fixed. Then under the conditions of Theorem* 1 *or* 2, $E^\alpha_\lambda f \to 0$ *if* $f(x) = 0$ *for* $|x - y| < \varepsilon$.

2. Asymptotics of Riesz kernels. In proving the main results, stated in § 1, we shall apply the method of the spectral function as in [3] and especially [4], where this method was used to find the uniform asymptotics of the Riesz kernels of order α in the case of Hermite series.

First we state the uniform asymptotics of the Riesz kernels (1.1) which we need. It is convenient to consider also the functions

$$e_\alpha(\lambda, x, y) = \lambda^\alpha I^\alpha e(\lambda, x, y), \quad E_\alpha(\lambda, x, y) = e_\alpha(\lambda, \sqrt{\lambda}x, \sqrt{\lambda}y). \tag{2.1}$$

For our purposes it is sufficient to consider only the cases: $0 < a \leq x < \infty$, $0 < c \leq y \leq d < \infty$. It is convenient to split the interval $[a, \infty)$ into the intervals $[a, b]$, $[A, (1-\varepsilon)\sqrt{\lambda}]$, $[(1-\varepsilon)\sqrt{\lambda}, (1+\varepsilon)\sqrt{\lambda}]$, $[(1+\varepsilon)\sqrt{\lambda}, \infty)$.

Theorem 3. *Let* $0 < a \leq x \leq b$ *and* $0 < c \leq y \leq d$. *Then,*

$$|I^\alpha e(\lambda, x, y) - (I^\alpha e^0(\lambda, x, y) + C_\delta I^\alpha e^0(\lambda, x, -y))| \leq C(1 + \sqrt{\lambda}|x-y|)^{-\alpha-1}, \tag{2.2}$$

where $\alpha > 0$, $I^\alpha e^0(\lambda, x, y)$ *is given by* (1.5) *and* C_δ *is a constant.*

Here and later on, C is a positive constant, not depending on λ, x, y.

Theorem 4. *Let* $\frac{A}{\sqrt{\lambda}} \le x \le 1-\varepsilon$, $\frac{c}{\sqrt{\lambda}} \le y \le \frac{d}{\sqrt{\lambda}}$. *Then for* $A > d$, $\varepsilon > 0$ *we have the uniform asymptotics*

$$E_\alpha(\lambda, x, y) = F_\alpha(\lambda, x, y) + C_\delta F_\alpha(\lambda, x, -y), \tag{2.3}$$

where

$$F_\alpha(\lambda, x, y) = \lambda^{-1/2} \sum_{k=1}^{4} b_k(\lambda, x, y) e^{i\lambda\psi_k} + x^{-1-\alpha} O(\lambda^{-1}) \tag{2.4}$$

and

$$|b_k| \le C x^{-1-\alpha}, \quad |\partial_x b_k| \le C x^{-2-\alpha} \tag{2.5}$$

$$|\partial_x \psi_k|^2 = 1 - x^2, \quad |\partial_x^2 \psi_k|^2 \le C(1-x^2)^{-1}. \tag{2.6}$$

Theorem 5. *If* $1-\varepsilon \le x \le 1+\varepsilon$, $\frac{c}{\sqrt{\lambda}} \le y \le \frac{d}{\sqrt{\lambda}}$, *then there exists a positive number* $\varepsilon > 0$ *such that the uniform asymptotics* (2.3) *is satisfied, where*

$$F_\alpha(\lambda, x, y) = \sum_{k=0}^{\infty} (a_{1k}(\lambda, x, y)\lambda^{-k-1/3} + b_{1k}(\lambda, x, y)\lambda^{-k-2/3}) \tag{2.7}$$

and

$$a_{1k} = (a_k e^{\lambda A} + b_k e^{\lambda \overline{A}}) Ai(\lambda^{2/3} B), \quad b_{1k} = (c_k e^{\lambda A} + d_k e^{\lambda \overline{A}}) Ai'(\lambda^{2/3} B).$$

The functions $\lambda \to a_k, b_k, d_k, c_k$ *or their derivatives with respect to* x *are bounded. Here* $Ai(s) = \frac{1}{2\pi} \int e^{i(st + t^3/3)}$ *is the Airy function,* $A = A(x, y)$, $B = B(x, y)$ *are smooth,* $ReA = 0$ *and* $B(x, y) \sim C(y)(x^2 - 1)$ *as* $x^2 \to 1$, $c(y) > 0$.

Analogously to Theorem 6 [3] we have

Theorem 6. *Let* $x > 1 + \varepsilon$ *for some* $\varepsilon > 0$. *Then*

$$|E_\alpha(\lambda, x, y)| \le C(x^2-1)^{-1/4} \lambda^{-1/2} \exp(-C\varepsilon(x^2-1)^{1/2}\lambda), \quad C > 0.$$

As a consequence of Theorems 5 and 6 it follows

Corollary 6. *If* $x^2 > \lambda + \lambda^{1/3+\varepsilon}$, $\varepsilon > 0$ *then*

$$|I^\alpha e(\lambda, x, y)| \le C\lambda^{-\alpha-1/3} \exp\left(-C\lambda^{1/3}\left(\frac{x^2}{\lambda} - 1\right)^{1/2}\right).$$

From Theorem 5 and the asymptotics of the Airy function it follows

Corollary 7. *If* $1 - \varepsilon_1 < x^2 < 1 - \lambda^{-2/3+\varepsilon}$, $\varepsilon > 0$ *and* $\frac{c}{\sqrt{\lambda}} \le y \le \frac{d}{\sqrt{\lambda}}$, *then we have the uniform asymptotics* (2.3), *where*

$$F_\alpha(\lambda, x, y) = \lambda^{-1/2} \sum_{k=1}^{4} \left(a_k(1-x^2)^{-1/4} + b_k(1-x^2)^{1/4}\right) e^{i\lambda\psi_k} + (1-x^2)^{-1} O(\lambda^{-1}),$$

the functions $\lambda \to a_k(\lambda, x, y), b_k(\lambda, x, y)$ *or their derivatives over* x *are bounded, and* ψ_k *satisfy* (2.6).

3. Proof of Theorem 1. First, according to [7], we have $|e(\lambda, x, y)| \le c$ if $0 < x, y < c$, $|x - y| \ge \varepsilon > 0$, and consequently

$$|I^\alpha e(\lambda, x, y)| \le c \quad \text{if} \quad 0 < x, y < c, \quad |x - y| > \varepsilon > 0.$$

Since

$$R^\alpha_\lambda f(y) = \left(\int_0^{y-\varepsilon} + \int_{y+\varepsilon}^{\infty}\right) f(x) I^\alpha e(\lambda, x, y) dx$$

we can write

$$|R^\alpha_\lambda f(y)| \le c\left[\int_0^A |f(x)|dx + |K(\lambda, y)|\right], \tag{3.1}$$

where $c \le y \le d$ and for some large $A > 0$,

$$K(\lambda, y) = \int_A^\infty f(x) I^\alpha e(\lambda, x, y) dx. \tag{3.2}$$

Now let $K_i(\lambda, y) = \int a_i(\lambda, x) f(x) I^\alpha e(\lambda, x, y) dx$, where $a_i(\lambda, x)$ is the characteristic function of the set A_i and

$A_1 = \{x \in R_+,\ A^2 < x^2 < (1-\varepsilon)\lambda\}$, $A_2 = \{x \in R_+,\ (1-\varepsilon)\lambda < x^2 < \lambda - \lambda^{1/3}\}$,
$A_3 = \{x \in R_+,\ |x^2 - \lambda| < \lambda^{1/3}\}$, $A_4 = \{x \in R_+,\ \lambda + \lambda^{1/3} < x^2 < \lambda + \lambda^{1/3+\varepsilon}\}$,
$A_5 = \{x \in R_+,\ x^2 > \lambda + \lambda^{1/3+\varepsilon}\}$

The estimates below are uniform with respect to $y \in [c,d]$ and the number A is large enough, say $A > d+1$.

a) Estimate of $K_1(\lambda, y)$. Using Theorem 4 we have

$$|I^\alpha e(\lambda, x, y)| \leq C\lambda^{-\alpha/2} x^{-1-\alpha}, \quad x \in A_1.$$

Hence by the Hölder inequality,

$$\begin{aligned} |K_1(\lambda, y)| &\leq c\lambda^{-\alpha/2} \int a_1(\lambda, x)|f(x)|x^{-1-\alpha}dx \\ &\leq c\lambda^{-\alpha/2}\|f\|_{m,p} J(\lambda), \end{aligned}$$

where

$$J(\lambda) = \left(\int_1^{\sqrt{\lambda}} \sigma^{-(1+\alpha+m)p'} d\sigma\right)^{\frac{1}{p'}}, \quad \frac{1}{p'} + \frac{1}{p} = 1.$$

Therefore

$$|K_1(\lambda, y)| \leq c\|f\|_{m,p} \quad \text{if} \quad m \geq -2\alpha - 1/p,\ 1 \leq p < \infty \tag{3.3}$$

$$|K_1(\lambda, y)| \leq \lambda^{-\gamma}\|f\|_{m,\infty} \quad \text{if} \quad m \geq -2\alpha,\ \alpha > 0 \quad \text{for some } \gamma > 0. \tag{3.4}$$

b) Estimate of $K_2(\lambda, y)$. Using Theorem 5 and the estimates $|Ai(s)| \leq c|s|^{-1/4}$, $|Ai'(s)| \leq c(1+|s|)^{1/4}$, we have

$$|I^\alpha e(\lambda, x, y)| \leq c\lambda^{-\alpha-1/2}\left(1 - \frac{x^2}{\lambda}\right)^{-1/4}, \quad x \in A_2.$$

Therefore

$$|K_2(\lambda, y)| \leq c\lambda^{-\alpha-1/2-m/2}\|f\|_{m,p} J(\lambda),$$

where

$$J(\lambda) = \mathrm{J}^{\frac{1}{2p'}} \left(\int_{\lambda^{-2/3}}^1 \sigma^{-p'/4} d\sigma\right)^{\frac{1}{p'}},$$

hence

$$|K_2(\lambda, y)| \leq c\lambda^{-(m+m_0)/2}\|f\|_{m,p}, \quad p \neq 4/3 \tag{3.5}$$

$$|K_2(\lambda, y)| \leq c\lambda^{-(m+m_0)/2}(\log\lambda)^{1/4}\|f\|_{m,p}, \quad p = 4/3. \tag{3.6}$$

Here m_0 is given by (1.2).

c) Estimate of $K_3(\lambda, y)$. According to Theorem 5 we have

$$|I^\alpha e(\lambda, x, y)| \le \lambda^{-\alpha-1/3}, \quad x \in A_3.$$

Hence

$$|K_3(\lambda, y)| \le c\lambda^{-\alpha-1/3}\|f\|_{m,p} J(\lambda),$$

where

$$J(\lambda) = \left(\int a_3(\lambda, x) x^{-mp'} dx\right)^{\frac{1}{p'}} \le c\lambda^{-\frac{m}{2}-\frac{1}{6p'}}.$$

Therefore

$$|K_3(\lambda, y)| \le c\lambda^{-(m+m_0)/2}\|f\|_{m,p}. \tag{3.7}$$

d) Estimate of $K_4(\lambda, y)$. Theorem 6 implies

$$|I^\alpha e(\lambda, x, y)| \le c\lambda^{-\alpha-1/2}\left(\frac{x^2}{\lambda} - 1\right)^{-\frac{1}{4}}, \quad x \in A_4.$$

Hence

$$|K_4(\lambda, y)| \le c\lambda^{-\alpha-\frac{1}{2p}-\frac{m}{2}}\|f\|_{m,p} J(\lambda),$$

where

$$J(\lambda) = \left(\int_{\lambda^{-2/3}}^{\lambda^{-2/3+\varepsilon}} \sigma^{-p'/4} d\sigma\right)^{\frac{1}{p'}}.$$

Therefore

$$|K_4(\lambda, y)| \le c\lambda^{-(m+m_0)/2}\|f\|_{m,p}, \quad 1 \le p < \frac{4}{3} \tag{3.8}$$

$$|K_4(\lambda, y)| \le c\lambda^{-(m+m_0)/2}(\log\lambda)^{1/4}\|f\|_{m,p}, \quad p = \frac{4}{3} \tag{3.9}$$

$$|K_4(\lambda, y)| \le c\lambda^{-(m+m_0)/2-\gamma}\|f\|_{m,p}, \quad \text{if } p > \frac{4}{3} \text{ for some } \gamma > 0. \tag{3.10}$$

f) Estimate of $K_5(\lambda, y)$. Corollary 6 gives

$$|I^\alpha e(\lambda, x, y)| \le c\lambda^{-\alpha-1/3}\exp(-c\lambda^{\varepsilon/2}), \quad \text{if } x \in A_5, \quad x < \lambda$$

$$|I^\alpha e(\lambda,x,y)| \le c\lambda^{-\alpha-1/3}\exp(-c\sqrt{x}), \quad \text{if } \ x>\lambda, \quad c>0.$$

Hence we obtain

$$(3.11) \qquad |K_5(\lambda,y)| \le c\lambda^{-\gamma}\|f\|_{m,p} \quad \text{for some } \ \gamma>0.$$

Thus the estimates (3.3)–(3.11) give

$$(3.12) \qquad |R^\alpha_\lambda f(y)| \le c\|f\|_{m,p}, \quad \text{if } \ m\ge -m_0, \ 1\le p<\infty, \ p\ne 4/3$$

$$R^\alpha_\lambda f(y)\to 0 \ \text{ if } m>-m_0(\alpha,p) \quad \text{and} \quad p=4/3 \ \text{ or } \ p=\infty.$$

On the other hand it is not hard to see that

$$(3.13) \qquad R^\alpha_\lambda f\to 0 \quad \text{uniformly on } [c,d] \quad \text{if } f\in C_0^\infty(0,\infty).$$

Finally, if $f\in L^p_m$, $1\le p<\infty$ or $f\in C_m$, then we can find $g\in C_0^\infty$ such that $\|f-g\|_{m,p}<\varepsilon$. Then (3.12) implies $|R^\alpha_\lambda f|\le c\varepsilon+|R^\alpha_\lambda g|$, whence (3.13) gives $R^\alpha_\lambda f\to 0$ locally uniformly.

4. Proof of Theorem 2. We start with (3.1) and (3.2), where $1\le i\le 4$, $a_i(\lambda,x)$ is the characteristic function of the set B_i and $B_1=A_1$,

$$B_2=\left\{x \ : \ (1-\varepsilon)\lambda<x^2<\lambda-\lambda^{\frac{1}{3}+\varepsilon}\right\}, \quad B_3=\left\{x \ : \ |x^2-\lambda|<\lambda^{\frac{1}{3}+\varepsilon}\right\}, \quad B_4=A_5.$$

Now, let $B_i(\lambda,y)=K_i(\lambda,\sqrt{\lambda}y)$, $i=1,2$. Then

$$B_i(\lambda,y)=\lambda^{1/2-\alpha}\int_0^\infty a_i(\lambda,\sqrt{\lambda}x)f(\sqrt{\lambda}x)E_\alpha(\lambda,x,y)dx.$$

a) Estimate of $K_1(\lambda,y)$. Using Theorem 4 we can write

$$B_1(\lambda,y)=I(\lambda,y)+C_\delta I(\lambda,-y),$$

where

$$(4.1) \qquad I(\lambda,y)=\lambda^{1/2-\alpha}\int a_1(\lambda,\sqrt{\lambda}x)f(\sqrt{\lambda}x)F_\alpha(\lambda,x,y)dx$$

and $F_\alpha(\lambda,x,y)$ is given by (2.4). It is enough to find the asymptotics of $I(\lambda,y)$. We have by (2.4),

$$(4.2) \qquad I(\lambda,y)=\lambda^{-\alpha}\sum_{k=1}^4\int_0^\infty a_1(\lambda,\sqrt{\lambda}x)b_k(\lambda,x,y)e^{i\lambda\psi_k}f(\sqrt{\lambda}x)dx+R_1O(\lambda^{-\alpha-1/2})$$

where, using $f(x) = O(x^\beta)$, $x \to \infty$,

$$R_1 = \int a_1(\lambda, \sqrt{\lambda}x)|f(\sqrt{\lambda}x)|x^{-1-\alpha}dx \le C\lambda^{\beta/2}J(\lambda), \tag{4.3}$$

$$J(\lambda) = \int_{\lambda^{-1/2}}^{1} x^{-1-\alpha+\beta}dx \le c \begin{cases} \lambda^{-\beta/2+\alpha/2}, & \beta < \alpha \\ \log\lambda, & \beta = \alpha \\ 1, & \beta > \alpha \end{cases}.$$

Then integrating by parts and using (2.5), (2.6), we get for $\beta \le 2\alpha + 1$,

$$|I(\lambda, y)| \le C\lambda^{-\alpha-1/2+\beta/2}J(\lambda) + C\lambda^{-1/2}.$$

If $\beta < 2\alpha + 1$, we see that $|I(\lambda, y)| \le \lambda^{-\gamma}$ for some $\gamma > 0$, hence

$$I(\lambda, y) \to 0 \ \text{ locally uniformly}$$

or

$$K_1(\lambda, y) \to 0 \ \text{ locally uniformly.} \tag{4.4}$$

b) Estimate of $K_2(\lambda, y)$. We shall use Corollary 7. Then analogously to (4.1), (4.2) and (4.3) we see that it suffices to estimate

$$B(\lambda, y) = \lambda^{-\alpha}\int a_2(\lambda, \sqrt{\lambda}x)a(\lambda, x, y)(1-x^2)^{-1/4}f(\sqrt{\lambda}x)e^{i\lambda\psi}dx + O(\lambda^{-1/2-\alpha})R_2 \tag{4.5}$$

where $a(\lambda, x, y) = a_k(\lambda, x, y)$ and

$$R_2 = \int a_2(\lambda, \sqrt{\lambda})|f(\sqrt{\lambda}x)|(1-x^2)^{-1}dx \le c\lambda^{\beta/2}\log\lambda. \tag{4.6}$$

Let

$$I(\lambda) = \int a_2(\lambda, \sqrt{\lambda})a(\lambda, x, y)(1-x^2)^{-1/4}f(\sqrt{\lambda}x)e^{i\lambda\psi}dx.$$

Integrating by parts and using (2.6) we get

$$\begin{aligned}|I(\lambda)| \le \ & C\lambda^{-1}\int a_2(\lambda, \sqrt{\lambda}x)\left[\lambda^{1/2}|f'(\sqrt{\lambda}x)|(1-x^2)^{-3/4} + \right.\\ & \left. |f(\sqrt{\lambda}x)|(1-x^2)^{-7/4}\right]dx + C\lambda^{-1/2}.\end{aligned}$$

Since $1 - x^2 > \lambda^{-2/3+\delta}$ we obtain for $\beta > 0$, $\varepsilon > 0$,

$$|I(\lambda)| \le C\lambda^{-1/2+\beta/2}. \tag{4.7}$$

Thus (4.5), (4.6) and (4.7) imply

$$|B(\lambda, y)| \le C\lambda^{-\alpha-1/2+\beta/2} + C\lambda^{-\alpha-1/2+\beta/2}\log\lambda \le C\lambda^{-\gamma} \quad \text{for some} \quad \gamma > 0$$

since $\beta < 2\alpha + 1$. In other words,

$$|K_2(\lambda, y)| \le C\lambda^{-\gamma} \to 0 \ \text{ locally uniformly.} \tag{4.8}$$

c) Estimate of $K_3(\lambda, y)$. Theorem 5 gives

$$|K_3(\lambda, y) \le C\lambda^{-\alpha-1/3}\int_0^\infty a_3(\lambda, x)|f(x)|dx.$$

Hence

$$|K_3(\lambda, y)| \le C\lambda^{-\alpha+\beta/2-1/2+\varepsilon} \to 0 \ \text{ if } \ 0 < \varepsilon < \alpha - \frac{\beta}{2} + \frac{1}{2}. \tag{4.9}$$

Finally it is easy to prove (see 3.11) that

$$K_4(\lambda, y) \to 0 \quad \text{locally uniformly.} \tag{4.10}$$

Thus (3.1) and the estimates (4.4), (4.8), (4.9), (4.10) give

$$|R_\lambda^\alpha f(y)| \le C\int_0^A |f(x)|dx + o(1), \quad \text{locally uniformly.}$$

Now the proof finishes analogously to the proof of Theorem 1. □

5. Proof of Corollaries 1–4.

P r o o f o f C o r o l l a r y 1. Let $y \in (0, \infty)$ is on the Lebesgue set of the function f and $0 < \varepsilon < y$. Comparing (1.3), (1.4) we have only to prove

$$I(\lambda, y) = \int_{y-\varepsilon}^{y+\varepsilon} f(x)[I^\alpha e(\lambda, x, y) - I^\alpha e^0(\lambda, x, y)]dx \to 0. \tag{5.1}$$

Let $\tilde{f}(x) = f(x)\chi(x)$ and let $\chi(x)$ be the characteristic function of the set $(y - \varepsilon, y + \varepsilon)$. According to theorem 3,

$$|I^\alpha e(\lambda, x, y) - I^\alpha e^0(\lambda, x, y)| \le C[\lambda^{-\alpha/2} + H_\alpha(\sqrt{\lambda}|x - y|)], \quad \alpha > 0, \tag{5.2}$$

$0 < y - \varepsilon \le x \le y + \varepsilon$, where $H_\alpha(s) = (1 + s)^{-\alpha-1}$, $s > 0$. Since $\alpha > 0$, then $H_\alpha(s) \in L^1(R)$, hence Theorem 1.25 [6] gives

$$\int \tilde{f}(x)\sqrt{\lambda}H_\alpha(\sqrt{\lambda}|x - y|)dx \to \tilde{f}(y),$$

or

$$\int_{y-\varepsilon}^{y+\varepsilon} f(x)H_\alpha(\sqrt{\lambda}|x-y|)dx \to 0. \tag{5.3}$$

Evidently (5.1) follows from (5.2), (5.3). □

P r o o f of C o r o l l a r y 2. According to Corollary 1 we have to prove

$$I(\lambda, y) = \int_{y-\varepsilon}^{y+\varepsilon} f(x)I^\alpha e^0(\lambda, x, y) \to f(y),$$

where y is on the Lebesgue set of the function f and $0 < \varepsilon < y$. Using (1.5) and $\tilde{f}$, $F_\alpha(s) \in L^1(R)$ for $\alpha > 0$, we see that Theorem 1.25 [6] implies

$$I(\lambda, y) = \int_{-\infty}^{+\infty} \tilde{f}(x)\lambda^{1/2}F_\alpha(\sqrt{\lambda}|x-y|)dx \to \tilde{f}(y) = f(y). \quad \square$$

P r o o f of C o r o l l a r y 3. First we have according to Theorem 1 or 2 $R_\lambda^\alpha f \to 0$ in $L^q_{loc}(0,\infty)$. Thus according to (1.3) it is sufficient to prove

$$I(\lambda, y) = \int_{y-\varepsilon}^{y+\varepsilon} f(x)I^\alpha e(\lambda, x, y)dx \to f(y) \ \text{ if } \ L^q[c, d], \tag{5.4}$$

$0 < c < d$, $0 < \varepsilon < c$. Let $\tilde{f}(x) = f(x)\chi(x)$, where χ is the characteristic function of $(c-\varepsilon, c+\varepsilon)$. Hence we can write

$$I(\lambda, y) = J_1(\lambda, y) - J_2(\lambda, y) \quad \text{for} \quad c \le y \le d, \tag{5.5}$$

where

$$J_1(\lambda, y) = \int_0^\infty \tilde{f}(x)I^\alpha e(\lambda, x, y)dx,$$

$$J_2(\lambda, y) = \int_M \tilde{f}(x)I^\alpha e(\lambda, x, y)dx, \quad M = \{x \ : \ |x-y| > \varepsilon\} \cap (c-\varepsilon, d+\varepsilon).$$

According to Theorem 3 we have $|I^\alpha e(\lambda, x, y)| \le C\lambda^{-\alpha/2}$ if $c \le y \le d$, $x \in M$. Since $\alpha > 0$ it follows

$$J_2(\lambda, y) \to 0 \quad \text{uniformly in} \ \ c \le y \le d. \tag{5.6}$$

On the other hand, Theorem 1.25 [6] gives

$$\int \tilde{f}(x)\sqrt{\lambda}H_\alpha(\sqrt{\lambda}|x-y|)dx \to \tilde{f}(y) \quad \text{in} \quad L^q \quad \text{if} \ \ 1 \le q < \infty, H_\alpha \in L^1(R).$$

Therefore using (5.2) and $\alpha > 0$, we get

$$\int_0^\infty \tilde f(x) I^\alpha e(\lambda,x,y)dx - \int_{-\infty}^\infty I^\alpha e^0(\lambda,x,y)\tilde f(x)dx \to 0$$

in $L^q(c,d)$, $1 \le q < \infty$.

The same Theorem 1.25 [6] and (1.5), (1.6) imply

$$\int \tilde f(x) I^\alpha e^0(\lambda,x,y)dx \to \tilde f(y) \ \text{ in } \ L^q \ \text{ if } \ 1 \le q < \infty.$$

Therefore,

$$J_1(\lambda,y) \to \tilde f(y) \ \text{ in } \ L^q(c,d). \tag{5.7}$$

Thus (5.4) follows from (5.5)–(5.7). □

P r o o f of C o r o l l a r y 4. Let $f \in C(0,\infty)$ and $0 < c \le y \le d$. Find a function $g \in C_0(R)$ such that $g(y) = f(y)$ for $c \le y \le d$. Further we can proceed as in the proof of Corollary 3. Thus we have again (5.4)–(5.7) but now the convergence is uniform for $y \in [c,d]$. □

6. Proof of Theorem 3. We shall use the formula

$$e_\alpha(\lambda,x,y) = \Gamma(\alpha+1)(2\pi i)^{-1}\int_S e^{\lambda p} V(p,x,y) H_\alpha(\lambda,p)\chi(p)dp, \tag{6.1}$$

where $S = \left(\varepsilon - i\frac{\pi}{2}, \varepsilon + i\frac{\pi}{2}\right)$, $\varepsilon > 0$, $\alpha > 0$, $\chi(p) \in C_0^\infty(S)$ and $s \to H_\alpha(s,p)$ is defined by

$$H_\alpha(s,p) = \sum_{k=-\infty}^{+\infty} e^{isk\pi/2}(p + ik\pi/2)^{-\alpha-1}, \quad p \in S, \quad \alpha > 0. \tag{6.2}$$

For proving (6.1) we notice that

$$e_\alpha(\lambda,x,y) = \lambda^\alpha I^\alpha e(\lambda,x,y) = \lambda_+^\alpha * de(\lambda,x,y)$$

and that the Laplace transform of λ_+^α is $\Gamma(\alpha+1)p^{-\alpha-1}$. Thus

$$\int_0^\infty e^{-\lambda p} e_\alpha(\lambda,x,y)d\lambda = \Gamma(\alpha+1)p^{-\alpha-1}V(p,x,y),$$

where

$$V(p,x,y) = \int_0^\infty e^{-\lambda p} de(\lambda,x,y), \quad \operatorname{Re} p > 0.$$

Since

$$V(p,x,y)=\sum e^{-\mu_k p}\phi_k^\delta(x)\phi_k^\delta(y),\quad \mu_k=4k+4$$

we have

$$V\left(p+ik\frac{\pi}{2},x,y\right)=V(p,x,y). \tag{6.3}$$

Further the inverse Laplace transform gives

$$e_\alpha(\lambda,x,y)=b\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^{\lambda p}p^{-\alpha-1}V(p,x,y)dp,\quad b=\Gamma(\alpha+1)(2\pi i)^{-1}$$

or using (6.2), (6.3) we get for $\alpha>0$,

$$e_\alpha(\lambda,x,y)=b\int_{S_1} e^{\lambda p}V(p,x,y)H_\alpha(s,p)dp, \tag{6.4}$$

where $S_1=\left(\varepsilon-i\frac{\pi}{4},\varepsilon+i\frac{\pi}{4}\right)$.

Noticing that $p\to g(p)=e^{\lambda p}V(p,x,y)H_\alpha(\lambda,p)$ is $i\frac{\pi}{2}$ - periodic function it is not hard to see that (6.4) implies (6.1) for some $\chi\in C_0^\infty(S)$, $\chi=1$ near $\varepsilon+i0$.

Now, we can write

$$e_\alpha(\lambda,x,y)=A_\lambda(x,y)+B_\lambda(x,y), \tag{6.5}$$

where

$$A_\lambda(x,y)=b\int_S e^{\lambda p}V(p,x,y)p^{-\alpha-1}\chi(p)dp,$$
$$B_\lambda(x,y)=b\int_S e^{\lambda p}V(p,x,y)h_\alpha(\lambda,p)\chi(p)dp$$

and the function $h_\alpha(\lambda,p)$ has no singularities on S. Further let the function $f(x)\in C_0^\infty(0,\infty)$ and consider the formula

$$\int_0^\infty e_\alpha(\lambda,x,y)f(x)dx=b\int e^{\lambda p}H_\alpha(\lambda,p)\chi(p)\left(\int_0^\infty V(p,x,y)f(x)dx\right)dp.$$

We want to take limit as $\varepsilon\to 0$. To this end we write

$$I(\lambda,y)=\int_0^\infty A_\lambda(x,y)f(x)dx=b\int e^{i\lambda t}V(it,y)(it+0)^{-\alpha-1}i\chi(t)dt,$$

where $\chi(t)\in C_0^\infty\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, $\chi(t)=1$, $|t|<\gamma$ for some $\gamma>0$ and $V(it,y)=\int_0^\infty V(it,x,y)f(x)dx$ is a smooth function. Since in the sense of distributions

$$\left((it+0)^{-\alpha-1},\varphi(t)\right)=\lim_{\varepsilon_1\to 0}C_1\int\int e^{-\varepsilon_1\eta^2-it\eta^2}\varphi(t)\eta^{2\alpha}dtd\eta,\quad \eta\in R^2,$$

where

$$\varphi(t) = ibe^{i\lambda t}V(it,y)\chi(t) \quad \text{and} \quad C_1 = \frac{1}{\pi\Gamma(\alpha+1)} \quad \text{we get}$$

$$I(\lambda,y) = \lim_{\varepsilon_1\to 0} ibC_1 \int e^{i\lambda t}V(it,y)e^{-i\eta^2 t-\varepsilon_1\eta^2}\chi(t)dtd\eta.$$

To represent $V(it,y)$ we shall use the generating function 1.1 46 [10]:

$$V(p,x,y) = (xy)^{1/2}e^{2p(\delta-1)}(\sinh 2p)^{-1}e^{-\left(\frac{x^2+y^2}{2}\right)\coth 2p}i^{-\delta}J_\delta\left(\frac{ixy}{\sinh 2p}\right).$$

Using the formula (1), p. 74, (6), p. 75 and 3, 4, p. 168 from [10] we can write

$$J_\delta(z) = z^{-1/2}(e^{iz}C_\delta^+ f(-z) + e^{-iz}C_\delta^- f(z)) \quad \text{if} \ \ \delta \geq -\frac{1}{2}, \tag{6.6}$$

where

$$f(z) = \begin{cases} \dfrac{1}{2}\left(\dfrac{2}{\pi}\right)^{\frac{1}{2}} \dfrac{1}{\Gamma(\delta+1/2)} \displaystyle\int_0^\infty e^{-u}u^{\delta-\frac{1}{2}}\left(1-\frac{iu}{2z}\right)^{\delta-\frac{1}{2}} du, & \delta > -\dfrac{1}{2} \\ \dfrac{1}{2}\left(\dfrac{2}{\pi}\right)^{\frac{1}{2}}, \quad \delta = -\dfrac{1}{2} \end{cases}$$

is a holomorphic function for $\operatorname{Re} z \neq 0$. Here $C_\delta^{\pm} = e^{\mp i\frac{\pi}{2}(\delta+\frac{1}{2})}$.

Note also the property for $f(t,u) = f(1/u\sin 2t)$,

$$\partial_t^k f(t,u)| \leq C_k \quad \text{uniformly in} \ \ u \in (0,c). \tag{6.7}$$

Therefore

$$\begin{aligned} V(p,x,y) \;\; = \;\; & (\sinh 2p)^{-1/2}e^{-\frac{(x^2+y^2)}{2}\coth 2p}(e^{xy/\sinh 2p}a(p,xy)+ \\ & +C_\delta e^{-xy/\sinh 2p}a(p,-xy)), \end{aligned} \tag{6.8}$$

where $C_\delta = e^{-\frac{i\pi}{2}(\delta+\frac{1}{2})}C_\delta^+$ and $a(p,x,y) = e^{2p(\delta-1)}f\left(\dfrac{ixy}{\sinh 2p}\right)$.

Now, since $-\dfrac{1}{2}(x^2+y^2)\coth 2p + \dfrac{xy}{\sinh 2p} = -\dfrac{(x-y)^2}{4p} + s(p,x,y)$, $s(0,x,y) = 0$ and s has no singularities as $|\operatorname{Im} p| < \dfrac{\pi}{2}$ we get

$$V(p,x,y) =$$
$$= (\sinh 2p)^{-1/2}\left(e^{-(x-y)^2/4p}b(p,x,y) + C_\delta e^{-(x+y)^2/4p}b(p,x,-y)\right)$$

where $b(p,x,y) = e^{s(p,x,y)}a(p,xy)$, $b(0,x,y) = (2\pi)^{-1/2}$.

Now using the equality

$$\lim_{\varepsilon_2\to 0}\int e^{-i\xi^2 t+i(x-y)\xi-\varepsilon_2\xi^2}\frac{d\xi}{2\pi} = \begin{cases} (4\pi it)^{-1/2}e^{-\frac{i(x-y)^2}{4t}}, & t\neq 0\\ \delta(x-y), & t=0\end{cases}$$

we obtain in $D'(R_+)$

$$V(it,x,y = \lim_{\varepsilon_2\to 0}[G(t,x,y,\varepsilon_2)+C_\delta G(t,x,-y,\varepsilon_2)],$$

where

$$G(t,x,y,\varepsilon_2) = \left(\frac{\sin 2t}{2t}\right)^{-1/2}(2\pi)^{-1/2}b(it,x,y)\int e^{-i\xi^2+i(x-y)\xi-\varepsilon_2\xi^2}d\xi.$$

Hence

$$I(\lambda,y) = J(\lambda,y)+C_\delta J(\lambda,-y), \tag{6.9}$$

where

$$\begin{aligned} J(\lambda,y) &= \lim_{\varepsilon_2\to 0}\int e^{-\varepsilon_1\eta^2+i\lambda t-i\eta^2 t}g_1(t,\eta,x,y)\times\\ &\times\lim_{\varepsilon_2\to 0}\int_0^\infty f(x)\left(\int e^{-i\xi^2 t+i(x-y)\xi-\varepsilon_2\xi^2}d\xi\right)dxdtd\eta\end{aligned}$$

and

$$g_1(t,\eta,x,y) = b(it,x,y)\left(\frac{\sin 2t}{2t}\right)^{-1/2}(2\pi)^{-1/2}\frac{1}{2\pi^2}\chi(it)\eta^{2\alpha}.$$

Since $f(x)\in C_0^\infty(0,\infty)$ and $h(\xi) = \int_0^\infty f(x)e^{ix\xi}dx$ is rapidly decreasing, the integral $h_{\varepsilon_2}(t) = \int e^{-i\xi^2 t-iy\xi-\varepsilon_2\xi^2}h(\xi)d\xi$ is absolutely convergent, and $|h_{\varepsilon_2}(t)|\leq\int|h(\xi)|d\xi$. Hence the Lebesgue theorem gives

$$J(\lambda,y) = \lim_{\substack{\varepsilon_1\to 0\\ \varepsilon_2\to 0}} G(\lambda,y,\varepsilon_1,\varepsilon_2),$$

where

$$G(\lambda,y,\varepsilon_1,\varepsilon_2) = \int_0^\infty\left[\int e^{i\lambda t-i\xi^2 t-i\eta^2 t+i(x-y)\xi-\varepsilon_1\eta^2-\varepsilon_2\xi^2}g_1(t,\eta,x,y)dtd\xi d\eta\right]f(x)dx$$

or

$$J(\lambda, y) = \lambda^{\alpha+3/2} \left[\int e^{i\lambda\psi} g(\xi,\eta) g_1(t,\eta,x,y) dt d\xi d\eta \right] f(x)dx + \\ + \lim_{\substack{\varepsilon_1\to 0\\ \varepsilon_2\to 0}} \int_0^\infty I_{\varepsilon_1\varepsilon_2}(\lambda,x,y) f(x)dx,$$

where

$$I_{\varepsilon_1\varepsilon_2}(\lambda,x,y) = \lambda^{\alpha+3/2} \int e^{i\lambda\psi_{\varepsilon_1\varepsilon_2}} g_1(t,\eta,x,y)(1-g(\xi,\eta)) dt d\xi d\eta,$$

$0 < a \le x \le b$, $0 < c \le y \le d$, $\operatorname{supp} f \subset [a,b]$,

$$\psi_{\varepsilon_1\varepsilon_2} = (1-\xi^2-\eta^2)t + \lambda^{-1/2}(x-y)\xi - \varepsilon_1\eta^2 - \varepsilon_2\eta^2, \quad \psi = \psi_{0,0}$$

and $g(\xi,\eta)$ is a cutoff function .

In the last integral we can integrate by parts. Namely, since $|\partial_t \psi_{\varepsilon_1\varepsilon_2}| > C(\xi^2 + \eta^2)$ for $\xi^2+\eta^2 > C_1$, then

$$|I_{\varepsilon_1\varepsilon_2}(\lambda,x,y)| \le \lambda^{-N+\alpha+3/2} \int |\partial_t^N g_1(t,\eta,x,y)| (\xi^2+\eta^2)^{-N}(1-g(\xi,\eta)) dt d\xi d\eta.$$

Using (6.7) we get $|\partial_t^N g_1(t,\eta,x,y)| \le C_N$. Hence,

$$|I_{\varepsilon_1\varepsilon_2}(\lambda,x,y)| \le C_N \lambda^{-N+\alpha+3/2} \text{ if } a \le x \le b, \quad 0 < c \le y \le d.$$

Finally we see that

$$J(\lambda,y) = \lambda^{\alpha+3/2} J_1(\lambda,y) + O(\lambda^{-\infty}), \tag{6.10}$$

where

$$J_1(\lambda,y) = \int e^{i\lambda\psi_1} g(\xi,\eta) g_1(t,\eta,x,y) dt d\xi d\eta \tag{6.11}$$

and

$$\psi_1(t,\xi,\eta,x,y) = (1-\xi^2-\eta^2)t + \lambda^{-1/2}(x-y)\xi. \tag{6.12}$$

The second term in (6.5), $B_\lambda(x,y)$, can by treated analogously. Thus (6.5), (6.10) give the basic formula

$$e_\alpha(\lambda,x,y) = F_\alpha(\lambda,x,y) + C_\delta F_\alpha(\lambda,x,-y) + O(\lambda^{-\infty}), \quad \delta \ge -\frac{1}{2}, \tag{6.13}$$

$$0 < a \le x \le b, \quad 0 < c \le y \le d$$

where

$$F_\alpha(\lambda,x,y)=\lambda^{\alpha+3/2}J_1(\lambda,y)+\lambda^{1/2}J_2(\lambda,y), \tag{6.14}$$

$J_1(\lambda,y)$ is given by (6.11), while $J_2(\lambda,y)=\int e^{i\lambda\psi_2}g(\xi)g_2(t,x,y)dtd\xi$ and ψ_1 is the function (6.12), while $\psi_2(t,\xi,x,y)=\psi_1(t,\xi,0,x,y)$.

Asymptotics of J_1, J_2. Since in polar coordinates $(\xi,\eta)=\sigma(w,\theta)$, $w\in R^1$, $\sigma>0$, $w^2+\theta^2=1$, ($w=\cos\varphi$, $\theta_1=\sin\varphi\cos\varphi_1$, $\theta_2=\sin\varphi\sin\varphi_1$, $0<\varphi<\pi$, $0<\varphi_1<2\pi$),

$$I=\int_{w^2+\theta^2=1}e^{i\sqrt{\lambda}(x-y)w\sigma}\theta^{2\alpha}d(w,\theta)=2\pi\int_{w^2<1}e^{i\lambda(x-y)w\sigma}(1-w^2)^\alpha dw,$$

then

$$I=C_\alpha(\sqrt{\lambda}|x-y|\sigma)^{-1/2-\alpha}J_{1/2+\alpha}(\sqrt{\lambda}|x-y|\sigma), \tag{6.15}$$

$C_\alpha=(2\pi)^{3/2}2^\alpha\Gamma(\alpha+1)$.

Therefore

$$J_1(\sqrt{\lambda}|x-y|)^{-1/2-\alpha}\int_0^\infty\int e^{i\lambda(1-\sigma^2)t}\sigma^{\alpha+3/2}J_{\alpha+1/2}(\sqrt{\lambda}|x-y|\sigma)q(t,\sigma)dtd\sigma,$$

$$q(0,1)=(2\pi)^{-3/2}2^{\alpha+1}\Gamma(\alpha+1). \tag{6.16}$$

For $\sigma\approx 0$ we can integrate by parts with respect to t, and hence we can suppose $q(t,\sigma)\in C_0^\infty(R\times(0,\infty))$. If $\sqrt{\lambda}|x-y|>1$ we use the formula (6.6). Thus we get

$$\begin{gathered}J_{\alpha+1/2}(\sqrt{\lambda}|x-y|\sigma)=(\sqrt{\lambda}|x-y|\sigma)^{-1/2}\times\\ \times\left[e^{-i\sqrt{\lambda}(x-y)\sigma}g(\sqrt{\lambda}|x-y|\sigma)+e^{i\sqrt{\lambda}|x-y|\sigma}g(-\sqrt{\lambda}|x-y|\sigma)\right],\end{gathered} \tag{6.17}$$

where

$$|\partial_\sigma^k g(\sqrt{\lambda}|x-y|\sigma)\le C_k \quad \text{if} \quad \sqrt{\lambda}|x-y|>1,\quad 0<C_1\le\sigma\le C_2.$$

Hence

$$J_1=(\sqrt{\lambda}|x-y|)^{-\alpha-1/2}[K_1+K_2+O(\lambda^{-\infty})],$$

$$K_1(\sqrt{\lambda}|x-y|)^{-1/2}\int e^{i\lambda\psi_i}q_i(t,\sigma,x,y)dtd\sigma,$$

$$\psi_1=(1-\sigma^2)t-|x-y|\lambda^{-1/2}\sigma,\quad \psi_2=(1-\sigma^2)t+|x-y|\lambda^{-1/2}\sigma,$$

$$|\partial^k q_i| \le C_k \quad \text{if} \quad \sqrt{\lambda}|x-y| > 1, \quad (t,\sigma) \to q_i \in C_0^\infty(R \times (0,\infty)).$$

To find the asymptotics of the integrals K_i we apply the stationary phase method [1].

The critical points $\sigma = 1$, $t_\pm = \pm\frac{1}{2}|x-y|\lambda^{-1/2}$ are nondegenerate and the Taylor formula gives.

(6.18) $$q(t_\pm, 1) = q(0,1) + |x-y|O(\lambda^{-1/2})$$

Therefore

$$K_1 = (\sqrt{\lambda}|x-y|)^{-1/2}\left[\frac{2\pi}{\lambda}\cdot\frac{1}{2}e^{-i\sqrt{\lambda}|x-y|}g(\sqrt{\lambda}|x-y|)q(t_+,1) + O(\lambda^{-2})\right]$$

$$K_2 = (\sqrt{\lambda}|x-y|)^{-1/2}\left[\frac{2\pi}{\lambda}\cdot\frac{1}{2}e^{i\sqrt{\lambda}|x-y|}g(-\sqrt{\lambda}|x-y|)q(t_-,1) + O(\lambda^{-2})\right]$$

or according to (6.16), (6.18)

(6.19) $$J_1 = (\sqrt{\lambda}|x-y|)^{-\frac{1}{2}-\alpha}\left[\frac{d_\alpha}{\lambda}J_{\alpha+\frac{1}{2}}(\sqrt{\lambda}|x-y| + O(\lambda^{-7/4}))\right] \quad \text{if} \quad \sqrt{\lambda}|x-y| > 1,$$

where d_α is given by (1.6).

Consider now the case $\sqrt{\lambda}|x-y| < 1$. Then analogously to (6.15) we get

$$J_1 = \int_0^\infty \int e^{i\lambda(1-\sigma^2)t} g(t,\sigma,\lambda)\,dt\,d\sigma + O(\lambda^{-\infty}),$$

where

$$g(t,\sigma,\lambda) = \int_{w^2<1}(1-w^2)^\alpha e^{i\sqrt{\lambda}(x-y)w\sigma}dw\,g_1(t,\sigma),$$

$$g_1(t,\sigma) \in C_0^\infty(R\times(0,\infty)), \quad g_1(0,1) = \frac{1}{2\pi^2}$$

and the method of stationary phase gives

(6.20) $$J_1 = \frac{1}{2\lambda\pi}\int_{w^2<1}(1-w^2)^\alpha e^{i\lambda(x-y)w\sigma}dw + O(\lambda^{-2}),$$

(6.21) $$J_1(\sqrt{\lambda}|x-y|)^{-\alpha-1/2}d_\alpha\lambda^{-1}J_{\alpha+1/2}(\sqrt{\lambda}|x-y|) + O(\lambda^{-2}) \quad \text{if} \quad \sqrt{\lambda}|x-y| < 1.$$

By the same method of stationary phase we have

(6.22) $$J_2(\lambda, x, y) = O(\lambda^{-1}).$$

Thus (6.14), (6.19), (6.21) and (6.22) imply

$$F_\alpha(\lambda,x,y)=d_\alpha\lambda^{1/2+\alpha}(\sqrt{\lambda}|x-y|)^{-1/2-\alpha}J_{\alpha+1/2}(\sqrt{\lambda}|x-y|)+R,$$

where

$$|R|\le\begin{cases}C(\sqrt{\lambda}|x-y|)^{-1/2-\alpha}\lambda^{\alpha-1/4} & \text{if}\quad \sqrt{\lambda}|x-y|>1\\ C\lambda^{\alpha-1/2} & \text{if}\quad \sqrt{\lambda}|x-y|<1.\end{cases}$$

This and (6.13), (2.1) give (2.2). Theorem (1) is completely proved.

7. Proof of Theorem 4. Starting with (6.1) and using (6.2) we can write

$$E_\alpha(\lambda,x,y)=E_1(\lambda,x,y)+E_2(\lambda,x,y),$$

$$E_i(\lambda,x,y)=b\int e^{\lambda p}V(p,\sqrt{\lambda}x,\sqrt{\lambda}y)H_\alpha(\lambda,p)K_i(p)dp,\quad i=1,2, \tag{7.1}$$

where for some $\gamma_1>0$,

$$K_i\in C_0^\infty(S),\quad \operatorname{supp}K_1(p)\subset\{|\operatorname{Im}p|<\gamma_1\},\quad K_1(p)=1\quad\text{for}\quad |\operatorname{Im}p|<\gamma<\gamma_1.$$

Further, analogously to the proof of Theorem 3,

$$E_1(\lambda,x,y)=A_\lambda(x,y)+B_\lambda(x,y),$$

where

$$A_\lambda(x,y)=b\int e^{\lambda p}V(p,\sqrt{\lambda}x,\sqrt{\lambda}y)p^{-\alpha-1}K_1(p)dp$$

and

$$B_\lambda(x,y)=b\int e^{\lambda p}V(p,\sqrt{\lambda}x,\sqrt{\lambda}y)h_\alpha(\lambda,p)K_1(p)dp.$$

Now instead of (6.8) we shall use

$$V(it,\sqrt{\lambda}x,\sqrt{\lambda}y)=\lim_{\varepsilon_2\to0}\Big[G(t,\sqrt{\lambda}x,\sqrt{\lambda}y,\varepsilon_2)+C_\delta G(t,\sqrt{\lambda}x,-\sqrt{\lambda}y,\varepsilon_2)\Big]$$

where

$$G(t,x,y,\varepsilon_2)=(4\pi)^{-1/2}e^{-\frac{i(x^2+y^2)}{2}\sin t}\int e^{-i\xi^2\frac{\sin 2t}{2}+i(x-y)\xi-\varepsilon_2\xi^2}a(it,x,y)d\xi.$$

Since in the sense of distributions

$$(i\sin 2t+o)^{-\alpha-1}=\lim_{\varepsilon_1\to0}C_\alpha\int e^{-\varepsilon_1\eta^2-i\eta^2\frac{\sin 2t}{2}}\eta^{2\alpha}d\eta,\quad \eta\in R^2,$$

we obtain, analogously to (6.13), (6.14),

$$E_1(\lambda, x, y) = F_{1\alpha}(\lambda, x, y) + C_\delta F_{1\alpha}(\lambda, x, -y) + O(\lambda^{-\infty}),$$

where

$$F_{1\alpha}(\lambda, x, y) = \lambda^{\alpha+3/2} I_1(\lambda, x, y) + \lambda^{1/2} I_2(\lambda, x, y), \tag{7.2}$$

$$I_1(\lambda, x, y) = \int e^{i\lambda\psi(t,x,y,\eta,\xi)} q_1(t, \lambda, x, y)\eta^{2\alpha} g_1(\xi, \eta) dt d\xi d\eta,$$

$$I_2(\lambda, x, y) = \int e^{i\lambda\psi(t,x,y,0,\xi)} q_2(t, \lambda, x, y) g_1(\xi) dt d\xi$$

and

$$q_1(t, \lambda, x, y) = \left(\frac{\sin 2t}{2t}\right)^{-\alpha-1} a(it, \lambda, x, y)(2\pi)^{\alpha+1}\sqrt{\pi} K_1(it),$$

$$q_2(t, \lambda, x, y) = h_\alpha(\lambda, t) a(it, \lambda, x, y)(2\pi)^{-3/2} 2^{-1/2} K_1(it),$$

$$\psi(t, x, y, \eta, \xi) = t - \frac{(\eta^2 + \xi^2)}{2}\sin 2t + (x - y)\xi - \frac{(x^2 + y^2)}{2}\sin t,$$

$g_1(\xi, \eta)$ and $g_1(\xi)$ are cutoff functions.

We can represent E_2 as follows:

$$E_2(\lambda, x, y) = F_{2\alpha}(\lambda, x, y) + C_\delta F_{2\alpha}(\lambda, x, -y),$$

where

$$F_{2\alpha}(\lambda, x, y) = b\int e^{i\lambda\varphi} q(t, \lambda, x, y) dt,$$

$$\varphi(t, y, x) = t + \frac{(x^2 + y^2)}{2}\cot 2t - \frac{xy}{\sin 2t} \tag{7.3}$$

and

$$q(t, \lambda, x, y) = (i\sin 2t)^{-1/2} H_\alpha(\lambda, it) iba(it, \lambda xy) K_2(it).$$

Note that $t \to q \in C_0^\infty\left(0 < |t| < \frac{\pi}{2}\right)$.

To find the uniform asymptotics of the integrals I_i $(i = 1, 2)$ in the domain $\left\{(x, y) \in R^2,\ \frac{A}{\sqrt{\lambda}} < x < 1 - \varepsilon,\ 0 < \frac{c}{\sqrt{\lambda}} \le y < \frac{d}{\sqrt{\lambda}}\right\}$ we shall apply the method of the stationary phase.

Asymptotics of I_1. Analogously to (6.20) we have

$$I_1 = (\lambda|x - y|)^{-1/2-\alpha}\int_0^\infty \int e^{i\lambda\psi_0}\sigma^{3/2+\alpha} J_{1/2+\alpha}(\lambda|x - y|\sigma) q(t, \sigma) dt d\sigma + O(\lambda^{-\infty}),$$

where

$$\psi_0 = t - \frac{\sigma^2}{2}\sin 2t - \frac{(x^2+y^2)}{2}\tan t, \quad q \in C_0^\infty(R \times (0,\infty)).$$

Let $s = \lambda|x-y|\sigma$. The asymptotics of $J_{1/2+\alpha}(s)$ gives

$$J_{1/2+\alpha}(s) = \sum_{k=0}^{2} s^{-1/2-k}(C_k e^{is} = \overline{C}_k e^{-is}) + O(s^{-7/2}). \tag{7.4}$$

Since $|x-y| \geq c|x| > c\lambda^{-1/2}$, then (7.2) and (7.4) imply

$$I_1(\lambda|x-y|)^{-1-\alpha}\sum_{k=0}^{2}(\lambda|x-y|)^{-k}M_k + x^{-1-\alpha}O(\lambda^{-5/2-\alpha}), \tag{7.5}$$

where

$$M_k = \int_0^\infty \int e^{i\lambda\psi}\sigma^{1+\alpha-k}q_k(t,\sigma)dtd\sigma, \quad q_k \in C_0^\infty(R\times(0,\infty)) \tag{7.6}$$

and

$$\psi = t - \frac{\sigma^2}{2}\sin 2t - \frac{(x^2+y^2)}{2}\tan t \pm |x-y|\sigma. \tag{7.7}$$

The critical points (t_i, σ_i) of ψ satisfy

$$\det\psi'' = \pm 4d, \quad d = \sqrt{(1-x^2)(1-y^2)} \tag{7.8}$$

$$\cos 2t_i = xy + (-1)^{i+1}d \quad (i=1,2), \quad t_3 = -t_1, \quad t_4 = -t_2, \quad \sigma_i \sin 2t_i = \pm|x-y| \tag{7.9}$$

and these critical points are nondegenerate for $x < 1-\varepsilon$, $y < 1-\varepsilon$.

Thus the method of the stationary phase gives

$$\begin{gathered} M_0 = \lambda^{-1}\sum_{i=1}^{4} e^{\lambda\psi_i}C_i(\lambda,x,y) + O(\lambda^{-2}), \quad |C_i(\lambda,x,y)| \leq C, \\ M_k = O(\lambda^{-1}), \quad k=1,2, \end{gathered} \tag{7.10}$$

$$\psi_i(x,y) = \psi(t_i,\sigma_i) = \varphi(t_i,x,y). \tag{7.11}$$

Then (7.5) and (7.10) imply

$$I_1 = \lambda^{-2-\alpha}\sum_{i=1}^{4} e^{i\lambda\psi_i}\tilde{b}_{1i}(\lambda,x,y) + x^{-1-\alpha}O(\lambda^{-5/2-\alpha}), \tag{7.12}$$

$$\tilde{b}_{1i}(\lambda, x, y) = |x-y|^{-1-\alpha} C_i(\lambda, x, y). \tag{7.13}$$

Asymptotics of I_2. The critical points of the phase function $\psi(t, x, y, 0, \xi)$ satisfy (7.9), where $\xi_i \sin 2t_i = x - y$. Therefore the stationary phase method gives

$$I_2 = \lambda^{-1} \sum_{i=1}^{4} e^{i\lambda\psi_i} \tilde{b}_{2i}(\lambda, x, y) + O(\lambda^{-2}), \tag{7.14}$$

where

$$\psi_i(x, y) = \psi(t_i, x, y, 0, \xi_i) \tag{7.15}$$

and

$$|\tilde{b}_{2i}| \le C, \quad |\partial_x \tilde{b}_{2i}| \le C. \tag{7.16}$$

Now (7.2), (7.12) and (7.14) show that

$$F_{1\alpha}(\lambda, x, y) = \lambda^{-1/2} \sum_{i=1}^{4} e^{i\lambda\psi} b_{1i}(\lambda, x, y) + x^{-1-\alpha} O(\lambda^{-1}), \tag{7.17}$$

where ψ_i satisfy (7.11), (7.15) and b_{1i} according to (7.13), (7.16) satisfy (2.5). To find the asymptotics of $F_{2\alpha}$, we first notice that the critical points t_i of the phase function $\varphi(t, x, y)$ given by (7.3) satisfy (7.9) and $\varphi''(t, x, y) = (-1)^{i+1} 4d(\sin 2t_i)^{-1}$, $1 \le i \le 4$. Thus the critical points are nondegenerate and

$$\begin{aligned} &F_{2\alpha}(\lambda, x, y) = \lambda^{-1/2} \sum_{i=1}^{4} e^{i\lambda\psi} b_{2i}(\lambda, x, y) + O(\lambda^{-3/2}), \\ &|b_{2i}| + |\partial_x b_{2i}| \le C, \quad \psi_i(x, y) = \varphi(t_i, x, y). \end{aligned} \tag{7.18}$$

Evidently (7.17) and (7.18) give (2.4).

8. Proof of Theorem 5. Starting with (6.4) and (2.1) we get formula (2.3), where

$$F_\alpha(\lambda, x, y) = \int_{S_1} e^{\lambda\varphi} q(p, \lambda) dp,$$

$$\varphi(p) = p - 2^{-1}(x^2 + y^2) \coth p + \frac{xy}{\sinh 2p},$$

$$q(p, \lambda) = b(\sinh 2p)^{-1/2} H_\alpha(\lambda, p) e^{2p(\delta-1)} f\left(\frac{i\lambda xy}{\sinh 2p}\right), \quad S_1 = \left(\varepsilon_1 - i\frac{\pi}{4}, \varepsilon_1 + i\frac{\pi}{4}\right), \quad \varepsilon_1 > 0.$$

Now we can apply the same method as in [3], which gives the asymptotics (2.7).

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